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Driven Classical System and Characteristic Functions

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A general dynamical equation of motion for the characteristic function associated with the phase density is derived from the Liouville equation. A similar procedure is applied to the linearized Liouville equation for a driven classical system. A general equation for the dynamical motion of the perturbed part of the characteristic function is obtained in which driving and relaxing terms are explicitly considered. This formalism is applied to the calculation of the electrical conductivity of a dilute gas of free, charged particles under the influence of a steady electric field and interacting with randomly distributed fixed scatterers.

INTRODUCTION

THE Liouville equation for a classical system with Hamiltonian H and phase density ρ may be written

$$(\partial\rho/\partial t)=[H,\rho], \tag{1.1}$$

where [ ] represents the Poisson bracket. In order to simplify our presentation, we confine ourselves to a single set of canonical variables $\mathbf{p}$ and $\mathbf{q}$, i.e., a one-particle problem. Equation (1.1) may be rewritten in the form

$$(\partial\rho/\partial t)=(\partial H/\partial\mathbf{q})\cdot(\partial\rho/\partial\mathbf{p})-(\partial H/\partial\mathbf{p})\cdot(\partial\rho/\partial\mathbf{q}), \tag{1.2}$$

where $(\partial/\partial\mathbf{q})$, $(\partial/\partial\mathbf{p})$ represent the gradient with respect to $\mathbf{q}$ and $\mathbf{p}$, respectively.

The phase density is so defined that

$$\int \rho d\mathbf{p}d\mathbf{q}=1, \tag{1.3}$$

where the integral is over all coordinate and momentum space, i.e., the limits are $-\infty$ to $+\infty$ for a given component. Some care must be taken in using infinite limits. For example, if ρ is a function of $\mathbf{p}$ alone we would first assume a very large finite volume to establish normalization and then consider infinite volume in the limit. With such reservations, we use the infinite limits on our integrals.

The expectation of the function $f(\mathbf{p},\mathbf{q})$ is

$$\langle f(\mathbf{p},\mathbf{q})\rangle=\int f\rho d\mathbf{p}d\mathbf{q}. \tag{1.4}$$

The expectation of the function $\exp[2\pi i(\boldsymbol{\tau}\cdot\mathbf{p}+\boldsymbol{\theta}\cdot\mathbf{q})]$ is known as the characteristic or moment-generating function,[1] $M(\boldsymbol{\tau},\boldsymbol{\theta})$.

$$M(\boldsymbol{\tau},\boldsymbol{\theta})=\int \exp[2\pi i(\boldsymbol{\tau}\cdot\mathbf{p}+\boldsymbol{\theta}\cdot\mathbf{q})]\rho d\mathbf{p}d\mathbf{q}, \tag{1.5a}$$

so that

$$\rho(\mathbf{p},\mathbf{q})=\int \exp[-2\pi i(\boldsymbol{\tau}\cdot\mathbf{p}+\boldsymbol{\theta}\cdot\mathbf{q})]M(\boldsymbol{\tau},\boldsymbol{\theta})d\boldsymbol{\tau}d\boldsymbol{\theta}. \tag{1.5b}$$

[1] See, e.g., M. S. Bartlett, *Stochastic Processes* (Cambridge University Press, New York, 1956), or J. E. Moyal, Proc. Roy. Stat. Soc. **11B**, 150 (1949).

This latter name follows since the process of taking the appropriate derivatives of $M(\boldsymbol{\tau},\boldsymbol{\theta})$ with respect to $2\pi i\boldsymbol{\tau}$ and $2\pi i\boldsymbol{\theta}$, followed by taking the limit of $\boldsymbol{\tau}$, $\boldsymbol{\theta}$ zero, provides the desired moment. For example,

$$\langle p_1 q_2^2\rangle=\lim_{\tau\to 0,\ \theta\to 0}\left[\frac{1}{(2\pi i)^3}\frac{\partial}{\partial\tau_1}\frac{\partial^2}{\partial\theta_2^2}M(\boldsymbol{\tau},\boldsymbol{\theta})\right], \tag{1.6}$$

where the subscripts are vector-component labels.

