Application of General Unitarity to Pion-Nucleon Scattering

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(Received May 23, 1961)

The $\pi-\pi-N$ intermediate state is included in the unitarity condition of pion-nucleon scattering. The general unitarity relations then couple the amplitudes for $\pi-N$ and $\pi-\pi-N$ scattering, and one-pion production or absorption. The kinematical structure of these amplitudes is discussed and a formal decomposition into partial waves is outlined. The dynamical approach is based on a system of coupled partial wave dispersion relations. The dynamical singularities of $\pi-N$ scattering can be taken from the Mandelstam representation, whereas perturbation-theoretic approximations must be made for the other amplitudes. The dispersion relations can be solved by the generalized N/D method. As an example, a one-pion exchange approximation for the production and absorption amplitudes is considered. It is shown that amplitudes can be chosen which do not have any kinematical singularities. The N/D method here takes a more simple form by setting up integral equations for the N functions rather than for the D functions. Integrations over contours in the complex plane are then avoided.

I. INTRODUCTION

IT is the goal of the dispersion-theoretic approach to strong-coupling problems to determine the scattering amplitudes in terms of the coupling constants involved, which are to be considered as external parameters. Unitarity and analyticity are the proper tools to be used in such a program. After the conjecture of Mandelstam[1] established a basis for formulating the analytic properties of two-particle scattering amplitudes, a first attack was made in the spirit of this program on the basic $\pi-\pi$[2] and $\pi-N$[3] scattering reactions. The $\pi-N$ scattering is especially challenging for a test of these ideas because the experimental situation in the low-energy region is very well known. In particular, it should be a first goal to predict at least the position of the 33 resonance. The success of the work of Frautschi and Walecka[3] in this respect is limited. This may have a twofold reason. The first is the purely technical one that it is actually quite difficult to perform a reliable approximation of the dynamical singularities. The second is more fundamental. It may prove to be necessary to include the next higher mass state into the unitarity condition which corresponds to a one-meson production process.

This can only be done if the four amplitudes for the $\pi-N$, $\pi-\pi-N$ scattering reactions and the one-meson absorption and production reactions are treated on the same level. This is possible, as far as unitarity is concerned, by using the generalized linear unitarity conditions, formulated by Blankenbecler.[4] But little is known at present about analytic properties of amplitudes for reactions, in which more than four particles are involved. So one is reduced to perturbation graphs as possible approximations for the dynamical singularities.

It is the purpose of this work to develop the methods to be used in such an approach as far as possible. In Sec. II we give a general kinematical analysis of the production and $\pi-\pi-N$ scattering amplitude. Section III contains the angular momentum decomposition and in Sec. IV the general dynamical approach is outlined. In Sec. V we discuss the single-meson exchange approximation as the simplest concept that has built in the $\pi-\pi$ interaction. Lumping together the interaction into the coupling constant λ makes the analysis very simple. Finally, in Sec. VI, we analyze the partial amplitudes in this approximation. It is shown that all kinematical singularities can be removed. The amplitudes are determined from coupled integral equations in the spirit of the general N/D method,[4–6] but integration along contours in the complex plane is avoided by setting up the integral equations for the N functions rather than for the D functions.

II. KINEMATICS

We begin with an analysis of the general kinematical structure of the three particle amplitudes. The S-matrix elements for the different processes to be considered are given by standard formulas:

$$p+q \to p'+q',$$

$$S_{22}=\delta_{22}+2\pi i\delta^{(4)}(p'+q'-p-q)\times\left(\frac{m^2}{p_0p_0'2q_02q_0'}\right)^{\frac{1}{2}}M_{22}, \quad (2.1)$$

$$p+q \to p'+q_1'+q_2',$$

$$S_{32}=\delta_{32}+2\pi i\delta^{(4)}(p'+q_1'+q_2'-p-q)\times\left(\frac{m^2}{p_0p_0'2q_02q_{10}'2q_{20}'\times 2}\right)^{\frac{1}{2}}M_{32}, \quad (2.2)$$

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$$p+q_1+q_2\rightarrow p'+q_1'+q_2',$$
$$S_{33}=\delta_{33}+2\pi i\delta^{(4)}(p'+q_1'+q_2'-p-q_1-q_2)\times\left(\frac{m^2}{p_0p_0'2q_{10}2q_{20}\times2\times2q_{10}'2q_{20}'\times2}\right)^{\frac{1}{2}}M_{33},\quad(2.3)$$

where we have used box normalization. Primed four-momenta label outgoing, the others incoming particles; p-momenta nucleons and q-momenta mesons. See Figs. 1–3. The factor 2 in the kinematical factors of the three-particle amplitudes appears because of the presence of two identical particles. S-matrix elements S_{ik} and causal amplitudes M_{ik} are labeled by the number of in- and outgoing particles in obvious notation. The amplitude for the meson absorption process can, of course, always be obtained from M_{32} by interchanging initial and final momenta. For later reference let us further write down the causal amplitudes with the incoming nucleon contracted:

$$M_{22}=[(p_0'/m)2q_0'2q_0]^{\frac{1}{2}}\langle p'q'|\bar{f}(0)|q\rangle,\quad(2.4)$$

$$M_{32}=[(p_0'/m)2q_02q_{10}'2q_{20}'\times2]^{\frac{1}{2}}\langle p'q_1'q_2'|\bar{f}(0)|q\rangle,\quad(2.5)$$

$$M_{33}=[(p_0'/m)2q_{10}2q_{20}\times2\times2q_{10}'\times2q_{20}'\times2]^{\frac{1}{2}}\times\langle p'q_1'q_2'|\bar{f}(0)|q_1q_2\rangle,\quad(2.6)$$

where $\bar{f}$ is the adjoint current, defined by

$$\left(-\gamma_\mu{}^T\frac{\partial}{\partial x_\mu}+m\right)\bar{\psi}=f,\quad(2.7)$$

spinor normalization shall be such that $\bar{u}(p)\gamma_4u(p)=p_0/m$.

We shall use standard variables for M_{22}:

$$\begin{aligned}s&=-(p+q)^2=-(p'+q')^2,\\ \bar{t}&=-(p'-q)^2=-(p-q')^2,\\ t&=-(p'-p)^2=-(q'-q)^2,\end{aligned}\quad(2.8)$$

where

$$s+\bar{t}+t=2m^2+2\mu^2,\quad(2.9)$$

but we will not chose a specific set of variables for M_{32} and M_{33}, because we shall work in an approximation, where these amplitudes depend only on a restricted number of variables, which will be specified later. As is well known a N-particle amplitude depends on $3N-10$ scalar invariants, that means 5 for M_{32} and 8 for M_{33}. Two of these, which will be used in any case, are the total center-of-mass energy s and the nucleon momentum transfer t:

$$\begin{aligned}s&=-(p+q)^2=-(p'+q_1'+q_2')^2:M_{32},\\ s&=-(p+q_1+q_2)^2=-(p'+q_1'+q_2')^2:M_{33},\\ t&=-(p'-p)^2.\end{aligned}\quad(2.10)$$

To analyze the form in the nucleon's spin space we write the amplitudes as follows:

$$\begin{aligned}M_{22}&=\bar{u}(p')\bar{M}_{22}u(p),\\ M_{32}&=\bar{u}(p')\gamma_5\bar{M}_{32}u(p),\\ M_{33}&=\bar{u}(p')\bar{M}_{33}u(p),\end{aligned}\quad(2.11)$$

where the $\bar{M}_{ik}$ are invariant matrices in spin space. They can be composed of the γ matrices in the usual way:

$$\bar{M}=\bar{M}_1+\bar{M}_\mu\gamma_\mu+\bar{M}_{\mu\nu}\sigma_{\mu\nu}+\bar{M}_{5\mu}\gamma_\mu\gamma_5+\bar{M}_5\gamma_5.\quad(2.12)$$