It is frequently useful to consider an intermediate characteristic function such as

$$F_\theta(\mathbf{p})=\int d\mathbf{q}\,\exp(2\pi i\boldsymbol{\theta}\cdot\mathbf{q})\rho(\mathbf{p},\mathbf{q}). \tag{1.7}$$

We note that $F_0(\mathbf{p})$ is the momentum distribution function and we may write

$$M(\boldsymbol{\tau},0)=\int \exp(2\pi i\boldsymbol{\tau}\cdot\mathbf{p})F_0(\mathbf{p})d\mathbf{p}. \tag{1.8}$$

Clearly, if we are interested only in the moments of $\mathbf{p}$, we may confine our attention to $M(\boldsymbol{\tau},0)$.

Similarly, we may define

$$\mathfrak{F}_\tau(\mathbf{q})=\int d\mathbf{p}\,\exp(2\pi i\boldsymbol{\tau}\cdot\mathbf{p})\rho(\mathbf{p},\mathbf{q}). \tag{1.9}$$

Here $\mathfrak{F}_0(\mathbf{q})$ is the spatial distribution function and

$$M(0,\boldsymbol{\theta})=\int d\mathbf{q}\,\exp(2\pi i\boldsymbol{\theta}\cdot\mathbf{q})\mathfrak{F}_0(\mathbf{q}) \tag{1.10}$$

generates the expectation values of the powers of $\mathbf{q}$.

The dynamical equation of motion for the characteristic function may be obtained by multiplying the Liouville equation, Eq. (1.2), by $\exp[2\pi i(\boldsymbol{\tau}\cdot\mathbf{p}+\boldsymbol{\theta}\cdot\mathbf{q})]$ and integrating over all coordinate and momentum space.

$$(\partial M/\partial t)=\int d\mathbf{p}d\mathbf{q}\,\exp[2\pi i(\boldsymbol{\tau}\cdot\mathbf{p}+\boldsymbol{\theta}\cdot\mathbf{q})]$$
$$\times[(\partial H/\partial\mathbf{q})\cdot(\partial\rho/\partial\mathbf{p})-(\partial H/\partial\mathbf{p})\cdot(\partial\rho/\partial\mathbf{q})]. \tag{1.11}$$

We denote $(\partial H/\partial\mathbf{q})$ by $\mathbf{g}_1(\mathbf{p},\mathbf{q})$ and $(\partial H/\partial\mathbf{p})$ by $\mathbf{g}_2(\mathbf{p},\mathbf{q})$.

Then using Eq. (1.5b) and the properties of the Dirac delta function, we obtain

$$\frac{\partial M(\boldsymbol{\tau},\boldsymbol{\theta},t)}{\partial t}=-2\pi i\mathbf{g}_1\left(\frac{1}{2\pi i}\frac{\partial}{\partial\boldsymbol{\tau}},\frac{1}{2\pi i}\frac{\partial}{\partial\boldsymbol{\theta}}\right)\cdot\boldsymbol{\tau}M(\boldsymbol{\tau},\boldsymbol{\theta},t)$$
$$+2\pi i\mathbf{g}_2\left(\frac{1}{2\pi i}\frac{\partial}{\partial\boldsymbol{\tau}},\frac{1}{2\pi i}\frac{\partial}{\partial\boldsymbol{\theta}}\right)\cdot\boldsymbol{\theta}M(\boldsymbol{\tau},\boldsymbol{\theta},t),\quad(1.12)$$

where the $\mathbf{g}_j[(1/2\pi i)(\partial/\partial\boldsymbol{\tau}),\ (1/2\pi i)(\partial/\partial\boldsymbol{\theta})]$ are vector operators which are obtained from the vector functions $\mathbf{g}_j(\mathbf{p},\mathbf{q})$ by replacing $\mathbf{p}$ by $(1/2\pi i)(\partial/\partial\boldsymbol{\tau})$ and $\mathbf{q}$ by $(1/2\pi i)(\partial/\partial\boldsymbol{\theta})$. Equation (1.12) provides the basis for the determination of M at any time and, subsequently, the determination of the moments of $\mathbf{p}$ and $\mathbf{q}$ at any time.

In the next section we consider generally the dynamical motion of M for a driven system using perturbation theory. In the third section we apply these general considerations to determine the electrical conductivity of a dilute gas of freely moving charged particles under the influence of an external electric field and interacting with randomly distributed fixed scatterers.