As is well known, there are only two invariant amplitudes for $\bar{M}_{22}$, which may be written as

$$\bar{M}_{22}=A_{22}-i\gamma\frac{(q+q')}{2}B_{22}.\quad(2.13)$$

In the production process we have 4 independent vectors. In constructing a basis of these we may use two orthogonal q vectors, say $Q_1=q_1'-q$ and $Q_2=q_1'+q$, and two orthogonal p vectors: $P_1=p-p'$ and $p+p'=P_2$. Taking into account the Dirac equation for the spinors $\bar{u}(p')$ and $u(p)$ it is clear that $\bar{M}_\mu$ contributes two amplitudes: $M^{(1)}Q_{1\mu}+M^{(2)}Q_{2\mu}$, and $\bar{M}_{\mu\nu}$ one amplitude $\bar{M}Q_{1\mu}Q_{2\nu}$. All possible pseudovector invariants can be reduced to these forms; i.e., $\gamma_\mu\gamma_5\epsilon_{\mu\nu\lambda\rho}P_{1\nu}Q_{1\lambda}Q_{2\rho}=\frac{1}{4}\{(\gamma P_1),[(\gamma Q_1),(\gamma Q_2)]\}\gamma_5{}^2$. The pseudoscalar part $\bar{M}_5\gamma_5$ can be reduced in the same fashion. So, in total we have four invariant amplitudes, and $\bar{M}_{32}$ may be written

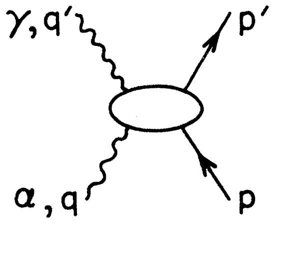

FIG. 1. Diagram for pion-nucleon scattering.

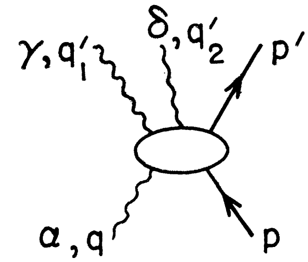

FIG. 2. Diagram for one-pion production.

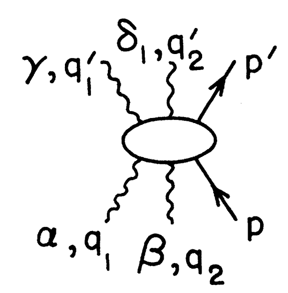

FIG. 3. Diagram for two-pion–nucleon scattering.

TABLE I. 2 meson−1 nucleon projection operators ${}^{TT'}\mathcal{P}^I$. I = total isospin; T = total isospin of the two outgoing, T' of ingoing mesons.

	$\delta_{\gamma\delta}\delta_{\alpha\beta}$	$\delta_{\gamma\beta}\delta_{\delta\alpha}$	$\delta_{\gamma\alpha}\delta_{\delta\beta}$	$F_{\gamma\delta}\delta_{\alpha\beta}$	$F_{\gamma\beta}\delta_{\delta\alpha}$	$F_{\gamma\alpha}\delta_{\beta\delta}$	$\delta_{\gamma\delta}F_{\alpha\beta}$	$\delta_{\gamma\beta}F_{\delta\alpha}$	$\delta_{\gamma\alpha}F_{\delta\beta}$
${}^{00}\mathcal{P}^{1/2}$	1/3	...	...	...	...	...	...	...	...
${}^{01}\mathcal{P}^{1/2}$	...	...	...	...	...	...	$-1/3\sqrt{2}$	...	...
${}^{10}\mathcal{P}^{1/2}$	...	...	...	$1/3\sqrt{2}$	...	...	...	...	...
${}^{11}\mathcal{P}^{1/2}$	...	−1/6	1/6	...	−1/6	1/6	...	−1/6	1/6
${}^{11}\mathcal{P}^{3/2}$	...	−1/3	1/3	...	1/6	−1/6	...	1/6	−1/6
${}^{12}\mathcal{P}^{3/2}$	...	...	...	$-2/3(5)^{1/2}$	$1/(20)^{1/2}$	$1/(20)^{1/2}$	...	$-1/(20)^{1/2}$	$-1/(20)^{1/2}$
${}^{21}\mathcal{P}^{3/2}$	...	...	...	...	$-1/(20)^{1/2}$	$1/(20)^{1/2}$	$2/3(5)^{1/2}$	$1/(20)^{1/2}$	$-1/(20)^{1/2}$
${}^{22}\mathcal{P}^{3/2}$	−2/15	1/5	1/5	...	1/10	1/10	...	1/10	1/10
${}^{22}\mathcal{P}^{5/2}$	−1/5	3/10	3/10	...	−1/10	−1/10	...	−1/10	−1/10

as:

$$\bar{M}_{32}=A_{32}-i(\gamma Q_1)B_{32}^{(1)}-i(\gamma Q_2)B_{32}^{(2)}+(\gamma Q_1)(\gamma Q_2)D_{32}.$$

All the arguments remain valid for M_{33} and in general for all two-nucleon multimeson amplitudes. They can be made up for four invariant amplitudes. The reduction from the maximum possible number 16 is due to the application of the Dirac equation to the two occurring nucleon momenta and spinors.

Next we consider the dependence on isospin variables. All amplitudes are nonisotropic tensors in the meson charge space, made up of the pseudovector τ_α and the unit tensor $\delta_{\alpha\beta}$ ($\alpha, \beta=1, 2, 3$). Therefore, we may write

$$\begin{aligned}
M_{\gamma\alpha}{}^{22}&=M^{(1)}\delta_{\gamma\alpha}+M^{(2)}\tau_\gamma\tau_\alpha,\\
M_{\gamma\delta,\alpha}{}^{32}&=M^{(1)}\delta_{\gamma\delta}\tau_\alpha+M^{(2)}\delta_{\delta\alpha}\tau_\gamma+M^{(3)}\delta_{\gamma\alpha}\tau_\delta\\
&\quad+M^{(4)}\tau_\gamma\tau_\delta\tau_\alpha,\\
M_{\gamma\delta,\alpha\beta}{}^{33}&=M^{(1)}\delta_{\gamma\delta}\delta_{\beta\alpha}+M^{(2)}\delta_{\gamma\beta}\delta_{\delta\alpha}+M^{(3)}\delta_{\gamma\alpha}\delta_{\beta\delta}\\
&\quad+M^{(4)}\tau_\gamma\tau_\delta\delta_{\alpha\beta}+M^{(5)}\tau_\gamma\tau_\beta\delta_{\delta\alpha}\\
&\quad+M^{(6)}\tau_\gamma\tau_\alpha\delta_{\beta\delta}+M^{(7)}\tau_\alpha\tau_\beta\delta_{\gamma\delta}\\
&\quad+M^{(8)}\tau_\delta\tau_\alpha\delta_{\gamma\beta}+M^{(9)}\tau_\beta\tau_\delta\delta_{\gamma\alpha}\\
&\quad+M^{(10)}\tau_\gamma\tau_\delta\tau_\alpha\tau_\beta.
\end{aligned}\tag{2.15}$$

If we use

$$\tau_\alpha\tau_\beta=\delta_{\alpha\beta}+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]=\delta_{\alpha\beta}+F_{\alpha\beta},$$

and

$$F_{\alpha\beta}=i\epsilon_{\alpha\beta\delta}\tau_\delta,$$

we see that $\tau_\gamma\tau_\delta\tau_\alpha\tau_\beta$ can be reduced to a combination of the other forms. Furthermore, all products of two τ matrices $\tau_\alpha\tau_\beta$, etc., may be replaced by the antisymmetric tensor $F_{\alpha\beta}$. We then have

$$\begin{aligned}
M_{\gamma\alpha}{}^{22}&=M^{(1)}\delta_{\gamma\alpha}+M^{(2)}F_{\gamma\alpha},\\
M_{\gamma\delta,\alpha}{}^{32}&=M^{(1)}\delta_{\gamma\delta}\tau_\alpha+M^{(2)}\delta_{\delta\alpha}\tau_\gamma+M^{(3)}\delta_{\gamma\alpha}\tau_\delta\\
&\quad+M^{(4)}F_{\gamma\delta}\tau_\alpha,\\
M_{\gamma\delta,\alpha\beta}{}^{33}&=M^{(1)}\delta_{\gamma\delta}\delta_{\alpha\beta}+M^{(2)}\delta_{\gamma\beta}\delta_{\delta\alpha}\\
&\quad+M^{(3)}\delta_{\gamma\alpha}\delta_{\beta\delta}+M^{(4)}F_{\gamma\delta}\delta_{\alpha\beta}\\
&\quad+M^{(5)}F_{\gamma\beta}\delta_{\delta\alpha}+M^{(6)}F_{\gamma\alpha}\delta_{\beta\delta}\\
&\quad+M^{(7)}\delta_{\gamma\delta}F_{\alpha\beta}+M^{(8)}\delta_{\gamma\beta}F_{\delta\alpha}\\
&\quad+M^{(9)}\delta_{\gamma\alpha}F_{\beta\delta}.
\end{aligned}\tag{2.16}$$

Thus there are 4 invariant amplitudes for M^{32} and 9 for M^{33} in the meson charge space. These amplitudes are now to be related to the amplitudes for definite isospin quantum numbers. This can be done conveniently by using isospin projection operators.

There are three different sets of projection operators corresponding to the amplitudes M_{22}, M_{32} and M_{33}. They are defined by

$$\begin{aligned}
\mathcal{P}^I&=\textstyle\sum_{I_z}|I,I_z\rangle\langle I_z,I|, && I=\tfrac{1}{2},\tfrac{3}{2};\\
{}^T\mathcal{P}^I&=\textstyle\sum_{I_z}|I,T,I_z\rangle\langle I_z,I|, && I=\tfrac{1}{2}(T=0,1),\\
& && I=\tfrac{3}{2}(T=1,2);\\
{}^{TT'}\mathcal{P}^I&=\textstyle\sum_{I_z}|I,T,I_z\rangle\langle I_z,T',T|, && \\
& && I=\tfrac{1}{2}(T,T'=0,1),\\
& && I=\tfrac{3}{2}(T,T'=1,2),\\
& && I=\tfrac{5}{2}(T=T'=2).
\end{aligned}\tag{2.17}$$

The number of operators of each kind corresponds, of course, exactly to the number of invariant amplitudes in charge space. We have used a $(\pi-\pi)-N$ coupling scheme where first the isospins of the two mesons are added and labeled by the total isospin quantum number T, before the nucleon's isospin is added. We could also have used a $(\pi-N)-\pi$ coupling scheme, but in our approximation later on we shall like to refer to states with a definite isospin of the two mesons involved.

We now represent the operators (2.17) as matrices in charge space, which are operators in the nucleon's isospin space. The first two operators are well known:

$$\begin{aligned}
\langle\gamma|\mathcal{P}^{\frac{1}{2}}|\alpha\rangle&=P_{\gamma\alpha}{}^{\frac{1}{2}}=\tau_\gamma\tau_\alpha/3,\\
\langle\gamma|\mathcal{P}^{\frac{3}{2}}|\alpha\rangle&=P_{\gamma\alpha}{}^{\frac{3}{2}}=\delta_{\gamma\alpha}-\tau_\gamma\tau_\alpha/3.
\end{aligned}\tag{2.18}$$

There is no easy way to determine the operators ${}^T\mathcal{P}^I$. They have to be calculated according to the definition (2.17). This procedure is tedious, but elementary. We, therefore, give only the result[7]:

$$\begin{aligned}
\langle\gamma\delta|{}^0\mathcal{P}^{\frac{1}{2}}|\alpha\rangle&=\tfrac{1}{3}\delta_{\gamma\delta}\tau_\alpha,\\
\langle\gamma\delta|{}^1\mathcal{P}^{\frac{1}{2}}|\alpha\rangle&=(1/3\sqrt{2})F_{\gamma\delta}\tau_\alpha,\\
\langle\gamma\delta|{}^1\mathcal{P}^{\frac{3}{2}}|\alpha\rangle&=(1/\sqrt{2})(\tau_\delta\delta_{\gamma\alpha}-\tau_\gamma\delta_{\delta\alpha})+(2/3\sqrt{2})F_{\gamma\delta}\tau_\alpha,\\
\langle\gamma\delta|{}^2\mathcal{P}^{\frac{3}{2}}|\alpha\rangle&=-[1/(10)^{\frac{1}{2}}](\tau_\delta\delta_{\gamma\alpha}+\tau_\gamma\delta_{\delta\alpha})\\
&\quad+[2/3(10)^{\frac{1}{2}}]\delta_{\gamma\delta}\tau_\alpha.
\end{aligned}\tag{2.19}$$

[7] The operators have also been given by P. Carruthers [thesis (unpublished); and to be published].

Operators of even T are symmetric in the two meson indices $\gamma\delta$, those of odd T antisymmetric. The corresponding operators for the absorption process are of course the Hermitian conjugates of (2.19). Next are the operators ${}^{TT'}\mathcal{P}^I$. Eight of these can be determined from (2.19) by observing that

$$ {}^{T}\mathcal{P}^I({}^{T'}\mathcal{P}^I)^\dagger = {}^{TT'}\mathcal{P}^I, \quad I=\tfrac{1}{2},\tfrac{3}{2}. \tag{2.20} $$

The last one ($T=T'=2$, $I=\frac{5}{2}$) can be obtained from the condition:

$$ \sum_{I,T} {}^{TT}\mathcal{P}^I = 1, \tag{2.21} $$

which expresses the completeness of the three-particle state vectors in the combined charge-nucleon isospin space. If one sums over the two meson-indices of (2.19), one obtains the operators

$$ ({}^{T}P_{\rho\sigma,\gamma}{}^{I})^\dagger({}^{T}P_{\rho\sigma,\alpha}{}^{I}) = P_{\gamma\alpha}{}^{I}, \tag{2.22} $$

independent of T. This relation may be used to check the operators calculated from (2.20) and (2.21). In Table I we have summarized all possible projection operators ${}^{TT'}\mathcal{P}^I$. Now it is not difficult to relate the invariant amplitudes of (2.16) to the amplitudes for definite isospin quantum numbers. To save space, we note the result only for M^{22} and M^{32}. The relations for M^{33} can easily be derived from Table I.

$$ \begin{pmatrix} M^{(1)} \\ M^{(2)} \end{pmatrix} = (Z_{22}) \begin{pmatrix} M^{\frac{1}{2}} \\ M^{\frac{3}{2}} \end{pmatrix}, \tag{2.23} $$

where

$$ Z_{22} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}; \quad (Z_{22})^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \tag{2.24} $$

and

$$ \begin{bmatrix} {}^{0}M^{\frac{1}{2}} \\ {}^{1}M^{\frac{1}{2}} \\ {}^{1}M^{\frac{3}{2}} \\ {}^{2}M^{\frac{3}{2}} \end{bmatrix} = (Z_{32}) \begin{bmatrix} M^{(1)} \\ M^{(2)} \\ M^{(3)} \\ M^{(4)} \end{bmatrix}, \tag{2.25} $$

where

$$ (Z_{32}) = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & \sqrt{2} & -\sqrt{2} & 3\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -(10)^{\frac{1}{2}}/2 & -(10)^{\frac{1}{2}}/2 & 0 \end{bmatrix}; $$

$$ (Z_{32})^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 2/3(10)^{\frac{1}{2}} \\ 0 & 0 & -1/\sqrt{2} & -1/(10)^{\frac{1}{2}} \\ 0 & 0 & 1/\sqrt{2} & -1/(10)^{\frac{1}{2}} \\ 0 & 1/3\sqrt{2} & 2/3\sqrt{2} & 0 \end{bmatrix}. \tag{2.26} $$

III. ANGULAR MOMENTUM DECOMPOSITION

The angular decomposition of the causal amplitudes can be done in a way completely analogous to the projecting out of isospin amplitudes. It is, however, quite dubious whether the corresponding series converges because there may be singularities in the physical region of angle variables of production amplitudes. Therefore, the choice of coupling scheme and quantum numbers must be adapted to the physical situation to be considered in such a way that a few terms of the decomposition series give a fair description of the physical problem and the quantum numbers can be considered as "good" quantum numbers.