DRIVEN CLASSICAL SYSTEM

Consider a classical system with Hamiltonian H such that

$$H=H_0{}^0+\lambda H_R+H_D=H_0+H_D.\quad(2.1)$$

Here H_0 is the undriven Hamiltonian which we have explicitly split up into two parts. $H_0{}^0$ is the unperturbed part which is most important in the discussion of equilibrium properties of the undriven system and λH_R is the relaxation perturbation which is usually ignored in the discussion of equilibrium properties of the undriven system. The driving term of the Hamiltonian H_D is time dependent and assumed to be of the form $H_D=\mathbf{A}(\mathbf{p},\mathbf{q})\cdot\mathbf{X}(t)$, where $\mathbf{X}(t)$ is an external force dependent only on time.

Now define

$$L_0\equiv(\partial H_0/\partial\mathbf{q})\cdot(\partial/\partial\mathbf{p})-(\partial H_0/\partial\mathbf{p})\cdot(\partial/\partial\mathbf{q})$$
$$\equiv\mathbf{g}_1{}^0\cdot(\partial/\partial\mathbf{p})-\mathbf{g}_2{}^0\cdot(\partial/\partial\mathbf{q}),\quad(2.2)$$

and

$$\Delta L_D\equiv(\partial H_D/\partial\mathbf{q})\cdot(\partial/\partial\mathbf{p})-(\partial H_D/\partial\mathbf{p})\cdot(\partial/\partial\mathbf{q})$$
$$\equiv\Delta\mathbf{g}_1\cdot(\partial/\partial\mathbf{p})-\Delta\mathbf{g}_2\cdot(\partial/\partial\mathbf{q}).\quad(2.3)$$

We assume the disturbance caused by H_D is sufficiently small that the phase density may be written as the sum of an unperturbed part plus a small perturbation, i.e.,

$$\rho=\rho_0+\Delta\rho,\quad(2.4)$$

where $L_0\rho_0=0$. The Liouville equation to the order of linear terms in the driving perturbation becomes

$$(\partial\Delta\rho/\partial t)=L_0\Delta\rho+\Delta L_D\rho_0.\quad(2.5)$$

We introduce the characteristic function

$$M(\boldsymbol{\tau},\boldsymbol{\theta})=\int d\mathbf{p}d\mathbf{q}\,\exp[2\pi i(\boldsymbol{\tau}\cdot\mathbf{p}+\boldsymbol{\theta}\cdot\mathbf{q})][\rho_0+\Delta\rho],\quad(2.6)$$

and its corresponding separation into two parts,

$$M_0=\int d\mathbf{p}d\mathbf{q}\,\exp[2\pi i(\boldsymbol{\theta}\cdot\mathbf{q}+\boldsymbol{\tau}\cdot\mathbf{p})]\rho_0,\quad(2.7a)$$

$$\Delta M=\int d\mathbf{p}d\mathbf{q}\,\exp[2\pi i(\boldsymbol{\theta}\cdot\mathbf{q}+\boldsymbol{\tau}\cdot\mathbf{p})]\Delta\rho.\quad(2.7b)$$

Using Eq. (2.5) and the methods used in Sec. I to derive (1.12), we obtain

$$(\partial\Delta M/\partial t)=g_{0p}\Delta M(\boldsymbol{\tau},\boldsymbol{\theta},t)+\Delta g_{0p}M_0(\boldsymbol{\tau},\boldsymbol{\theta}),\quad(2.8)$$

where

$$g_{0p}=-2\pi i\mathbf{g}_1{}^0[(1/2\pi i)(\partial/\partial\boldsymbol{\tau}),\ (1/2\pi i)(\partial/\partial\boldsymbol{\theta})]\cdot\boldsymbol{\tau}$$
$$+2\pi i\mathbf{g}_2{}^0[(1/2\pi i)(\partial/\partial\boldsymbol{\tau}),\ (1/2\pi i)(\partial/\partial\boldsymbol{\theta})]\cdot\boldsymbol{\theta},\quad(2.9a)$$

and

$$\Delta g_{0p}=-2\pi i\Delta\mathbf{g}_1[(1/2\pi i)(\partial/\partial\boldsymbol{\tau}),\ (1/2\pi i)(\partial/\partial\boldsymbol{\theta})]\cdot\boldsymbol{\tau}$$
$$+2\pi i\Delta\mathbf{g}_2[(1/2\pi i)(\partial/\partial\boldsymbol{\tau}),\ (1/2\pi i)(\partial/\partial\boldsymbol{\theta})]\cdot\boldsymbol{\theta}.\quad(2.9b)$$