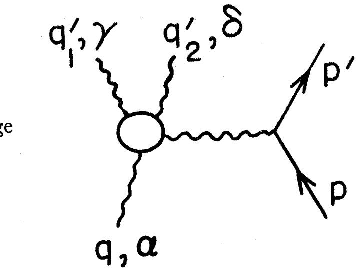

FIG. 4. One-pion exchange diagram.

We shall always work in the center-of-mass frame of the s channel. The total momentum of the three-particle states is, therefore, zero.

$$ 0 = \mathbf{q}_1' + \mathbf{q}_2' + \mathbf{p}'. \tag{3.1} $$

As far as the coupling scheme is concerned, we can either first add the two orbital angular momenta or one orbital angular momentum and the nucleon spin. Now, what we shall do is to approximate the negative singularities of the production amplitude by the single meson exchange diagram (Fig. 4). In this case, scattering of the outgoing mesons with the nucleon is neglected and the mesons have the phase of $\pi-\pi$ scattering, if the exchanged meson is on the mass shell. It is, of course, then most convenient to use a $\pi-\pi$ coupling scheme and to decompose the total orbital angular momentum $\mathbf{L}$ of the two mesons in the angular momenta of their relative motion $\mathbf{L}_r$ and their center of mass $\mathbf{L}_c$:

$$ \mathbf{L} = \mathbf{L}_c + \mathbf{L}_r = [(\mathbf{q}_1' - \mathbf{q}_2')/2, \mathbf{r}_1 - \mathbf{r}_2] + [\mathbf{q}_1' + \mathbf{q}_2', (\mathbf{r}_1 + \mathbf{r}_2)/2] \tag{3.2} $$

($\mathbf{r}_1, \mathbf{r}_2$ position vectors of mesons in the three-particle center-of-mass system). We then have the following quantum numbers for the three-particle state: J (total), L (total orbital), l_c (2 meson-center of mass), l_r (2 meson relative).

However, if one allows the outgoing pions to scatter with the nucleon, then the situation is quite different, as has also been pointed out by Carruthers.[7] We could, e.g., take into account the diagram of Fig. 5. This

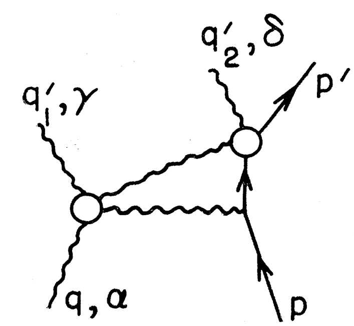

FIG. 5. Final-state interaction of one pion and the nucleon.

situation can hardly be described in a $(\pi-\pi)$ coupling scheme, because the interaction of one of the outgoing mesons occurs essentially in 33 state. Roughly speaking, this may be considered as the principal final state interaction,[7] so that using the diagram of Fig. 5 instead of Fig. 4 should be a better approximation. But as we shall see, this introduces additional variables into the problem, which make the situation hopelessly complicated. We, therefore, use the $(\pi-\pi)N$ coupling scheme and stress that is useful only in the approximation to be considered. Nevertheless, the following formal operations can, of course, be performed in the same way for the $(\pi-N)\pi$ coupling scheme.

Angular momentum projection operators can be conveniently represented as matrices in Pauli spin space. We, therefore, rewrite the causal amplitudes as matrix elements between Pauli spinors by introducing:

$$u(p)=\frac{1}{[2m(E+m)]^{\frac{1}{2}}}\begin{pmatrix}E+m\\(\sigma\mathbf{p})\end{pmatrix}\chi;\qquad \bar{u}(p')=\frac{\chi'^{*}}{[2m(E'+m)]^{\frac{1}{2}}}\begin{pmatrix}E'+m\\-(\sigma\mathbf{p}')\end{pmatrix},\tag{3.3}$$

where χ, χ' are two component Pauli spinors. Let us work always in the center-of-mass frame of in- and outgoing particles. For two-particle scattering we then have only two independent vectors in three-dimensional space, which may be chosen as $\mathbf{p}$, $\mathbf{p}'$. From these we construct the scalar

$$i\boldsymbol{\sigma}\cdot\mathbf{p}'\times\mathbf{p}=(\boldsymbol{\sigma}\cdot\mathbf{p}')(\boldsymbol{\sigma}\cdot\mathbf{p})-\mathbf{p}'\cdot\mathbf{p},\tag{3.4}$$

M_{22} then may be written in the conventional form:

$$M_{22}=\chi'^{*}\{f_1+f_2(\boldsymbol{\sigma}\cdot\mathbf{p}')(\boldsymbol{\sigma}\cdot\mathbf{p})\}\chi,\tag{3.5}$$

f_1 and f_2 can be expressed in terms of A and B of (2.13) by the well-known relations

$$\begin{aligned}f_1&=[(E+m)/2m]\{A+(W-m)B\},\\ f_2&=[(E-m)/2m]\{-A+(W+m)B\},\end{aligned}\tag{3.6}$$

where W is the total energy of the meson-nucleon-system. ($E'=E$ in c.m. system.) In the production process the matrix element must be pseudoscalar and there are three independent vectors, say $\mathbf{p}$, $\mathbf{p}'$, $\mathbf{q}_1$. Then we have the general form:

$$\begin{aligned}M_{32}=\chi'^{*}\{g_1[\mathbf{q}_1\cdot[\mathbf{p}'\times\mathbf{p}]]+g_2(\boldsymbol{\sigma}\cdot\mathbf{p}')\\ +g_3(\boldsymbol{\sigma}\cdot\mathbf{p})+g_4(\boldsymbol{\sigma}\cdot\mathbf{q}_1)\}\chi.\end{aligned}\tag{3.7}$$

There is no difficulty to express the four invariant g_i in terms of A, $B^{(1)}$, $B^{(2)}$, D of (2.14), but we shall not do so. Similarly, the scalar M_{33} can be expressed by four amplitudes:

$$\begin{aligned}M_{33}=\chi'^{*}\{h_1+h_2\boldsymbol{\sigma}\cdot[\mathbf{p}'\times\mathbf{p}]+h_3\boldsymbol{\sigma}\cdot[\mathbf{p}\times\mathbf{q}_1]\\ +h_4\boldsymbol{\sigma}\cdot[\mathbf{q}_1\times\mathbf{p}']\}\chi.\end{aligned}\tag{3.8}$$

Let us now define a set of projection operators in a fashion completely analogous to (2.17) and observing the conservation of angular momentum and parity:

$$\begin{aligned}\mathcal{P}_{Jl}&=\sum_{M=-J}^{+J}|JlM\rangle\langle MlJ|,\\ \mathcal{P}_{JLl_cl_rl}&=\sum_{M=-J}^{+J}|JLl_cl_rM\rangle\langle MlJ|,\qquad l+l_c+l_r=\text{odd},\\ \mathcal{P}_{JLl_cl_rL'l_c'l_r'}&=\sum_{M=-J}^{+J}|JLl_cl_rM\rangle\langle Ml_r'l_c'L'J|,\qquad l_r'+l_c'+l_r+l_c=\text{even}.\end{aligned}\tag{3.9}$$

The quantum numbers are: J= total angular momentum, M= total z component of angular momentum, L= total angular momentum of two mesons, l_c= center-of-mass angular momentum of two mesons, l_r= relative angular momentum of two mesons. The sum rules express parity conservation. Of course, the vector addition of l_c and l_r must always result in the corresponding L: $\mathbf{L}=\mathbf{l}_c+\mathbf{l}_r$.