Now recall that $g_1{}^0$ and $g_2{}^0$ are derived from H_0 which in turn is made up of $H_0{}^0$ and H_R. This means that Eq. (2.8) while linearized in the driving perturbation, has no similar restrictions on the relaxing perturbation. To remedy this, we write $g_{0p}=g_{0p}{}^0+\lambda g_{0p}{}'$, where $g_{0p}{}^0$ is derived from $H_0{}^0$ and $\lambda g_{0p}{}'$ is derived from λH_R. Equation (2.8) may be rewritten in the form

$$(\partial\Delta M/\partial t)=g_{0p}{}^0\Delta M(\tau,\theta,t)+\lambda g_{0p}{}'\Delta M(\boldsymbol{\tau},\boldsymbol{\theta},t)$$
$$+\Delta g_{0p}M_0(\boldsymbol{\tau},\boldsymbol{\theta}).\quad(2.10)$$

Our procedure then is to solve this equation to lowest order in λ which, in practice, is usually λ^2. To proceed, we treat Eq. (2.10) as an inhomogeneous differential equation with the last two terms on the right-hand side the inhomogeneous contribution. We solve under the assumption that $\mathbf{X}(t)$ is turned on with a Heaviside unit function at some time, say $t=0$, so that

$$\Delta M(\boldsymbol{\tau},\boldsymbol{\theta},t)=\int_0^t dt'\,\exp[(t-t')g_{0p}{}^0][\Delta g_{0p}M_0(\boldsymbol{\tau},\boldsymbol{\theta})]$$
$$+\lambda\int_0^t dt'\,\exp[(t-t')g_{0p}{}^0]g_{0p}{}'\Delta M(\boldsymbol{\tau},\boldsymbol{\theta},t').\quad(2.11)$$

Equation (2.11) is a solution of Eq. (2.10) as may be verified by direct substitution. Using Eq. (2.11), we

may rewrite Eq. (2.10) in the following form:

$$(\partial\Delta M/\partial t)=g_{0p}{}^0\Delta M+\Delta g_{0p}M_0$$
$$+\lambda g_{0p}'\int_0^t dt'\exp[(t-t')g_{0p}{}^0]\Delta g_{0p}(t')M_0$$
$$+\lambda^2 g_{0p}'\int_0^t dt'\exp[(t-t')g_{0p}{}^0]g_{0p}'\Delta M(\boldsymbol{\tau},\boldsymbol{\theta},t'). \quad (2.12)$$

Equation (2.12) is exactly equivalent to Eq. (2.10) but is more convenient for considering the lowest order contributions from the relaxing perturbation. Indeed it frequently happens that the λ term in Eq. (2.12), in which the operators act on M_0, vanishes or is negligible because of the nature of the relaxing potential, i.e., because it is some random function. This is the case for our example in the next section, and we shall assume it so in our general development. Then

$$(\partial\Delta M/\partial t)=g_{0p}{}^0\Delta M+\Delta g_{0p}M_0$$
$$+\lambda^2 g_{0p}'\int_0^t dt'\exp[(t-t')g_{0p}{}^0]g_{0p}'\Delta M(\boldsymbol{\tau},\boldsymbol{\theta},t'). \quad (2.13)$$

Equation (2.13) is our working equation. We solve it to lowest order in λ^2 by iteration and use the result to determine the desired moments for the perturbed part of the phase density. It is difficult to carry out this procedure in a general fashion because of the various possibly noncommuting operators appearing in Eq. (2.13). However, for specific cases in certain long-time limits when we are close to a steady state, such a procedure can be carried out as will be shown by example in the next section.

ELECTRICAL CONDUCTIVITY

We consider a dilute gas of free charged particles subject to a steady electric field turned on at $t=0$ with a Heaviside unit function, and interacting with N fixed scatterers distributed randomly throughout space. The gas is assumed sufficiently dilute that particle-particle interactions may be neglected and a one-particle approximation is justified. The Hamiltonian for the single particle may be written

$$H=(p^2/2m)+\lambda H_R+e\mathbf{E}\cdot\mathbf{q}u(0)=H_0+e\mathbf{E}\cdot\mathbf{q}u(0), \quad (3.1)$$

where

$$\lambda H_R=\lambda\sum_{\mathbf{l}\neq 0}\sum_j V_{\mathbf{l}}\exp[2\pi i\mathbf{l}\cdot(\mathbf{q}-\mathbf{Q}_j)] \quad (3.2)$$

is the scattering or relaxation term. $\mathbf{Q}_j$ is the coordinate of the jth random scatterer and e is the charge on the free particle. The external force is $e\mathbf{E}u(0)$, where $u(0)$ is the Heaviside unit function and $\mathbf{E}$ is a constant vector.

The unperturbed phase density is $\rho_0=e^{-\beta H_0}/Z_0$, where $Z_0=\int d\mathbf{p}d\mathbf{q}\,e^{-\beta H_0}$ and $\beta=1/kT$. We expand $e^{-\beta H_0}$ in powers of λ, i.e.,

$$e^{-\beta H_0}=\exp(-\beta p^2/2m)$$
$$\times\Big\{1+\lambda\sum_{\mathbf{l}\neq 0}\sum_j V_{\mathbf{l}}\exp[2\pi i\mathbf{l}\cdot(\mathbf{q}-\mathbf{Q}_j)]$$
$$+\frac{\lambda^2}{2!}\sum_{\mathbf{l},\mathbf{l}'\neq 0}\sum_{j,k}V_{\mathbf{l}}V_{\mathbf{l}'}\exp[2\pi i\mathbf{l}\cdot(\mathbf{q}-\mathbf{Q}_j)]$$
$$\times\exp[2\pi i\mathbf{l}'\cdot(\mathbf{q}-\mathbf{Q}_k)]+\cdots\Big\}. \quad (3.3)$$

The random distribution of the scatterers may be used to simplify Eq. (3.3). For example the λ term may be written

$$\lambda\sum_{\mathbf{l}\neq 0}V_{\mathbf{l}}\int d\mathbf{Q}\exp[2\pi i\mathbf{l}\cdot(\mathbf{q}-\mathbf{Q})]\sum_j\delta(\mathbf{Q}-\mathbf{Q}_j). \quad (3.4)$$

We note that $\sum_j\delta(\mathbf{Q}-\mathbf{Q}_j)$ is just the density function for the random scatters and $\int d\mathbf{Q}\sum_j\delta(\mathbf{Q}-\mathbf{Q}_j)=N$. As a suitable random density function we assume a Gaussian distribution with a very large isotropic variance σ^2 and mean zero so that the density is nearly a constant. Then

$$N\int d\mathbf{Q}\exp(-2\pi i\mathbf{l}\cdot\mathbf{Q})\exp(-Q^2/2\sigma^2)/(2\pi\sigma^2)^{\frac{3}{2}}$$
$$=N\exp(-2\pi^2\sigma^2 l^2). \quad (3.5)$$

Since $\mathbf{l}\neq 0$ and σ^2 is very large, the expression (3.4) becomes vanishingly small when the right-hand side of Eq. (3.5) is used in the sum over $\mathbf{l}$. We therefore neglect the λ contribution in Eq. (3.3) on the basis of the random distribution of the scatterers. A similar situation holds in the λ^2 term if we assume the j and k summations are uncorrelated. However, when $j=k$ and also $\mathbf{l}+\mathbf{l}'=0$ we obtain a nonvanishing contribution. Thus

$$e^{-\beta H_0}\simeq\exp(-\beta p^2/2m)[1+(\lambda^2/2!)\textstyle\sum_{\mathbf{l}}V_{\mathbf{l}}V_{-\mathbf{l}}]. \quad (3.6)$$