The matrix elements of these operators in momentum space may be labeled by the spatial momenta of the in- and outgoing mesons. In the special frame, where the incoming meson momentum $\mathbf{q}$ is parallel to the z direction, we have

$$P_{Jl}=\langle\mathbf{q}'|\mathcal{P}_{Jl}|\mathbf{q}\rangle=\begin{cases}\dfrac{l+1+(\boldsymbol{\sigma}\cdot\mathbf{l})}{[4\pi(2l+1)]^{\frac{1}{2}}}Y_{l0}(\hat{q}'\cdot\hat{q}), & J=l+\frac{1}{2},\\[2ex] \dfrac{l-(\boldsymbol{\sigma}\cdot\mathbf{l})}{[4\pi(2l+1)]^{\frac{1}{2}}}Y_{l0}(\hat{q}'\cdot\hat{q}), & J=l-\frac{1}{2},\end{cases}\tag{3.10}$$

$$\mathbf{l}=(1/i)[\hat{q}',\boldsymbol{\nabla}_{q'}].$$

$$\begin{aligned}P_{JLl_cl_rl}&=\langle\mathbf{q}_1'\mathbf{q}_2'|\mathcal{P}_{JLl_cl_rl}|\mathbf{q}\rangle\\ &=\frac{L+1+(\boldsymbol{\sigma}\mathbf{L})}{[4\pi(2L+1)]^{\frac{1}{2}}}\mathcal{Y}_{L0}(\hat{q}_c'\cdot\hat{q},\hat{q}_r'\cdot\hat{q}),\qquad J=L+\tfrac{1}{2},\\ &=\frac{L-(\boldsymbol{\sigma}\mathbf{L})}{[4\pi(2L+1)]^{\frac{1}{2}}}\mathcal{Y}_{L0}(\hat{q}_c'\cdot\hat{q},\hat{q}_r'\cdot\hat{q}),\qquad J=L-\tfrac{1}{2},\end{aligned}\tag{3.11}$$

$$\mathbf{L}=(1/i)\{[\hat{q}_c',\boldsymbol{\nabla}_{q_c'}]+[\hat{q}_r',\boldsymbol{\nabla}_{q_r'}]\},$$

$\mathcal{Y}_{L0}$ is a two-particle spherical harmonic:

$$\mathcal{Y}_{L0}(\hat{q}_c'\cdot\hat{q},\hat{q}_r'\cdot\hat{q})=\sum_m C_{L0}{}^{l_c,m;\,l_r,-m}Y_{l_c,m}(\hat{q}_c'\cdot\hat{q})Y_{l_r,-m}(\hat{q}_r'\cdot\hat{q}),$$

where $C_{L0}{}^{l_c,m;\,l_r,-m}$ are the corresponding Clebsch-Gordan coefficients. The $\hat{q}$ vectors are unit vectors. The three-particle projection operators (3.11) have been

evaluated by Ciulli and Fischer[8] and may be taken directly from the tables of these authors. There is further no difficulty to extend their methods also to the $P_{JLl_cl_rL'l_c'l_r'}$. The normalization is such that

$$\underset{\text{spin}}{\mathrm{Tr}}\int d\Omega_1\cdots\int d\Omega_n P_{J'S'}^{\dagger}P_{JS}=(2J+1)\delta_{J'J}\delta_{S'S}, \quad (3.12)$$

where S' and S stand for the corresponding indices. The partial amplitudes are, therefore, given by:

$$M_{JS}=\frac{1}{2J+1}\underset{\text{spin}}{\mathrm{Tr}}\int d\Omega_1\cdots\int d\Omega_n P_{JS}^{\dagger}M, \quad (3.13)$$

where M is a matrix in two-dimensional Pauli spin space. There is a relation between the relative angular momentum l_r and the isospin T of the two mesons in the three-particle states. This can be seen easily from the Pauli principle. Each amplitude containing a three-particle state must be invariant under exchange of the two mesons. Consequently,

$$M=(-1)^{T+l_r}M. \quad (3.14)$$

Only those M_{23} and M_{32} are different from zero for which $T+l_r=$ even. For M_{33} we have the two conditions $T+l_r=$ even, $T'+l_r'=$ even.

IV. DYNAMICS

At the present stage the dynamical treatment of strong interactions is based on the assumption that the scattering amplitudes can be determined from their analytic properties and the unitarity condition in terms of the coupling constants involved. The latter may enter the theory as the residue of a pole ($\pi-N$) or an arbitrary constant ($\pi-\pi$) of the scattering amplitude. The coupling constants are external parameters in the theory and have to be taken from experiment. For practical calculations one has further to believe that the low-energy behavior of the amplitudes in the physical region is dominated by the nearby singularities.

For two-particle scattering processes one has a tool for determining the singularities of the scattering amplitudes in Mandelstam's conjecture.[1] It is then, in principle, possible to carry out the indicated program, if one neglects the inelastic contributions in the unitarity condition. This has been done by Chew and Mandelstam[2] for $\pi-\pi$ scattering and by Frautschi and Walecka[3] for $\pi-N$ scattering. The next step would be to take into account a three-particle intermediate state in the unitarity condition for a two-particle scattering process. But then all the amplitudes involved: M_{22}, M_{32}, M_{33}, must be treated simultaneously on the same level. Up to now little is known about the analytic properties of a production amplitude like M_{32}, although some results have been obtained[9]; nothing is known about M_{33}. However, one can make the reasonable assumption[4] that M_{32} and M_{33} have a physical cut in each of its variables with the other variables fixed at physical values. Furthermore, one may introduce the singularities of some low-order diagrams, based on physical arguments.

A frame for putting all these pieces together are the general linear unitarity conditions, formulated by Blankenbecler[4] on the basis of the generalized N/D method given by Bjorken[5] and Nauenberg.[6] Let us first write down the unitarity condition in the s channel $[s\geq(m+\mu)^2]$ for the amplitudes to be considered:

$$\begin{aligned}
&M_{22}(s+i\epsilon)-M_{22}(s-i\epsilon)\\
&\quad=2\pi i\textstyle\sum_2 M_{22}(s-i\epsilon)M_{22}(s+i\epsilon)\\
&\qquad\qquad+2\pi i\textstyle\sum_3 M_{23}(s-i\epsilon)M_{32}(s+i\epsilon),\\
&M_{23}(s+i\epsilon)-M_{23}(s-i\epsilon)\\
&\quad=2\pi i\textstyle\sum_2 M_{22}(s-i\epsilon)M_{23}(s+i\epsilon)\\
&\qquad\qquad+2\pi i\textstyle\sum_3 M_{23}(s-i\epsilon)M_{33}(s+i\epsilon),\\
&M_{33}(s+i\epsilon)-M_{33}(s-i\epsilon)\\
&\quad=2\pi i\textstyle\sum_2 M_{32}(s-i\epsilon)M_{23}(s+i\epsilon)\\
&\qquad\qquad+2\pi i\textstyle\sum_3 M_{33}(s-i\epsilon)M_{33}(s+i\epsilon).
\end{aligned} \quad (4.1)$$

These relations can easily be derived from (2.4)–(2.6) by contracting the outgoing nucleon p'. In (4.1) we have suppressed all other variables beside the channel energy s. The sums run over all two- or three-particle intermediate states and may be written as phase space integrals:

$$\sum_2=\frac{2m}{32\pi^3}\int d\Omega_r\frac{p(s)}{\sqrt{s}}\Theta[s-(m+\mu)^2] \quad (4.2)$$

(the factor $2m$ arises from the intermediate nucleon), where s is the center-of-mass energy of meson and nucleon and p the momentum in this system:

$$p^2(s)=[s-(m+\mu)^2][s-(m-\mu)^2]/4s. \quad (4.3)$$

The angular integration may be performed over the direction of a unit vector $\hat{q}'$, with $\mathbf{q}'$ the c.m. momentum of the intermediate state, by which the projection operators in Sec. III have been labeled.

The three-particle phase space integral can be written as[4]:

$$\sum_3=\frac{2m}{(2\pi)^3}\int d^4p'\,\Theta(p_0')\delta(p'^2+m^2)\times\sum_2(s+m^2-2p_0'\sqrt{s}), \quad (4.4)$$

where p' is the four momentum of the intermediate nucleon and the argument of $\sum_2$ is the c.m. energy w of the two intermediate mesons, evaluated in the c.m. frame of the ingoing meson and nucleon:

$$w=-(q_1'+q_2')^2=-(p-p'+q)^2=s+m^2-2p_0'\sqrt{s}, \quad (4.5)$$

$\sum_2$ is, therefore, a phase space integral for two particles

[8] S. Ciulli and J. Fischer, Nuovo cimento **12**, 264 (1959).

[9] L. Cook and J. Tarski (to be published); Y. S. Kim (to be published); S. B. Treiman and P. Landshoff (to be published).

of equal mass:

$$\sum_2(w)=\frac{1}{32\pi^3}\int d\Omega_r \frac{q(w)}{\sqrt{w}}\Theta(w-4\mu^2), \tag{4.6}$$

where

$$q^2(w)=(w-4\mu^2)/4, \tag{4.7}$$

and $\mathbf{q}_r$ is the relative momentum of the intermediate mesons. Finally, we have for $\sum_3$

$$\sum_3=\frac{2m}{8(2\pi)^6}\int d\Omega_c\int d\Omega_r \times\int dp_0'\left(\frac{p'^2q^2(w)}{w}\right)^{\frac{1}{2}}\Theta(p_0'-m)\Theta(w-4\mu^2), \tag{4.8}$$

where

$$p'^2=p_0'^2-m^2. \tag{4.9}$$

Instead of p_0' we can also use w as integration variable. The p' has to be expressed by w, which can easily be done from (4.5):

$$p'^2(s,w)=[s-(m+\sqrt{w})^2][s-(m-\sqrt{w})^2]/4s. \tag{4.10}$$

Then all variables correspond to the angular decomposition of Sec. III ($\mathbf{p}'=-\mathbf{q}_1'-\mathbf{q}_2'$) and decomposition of (4.1) into partial waves becomes extremely simple. For abbreviation we define the densities:

$$\rho_2(s)=\frac{2m}{4(2\pi)^3}\left(\frac{p^2(s)}{s}\right)^{\frac{1}{2}}\Theta(s-(m+\mu)^2),$$

$$\rho_3(s,p_0')=\frac{2m}{8(2\pi)^6}\left(\frac{(p_0'^2-m^2)(s+m^2-2p_0's^{\frac{1}{2}}-4\mu^2)}{4(s+m^2-2p_0's^{\frac{1}{2}})}\right)^{\frac{1}{2}} \times\Theta(p_0'-m)\Theta(s+m^2-2p_0's^{\frac{1}{2}}-4\mu^2),$$

or

$$\rho_3(s,w)=\rho_3(s,p_0(s,w))\left|\frac{dp_0'}{dw}\right| =\frac{2m}{8(2\pi)^6}\left(\frac{p^2(s,w)q^2(w)}{4sw}\right)^{\frac{1}{2}} \times\Theta([p^2(s,w)+m^2]^{\frac{1}{2}}-m)\Theta(w-4\mu^2). \tag{4.11}$$

We now write down the linear unitarity relations which have been formulated by Blankenbecler[4]:

$$\begin{aligned}
&\sum_2 M_{22}(s)\frac{1}{\rho_2}D_{22}(s)+\sum_3 M_{23}(s)\frac{1}{\rho_3}D_{32}(s)=N_{22}(s),\\
&\sum_2 M_{22}(s)\frac{1}{\rho_2}D_{23}(s)+\sum_3 M_{23}(s)\frac{1}{\rho_3}D_{33}(s)=N_{23}(s),\\
&\sum_2 M_{32}(s)\frac{1}{\rho_2}D_{22}(s)+\sum_3 M_{33}(s)\frac{1}{\rho_3}D_{32}(s)=N_{32}(s),\\
&\sum_2 M_{32}(s)\frac{1}{\rho_2}D_{23}(s)+\sum_3 M_{33}(s)\frac{1}{\rho_3}D_{33}(s)=N_{33}(s),
\end{aligned} \tag{4.12}$$

where again only the center-of-mass energy s has been indicated, because we are working in the s channel. The solution of (4.12) for the causal amplitudes M_{ik} fulfills the unitarity relations (4.1) for N_{ik} with arbitrary "dynamical" cuts [cuts, beside the physical cut $s\geq(m+\mu)^2$], if the D_{ik} are defined by (4):

$$D_{ik}(s)=\delta_{ik}-\int^{\infty}\frac{ds'}{s'-s}\rho_i(s')N_{ik}(s'), \tag{4.13}$$

where the lower limit of integration is $(m+\mu)^2(i=2)$ or $(m+2\mu)^2(i=3)$. The δ_{kk} are δ functions. In our variables,

$$\begin{aligned}
&\delta_{22}=\delta(\Omega_{r'}-\Omega_r),\\
&\delta_{33}=\delta(\Omega_{c'}-\Omega_c)\delta(\Omega_{r'}-\Omega_r)\delta(p_0'-p_0).
\end{aligned} \tag{4.14}$$

If we define the absorptive parts across the dynamical cuts by

$$M_{ik}^{(+)}-M_{ik}^{(-)}=2\pi i A_{ik}, \tag{4.15}$$

we can express the N_{ik} in terms of D_{ik}:

$$N_{ik}(s)=\int_{C_{i2}}\frac{ds'}{s'-s}\sum_2 A_{i2}\frac{1}{\rho_2}D_{2k} +\int_{C_{i3}}\frac{ds'}{s'-s}\sum_3 A_{i3}\frac{1}{\rho_3}D_{3k}, \tag{4.16}$$

where the C_{ik} are the dynamical cuts of M_{ik} in the s plane.

Now there are two different ways of handling the problem. In any case one has to consider the $A_{ik}(s)$ as given functions. Then one can either put (4.16) in (4.13) to get a set of linear integral equations for the D_{ik} or insert (4.13) in (4.16) and get equations for the N_{ik}. If the contours C_{ik} are not too unwieldy, there is no special advantage in taking either way. If one neglects inelastic contributions, e.g., only the dynamical cuts of M_{22} appear, which all lie on the real axis, besides from kinematical cuts. But the three-particle amplitudes should have dynamical cuts anywhere in the s plane, so that it is more advantageous to treat the integral equations for the N_{ik}, in which only integrals over the real axis appear.

This can be seen easily. Suppose we put in N_{ik} the dynamical cuts of certain graphs $M_{ik}=G_{ik}$. Then we have

$$G_{ik}(s)=\int_{C_{ik}}\frac{ds'}{s'-s}A_{ik}(s'). \tag{4.17}$$

Inserting now (4.13) in (4.16) and using (4.17) we obtain:

$$N_{ik}(s)=G_{ik}(s) +\int_{(m+\mu)^2}^{\infty}ds'\sum_2\frac{G_{i2}(s')-G_{i2}(s)}{s'-s}\rho_2(s')N_{2k}(s') +\int_{(m+2\mu)^2}^{\infty}ds'\sum_3\frac{G_{i3}(s')-G_{i3}(s)}{s'-s} \times\rho_3(s')N_{3k}(s'). \tag{4.18}$$

Therefore, the most convenient procedure is to choose some G_{ik} from perturbation theory, solve (4.16) for the N_{ik}, determine D_{ik} from (4.13) and construct the unitary amplitudes which have the dynamical cuts of the chosen graphs G_{ik} from the linear unitarity conditions (4.12).