For central forces $V_{\mathbf{l}}=V_{-\mathbf{l}}$, and we note that $\sum_{\mathbf{l}}\rightarrow(1/\Lambda)\int d\mathbf{l}$ in the limit of continuous $\mathbf{l}$, where Λ is the reciprocal volume. Here the integral over $\mathbf{l}$ excludes the value $\mathbf{l}=0$. We assume that $c=N/\Lambda$ remains a constant as both N and Λ are allowed to take infinite values and write

$$e^{-\beta H_0}\simeq\exp(-\beta p^2/2m)\Big[1+(\lambda^2/2!)c\int d\mathbf{l}\,|V_{\mathbf{l}}|^2\Big], \quad (3.7)$$

so that $e^{-\beta H_0}$ remains independent of $\mathbf{q}$ to lowest order in λ. We rewrite

$$\rho_0=(\beta/2\pi m)^{\frac{3}{2}}\exp(-\beta p^2/2m)/\Omega. \quad (3.8)$$

Then

$$M_0(\boldsymbol{\tau},\boldsymbol{\theta})=(1/\Omega)\exp(-2\pi^2 m\tau^2/\beta)\Delta(\boldsymbol{\theta}), \quad (3.9)$$

where $\lim_{\theta\to 0}\Delta(\boldsymbol{\theta})=\Omega$ so that

$$M_0(\boldsymbol{\tau},0)=\exp(-2\pi^2 m\tau^2/\beta). \quad (3.10)$$

Also using Eq. (3.1) and definitions from Sec. II, we have

$$g_{0p}{}^0=(\boldsymbol{\theta}/m)(\partial/\partial\boldsymbol{\tau}), \tag{3.11a}$$

$$g_{0p}{}'=-\lambda\sum_{\mathbf{l}\neq 0}\sum_j(2\pi i)^2V_{\mathbf{l}}\mathbf{l}\cdot\boldsymbol{\tau} \times\exp(2\pi i\mathbf{l}\cdot\mathbf{Q}_j)\exp(\mathbf{l}\cdot\partial/\partial\boldsymbol{\theta}), \tag{3.11b}$$

$$\Delta g_{0p}=-2\pi ie\mathbf{E}\cdot\boldsymbol{\tau}u(0). \tag{3.11c}$$

Again using the random distribution of the scatterers, we are justified in using Eq. (2.13) which may be specifically written here in the following form.

$$(\partial\Delta M/\partial t)=(\boldsymbol{\theta}/m)\cdot(\partial/\partial\boldsymbol{\tau})\Delta M(\boldsymbol{\tau},\boldsymbol{\theta},t) -2\pi ie\mathbf{E}\cdot\boldsymbol{\tau}u(0)M_0-\lambda^2(2\pi i)^4c\int d\mathbf{l}\,|V_{\mathbf{l}}|^2\mathbf{l}\cdot\boldsymbol{\tau} \times\int_0^t dt'\exp\{(t-t')[(\boldsymbol{\theta}+\mathbf{l})/m]\cdot(\partial/\partial\boldsymbol{\tau})\} \times[\mathbf{l}\cdot\boldsymbol{\tau}\Delta M(\boldsymbol{\tau},\boldsymbol{\theta},t')]. \tag{3.12}$$

Here we have used the fact that $\exp(\mathbf{l}\cdot\partial/\partial\boldsymbol{\theta})$ is a shift operator on $\boldsymbol{\theta}$ by the amount $\mathbf{l}$. Again using the properties of shift operators, we may rewrite Eq. (3.12):

$$\partial\Delta M(\boldsymbol{\tau},\boldsymbol{\theta},t)/\partial t=(\boldsymbol{\theta}/m)\cdot(\partial/\partial\boldsymbol{\tau})\Delta M(\boldsymbol{\tau},\boldsymbol{\theta},t) -2\pi ie\mathbf{E}\cdot\boldsymbol{\tau}u(0)M_0(\boldsymbol{\tau},\boldsymbol{\theta}) -\lambda^2\Gamma_{\mathbf{l}}\mathbf{l}\cdot\boldsymbol{\tau}\int_0^t dt'\,\mathbf{l}\cdot[\boldsymbol{\tau}+(t-t')(\boldsymbol{\theta}+\mathbf{l})/m] \times\Delta M[\boldsymbol{\tau}+(t-t')(\boldsymbol{\theta}+\mathbf{l})/m,\,\boldsymbol{\theta},\,t'], \tag{3.13}$$

where

$$\Gamma_{\mathbf{l}}=(2\pi i)^4c\int d\mathbf{l}\,|V_{\mathbf{l}}|^2 \tag{3.14}$$

is an integral operator operating on everything to its right.