In the present formulation this program is far too complicated. We, therefore, now decompose (4.12) in partial waves. The remaining amplitudes then depend only on the energy variables s, w, w'. The decomposition of the linear unitarity conditions (4.12) with respect to angular momentum and isotopic spin can easily be carried out with the corresponding projection operators. We write the partial wave relations down in a fashion, where the energy $w=-(q_1'+q_2')^2$ of the meson pair is used as a second energy variable for the three particle amplitudes. All indices are displayed, but we use the abbreviation $\mathrm{L}=\{L,l_c,l_r\}$ for the angular momentum quantum numbers of the two-meson system. The sums are, therefore, to be extended over all l_c and l_r, which fulfill $\mathbf{l}_c+\mathbf{l}_r=\mathbf{L}$. Observing (3.12) and (3.13), we obtain from (4.12):

$$\begin{aligned}
{}^{I}N_{J\,l}{}^{22}(s)
&={}^{I}M_{J\,l}{}^{22}(s)\;{}^{I}D_{J\,l}{}^{22}(s)+\sum\int_{4\mu^2}^{\infty}dw'\;{}_{T'}{}^{I}M_{J\mathrm{L}'\,l}{}^{23}(s,w')\\
&\qquad\times{}_{T'}{}^{I}D_{J\mathrm{L}'\,l}{}^{32}(s,w'),\\
{}_{T}{}^{I}N_{J\mathrm{L}\,l}{}^{23}(s,w)
&={}^{I}M_{J\,l}{}^{22}(s)\;{}_{T}{}^{I}D_{J\mathrm{L}\,l}{}^{23}(s,w)\\
&\quad+\sum\int_{4\mu^2}^{\infty}dw'\;{}_{T'}{}^{I}M_{J\mathrm{L}'\,l}{}^{23}(s,w')\\
&\qquad\times{}_{T'T}{}^{I}D_{J\mathrm{L}'\mathrm{L}}{}^{33}(s,w',w),\\
{}_{T}{}^{I}N_{J\mathrm{L}\,l}{}^{32}(s,w)
&={}_{T}{}^{I}M_{J\mathrm{L}\,l}{}^{32}(s,w)\;{}^{I}D_{J\,l}{}^{22}(s)\\
&\quad+\sum\int_{4\mu^2}^{\infty}dw'\;{}_{TT'}{}^{I}M_{J\mathrm{L}\mathrm{L}'}{}^{33}(s,w,w')\\
&\qquad\times{}_{T'}{}^{I}D_{J\mathrm{L}'\,l}{}^{32}(s,w'),\\
{}_{TT''}{}^{I}N_{J\mathrm{L}\mathrm{L}''}{}^{33}(s,w,w'')
&={}_{T}{}^{I}M_{J\mathrm{L}\,l}{}^{32}(s,w)\;{}_{T''}{}^{I}D_{J\mathrm{L}''\,l}{}^{23}(s,w'')\\
&\quad+\sum\int_{4\mu^2}^{\infty}dw'\;{}_{TT'}{}^{I}M_{J\mathrm{L}\mathrm{L}'}{}^{33}(s,w,w')\\
&\qquad\times{}_{T'T''}{}^{I}D_{J\mathrm{L}'\mathrm{L}''}{}^{33}(s,w',w'')
\end{aligned}\tag{4.19}$$

[the sum is always to be extended over $T'(I)$ and $\mathbf{L}'+\boldsymbol{\sigma}/2=\mathbf{J}$]. By parity conservation only those production or absorption amplitudes appear for which $l_c+l_r+l=\text{odd}$, whereas for the three-particle scattering amplitude we have $l_c+l_r+l_c''+l_r''=\text{even}$.

As an example we specify the three-particle contribution of the first equation in more detail. According to (4.13), D_{32} is defined by:

$$D_{32}=-\int_{(m+2\mu)^2}^{\infty}\frac{ds'}{s'-s}\rho_3N_{32}.\tag{4.20}$$

Let us drop all indices now and first use the nucleon energy p_0' as the second energy variable, which is the original version. If we observe the theta functions in ρ_3 [see Eq. (4.11)], we obtain the following three-particle contribution:

$$-\int_{(m+2\mu)^2}^{\infty}\frac{ds'}{s'-s}\int_{m}^{(s'+m^2-4\mu^2)/2\surd s'}dp_0{}'M_{23}(s,\,p_0'+i\epsilon)\times N_{32}(s,\,p_0'-i\epsilon)\rho_3(s,p_0').\tag{4.21}$$

It must be emphasized[4] that we have to use the boundary values on different sides of the physical cut in the p_0 plane. We can also interchange the order of integration in (4.21) and obtain:

$$-\int_{m}^{\infty}dp_0'\int_{(p_0'+[4\mu^2-m^2+p_0'^2]^{\frac{1}{2}})^2}^{\infty}\frac{ds'}{s'-s}\times M_{23}(s,\,p_0'+i\epsilon)N_{32}(s',\,p_0'-i\epsilon)\rho_3(s',p_0').\tag{4.22}$$

Next we introduce w as variable (4.5):

$$w'=s'+m^2-2(s')^{\frac{1}{2}}p_0'.\tag{4.23}$$

Then (4.21) has the form:

$$-\int_{(m+2\mu)^2}^{\infty}\frac{ds'}{s'-s}\int_{4\mu^2}^{[(s')^{\frac{1}{2}}-m]^2}dw'\,M_{23}(s,\,w'-i\epsilon)\times\rho_3(w',s')N_{32}(s,\,w'+i\epsilon),\tag{4.24}$$

or with the order of integration interchanged:

$$-\int_{4\mu^2}^{\infty}dw'\int_{(m+\surd w')^2}^{\infty}\frac{ds'}{s'-s}M_{23}(s,\,w'-i\epsilon)\times N_{32}(s',\,w'+i\epsilon)\rho_3(w',s').\tag{4.25}$$

The latter form is to be used in the formulation (4.19). Therefore, the lower limits occurring under an integral sign are actually w' dependent. The next point is now to specify the integral equations in the single-meson exchange approximation.

V. SINGLE-MESON EXCHANGE APPROXIMATION

The single-meson exchange approximation is the simplest one which can be made in analyzing the effect of the one-meson production process on meson nucleon scattering. We do not consider here any dynamical singularities of the three-particle scattering amplitude M_{33}. We assume that the singularities of the single-meson exchange diagram (Fig. 4) dominate the low-energy behavior of the production amplitude M_{32}. They enter the functions N_{32}, (N_{23}), and N_{22}. The latter con-

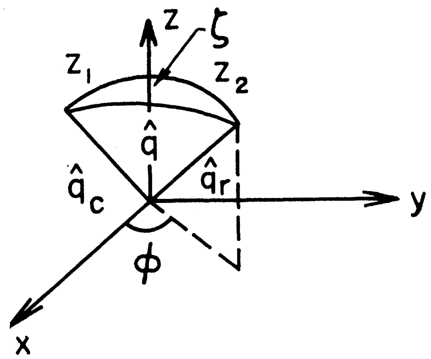

FIG. 6. Angular variables for M_{32}.

tains of course also all singularities of M_{22}, which are given by the Mandelstam representation. The partial-wave amplitude singularities of M_{22} have given by Frazer and Fulco.[10] Let us now first analyze the singularities of the diagram Fig. 4 and then set up the corresponding integral equations.

The M matrix element of the diagram (Fig. 4) is given by

$$M_{\gamma\delta\alpha}{}^{32}=\frac{\bar{u}(p')\gamma_5 u(p)}{(p-p')^2+\mu^2}g(-i)\tau_\beta M_{\gamma\delta,\alpha\beta}{}^{\pi\pi}, \tag{5.1}$$

where $M^{\pi\pi}$ is the $\pi-\pi$ amplitude, analytically continued in the mass of the exchanged meson $-(p-p')^2$. $M^{\pi\pi}$ can be written as[2]

$$M^{\pi\pi}=\sum_{T=0}^{2} M^T\mathcal{P}^T. \tag{5.2}$$

The projection operators for the states of total isospin T are

$$\begin{aligned}
P_{\gamma\delta,\alpha\beta}{}^0&=\tfrac{1}{3}\delta_{\gamma\delta}\delta_{\alpha\beta},\\
P_{\gamma\delta,\alpha\beta}{}^1&=\tfrac{1}{2}(\delta_{\gamma\alpha}\delta_{\delta\beta}-\delta_{\gamma\beta}\delta_{\delta\alpha}),\\
P_{\gamma\delta,\alpha\beta}{}^2&=\tfrac{1}{2}(\delta_{\gamma\alpha}\delta_{\delta\beta}+\delta_{\gamma\beta}\delta_{\delta\alpha})-\tfrac{1}{3}\delta_{\gamma\delta}\delta_{\alpha\beta}.
\end{aligned} \tag{5.3}$$

Using (2.19), we have

$$\begin{aligned}
\tau_\beta P_{\gamma\delta,\alpha\beta}{}^0&={}^0P_{\gamma\delta,\alpha}{}^{\frac{1}{2}},\\
\tau_\beta P_{\gamma\delta,\alpha\beta}{}^1&=(1/\sqrt{2})\,{}^1P_{\gamma\delta,\alpha}{}^{\frac{3}{2}}-\sqrt{2}\,{}^1P_{\gamma\delta,\alpha}{}^{\frac{1}{2}},\\
\tau_\beta P_{\gamma\delta,\alpha\beta}{}^2&=-\tfrac{1}{2}(10)^{\frac{1}{2}}\,{}^2P_{\gamma\delta,\alpha}{}^{\frac{3}{2}},
\end{aligned} \tag{5.4}$$

from which the partial amplitudes M_{32} of the matrix element (5.1) can be obtained.