Since we are interested in conductivity, i.e., the first moment of $\mathbf{p}$, and there are no derivatives with respect to $\boldsymbol{\theta}$, we may simplify Eq. (3.13) by taking the limit of $\boldsymbol{\theta}$ vanishing, i.e.,

$$\partial\Delta M(\boldsymbol{\tau},0,t)/\partial t=-2\pi i\mathbf{E}\cdot\boldsymbol{\tau}u(0)M_0(\tau,0) -\lambda^2\Gamma_{\mathbf{l}}\mathbf{l}\cdot\boldsymbol{\tau}\int_0^t dt'\,\mathbf{l}\cdot[\boldsymbol{\tau}+(t-t')\mathbf{l}/m] \times\Delta M[\boldsymbol{\tau}+(t-t')\mathbf{l}/m,\,0,\,t']. \tag{3.15}$$

Now we solve Eq. (3.15) by an iterative scheme in powers of λ^2. The first iteration yields

$$\Delta M_0(\tau,0,t)=-2\pi ieE_1\tau_1M_0(\tau,0)t, \tag{3.16}$$

where we have assumed $\mathbf{E}$ to be in the 1 direction. Inserting Eq. (3.16) in the λ^2 term in Eq. (3.15) and performing the time integral assuming t is very large, we obtain

$$\partial\Delta M_1(\boldsymbol{\tau},0,t)/\partial t=-2\pi ieE_1\tau_1u_0\exp(-2\pi^2m\tau^2/\beta) -\lambda^2\Gamma_{\mathbf{l}}\mathbf{l}\cdot\boldsymbol{\tau}\left\{\frac{\beta}{4\pi^2m}+\frac{\beta l_1}{8\pi^2m\tau_1}\left[\left(\frac{\beta}{2\pi^2ml^2}\right)^{\frac{1}{2}} -2\boldsymbol{\tau}\cdot\mathbf{l}+\cdots\right]\right\}\Delta M_0(\boldsymbol{\tau},0,t). \tag{3.17}$$

Now to order λ^2 we may cut off the iteration in Eq. (3.17) by replacing $\Delta M_0(\tau,0,t)$ by $\Delta M_1(\tau,0,t)$ in the λ^2 term. Then

$$\Delta M_1(\tau,0,t)=\int_0^t dt'\exp[-(t-t')\lambda^2\Gamma] \times 2\pi ieE_1\tau_1\exp(-2\pi^2m\tau^2/\beta), \tag{3.18}$$

where

$$\Gamma=\Gamma_{\mathbf{l}}\mathbf{l}\cdot\boldsymbol{\tau}\left\{\frac{\beta}{4\pi^2m}+\frac{\beta l_1}{8\pi^2m\tau_1}\left[\left(\frac{\beta}{2\pi^2ml^2}\right)^{\frac{1}{2}}-2\boldsymbol{\tau}\cdot\mathbf{l}+\cdots\right]\right\}.$$

Equation (3.18) represents the desired characteristic function to order λ^2. The current density in the 1 direction is given by $J_1=(ne/m)\langle\Delta p_1\rangle$, where n is the number of free charged particles per unit volume. Using the fact that odd functions of $\mathbf{l}$ vanish under the operation of $\Gamma_{\mathbf{l}}$, we obtain

$$\langle\Delta p_1\rangle=\lim_{\tau\to 0}(1/2\pi i)(\partial/\partial\tau_1)\Delta M(\boldsymbol{\tau},0,t) =\int_0^t dt'\exp[-(t-t')\lambda^2\Gamma_0]eE_1, \tag{3.19}$$

where

$$\lambda^2\Gamma_0=\Gamma_{\mathbf{l}}(l_1^2\beta/8\pi^2m)(\beta/2\pi^2ml^2)^{\frac{1}{2}} =(2\pi mkT)^{-\frac{3}{2}}\Gamma_l(l_1^2/4|l|), \tag{3.20}$$

may be identified with the reciprocal of the relaxation time. The current density in the 1 direction is

$$J_1=[ne^2/m\lambda^2\Gamma_0]E_1[1-\exp(-\lambda^2t\Gamma_0)], \tag{3.21}$$

where the quantity in brackets is the conductivity. The mobility, $u=(e/m\lambda^2\Gamma_0)$, has a $T^{\frac{3}{2}}$ dependence which is consistent with the usual result[2] for impurity scattering using the Boltzmann equation.

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[2] See, e.g., F. J. Blatt, *Solid-State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, Inc., New York, 1957), Vol. 4, p. 343 ff.