Let us now introduce angular variables for the angular momentum projection operators. They depend on the three unit vectors $\hat{q}$, $\hat{q}_c$, $\hat{q}_r$ [see Eq. (3.11)]. We may choose (Fig. 6)

$$\begin{aligned}
\hat{q}&=(0,0,1),\\
\hat{q}_c&=(\sin\vartheta_1,0,\cos\vartheta_1),\\
\hat{q}_r&=(\sin\vartheta_2\cos\varphi,\sin\vartheta_2\sin\varphi,\cos\vartheta_2).
\end{aligned} \tag{5.5}$$

We have now a complete set of 5 variables for the amplitudes M_{32} and M_{23}, which are

$$s,\ w,\ Z_1,\ Z_2,\ \zeta,$$

where $Z_1=\cos\vartheta_1$, $Z_2=\cos\vartheta_2$, $\zeta=\cos\vartheta_1\cos\vartheta_2+\sin\vartheta_1\sin\vartheta_2\times\cos\varphi$.

In the residue of (5.1) at the pole $(p-p')^2=-\mu^2$, $M^{\pi\pi}$ is the $\pi-\pi$ scattering amplitude. We can, therefore, in principle use the results of Chew and Mandelstam[2] to describe $M^{\pi\pi}$. But this is actually quite hard, because the $\pi-\pi$ variables are complicated functions of s, w, Z_1, Z_2, and ζ. We have the following variables for $M^{\pi\pi}$:

$$\begin{aligned}
w&=-(q_1'+q_2')^2=-(p-p'+q)^2,\\
x&=-(q-q_2')^2\ =-(p-p'-q_1')^2,\\
y&=-(q-q_1')^2\ =-(p-p'-q_2')^2,
\end{aligned} \tag{5.6}$$

$$w+x+y=4\mu^2. \tag{5.7}$$

w belongs already to our five variables. Let us determine y as a function of s, w, Z_1, Z_2, and ζ. Then x is given by (5.7).

$$y=-(q-q_1')^2=(q_1'-q)^2+(q_{10}'-q_0)^2. \tag{5.8}$$

The spatial part may be written as (in the s channel c.m. system)

$$\begin{aligned}
(\mathbf{q}_1-\mathbf{q})^2&=[(\mathbf{q}_1'-\mathbf{q}_2')/2-(\mathbf{q}+\mathbf{p}'/2)]^2=[\mathbf{q}_r-(\mathbf{q}+\mathbf{p}'/2)]^2\\
&=q_r^2+(p^2+p'^2/4-pp'Z_1)\\
&\qquad+q_rp'\zeta/2-q_rpZ_2.
\end{aligned} \tag{5.9}$$

There, p is given as a function of s by (4.3) and p' as a function of s and w by (4.10). The magnitude q_r of the vector $\mathbf{q}_r$ is:

$$q_r^2=\left(\frac{w}{4}-\mu^2\right)\frac{w+p'^2}{w+p'^2(1-\zeta^2)}. \tag{5.10}$$

In the time part of (5.8) we have: $q_0=(\mu^2+p^2)^{\frac{1}{2}}$ and $q_{10}'=(\mu^2+q_1'^2)^{\frac{1}{2}}$, where

$$q_1'^2=q_r^2+p'^2/4+q_rp'\zeta. \tag{5.11}$$

Putting all these pieces together we obtain

$$y=y(s,w,Z_1,Z_2,\zeta).$$

The cosine of the scattering angle in the $\pi-\pi$ center-of-mass frame is then given by

$$\cos\Theta=1+y(s,w,Z_1,Z_2,\zeta)/2q^2(w). \tag{5.12}$$

If the Legendre expansion of $M^{\pi\pi}$ converges for those values of s, w, Z_1, Z_2, ζ which correspond to the pole $(p-p')^2=-\mu^2$, then one can approximate the isospin amplitudes $M^{\pi\pi}$ by the first terms of the expansion[2]:

$$\begin{aligned}
{}^0M^{\pi\pi}&\approx{}^0F_0(w),\\
{}^1M^{\pi\pi}&\approx{}^1F_1(w)\ 3\cos\Theta,\\
{}^2M^{\pi\pi}&\approx{}^2F_0(w),
\end{aligned} \tag{5.13}$$

where

$${}^TF(w)=[q^2(w)+\mu^2]^{\frac{1}{2}}/\mu q\exp(i\delta_l{}^T)\sin\delta_l{}^T. \tag{5.14}$$

[10] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960).

Furthermore, one may approximate the S amplitudes by

$$^{T}F_0(w)=a_T \exp\left\{(w-w_0)\int_{4\mu^2}^{\infty}\frac{\delta_0{}^T(w')}{(w'-w)(w'-w_0)}\right\}, \quad (5.15)$$

a_T is the value at the "symmetry point", $w_0=(4/3)\mu^2$ and expressible by the rationalized $\pi-\pi$ coupling constant[2] λ:

$$a_1=-5\lambda;\quad a_1=0;\quad a_2=-2\lambda. \quad (5.16)$$

The P-wave amplitude $^1F_1$ may be approximated by the resonant solution of Frazer and Fulco.[11]

Now, besides that it is difficult to show that we do not leave the domain of convergence, if s, w, z_1, z_2, and ζ run over all values for which $(p-p')^2=-\mu^2$, we do not believe that it is a good approximation to associate a $\pi-\pi$ phase with the outgoing three-particle state of M_{32}. As has already been pointed out in Sec. III, it is much more probable that the final-state behavior is dominated by a $\pi-N$ interaction in the 33 state (Fig. 5). (See, in this respect, also the work of Carruthers[7].) It seems, therefore, to be just as good, if we put in simply the $\pi-\pi$ coupling constant for $M^{\pi\pi}$, which may be defined as the value at the symmetry point[2]:

$$\begin{aligned} M^0&=-5\lambda;\\ M^1&=0; \\ M^2&=-2\lambda \quad \text{at} \quad x=y=w=\tfrac{4}{3}\mu^2. \end{aligned} \quad (5.17)$$

We then have, observing (5.4):

$$\begin{aligned} {}_0{}^{\frac{1}{2}}M_{32}&=\frac{\bar{u}(p')\gamma_5 u(p)}{(p-p')^2+\mu^2}i5g\lambda,\\ {}_2{}^{\frac{3}{2}}M_{32}&=\frac{\bar{u}(p')\gamma_5 u(p)}{(p-p')^2+\mu^2}(-i)g\lambda(20)^{\frac{1}{2}}. \end{aligned} \quad (5.18)$$

The other amplitudes M_{32} are zero.

Next we project out the partial amplitudes of

$$\frac{\bar{u}(p')\gamma_5 u(p)}{(p-p')^2+\mu^2}.$$

Using Hermitian γ matrices and the spinors (3.3), we have

$$\bar{u}(p')\gamma_5 u(p)=\chi'^*\frac{1}{2m}\{\epsilon p(\sigma\hat{p})-\epsilon^{-1}p'(\sigma\hat{p}')\}\chi, \quad (5.19)$$

where $\epsilon=[(E'+m)/(E+m)]^{\frac{1}{2}}$ and $p(p')$ is the magnitude of the ingoing (outgoing) nucleon's momentum. Because the two outgoing mesons are now in an S state relative to each other, we have $l_r=0$, $l_c=L$. Parity conservation tells us that L and l must differ by ±1. We have, therefore, two different sets of amplitudes:

$$\text{(a)}\quad J=l-\tfrac{1}{2}=L+\tfrac{1}{2},$$
$$\text{(b)}\quad J=l+\tfrac{1}{2}=L-\tfrac{1}{2}.$$

[11] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).

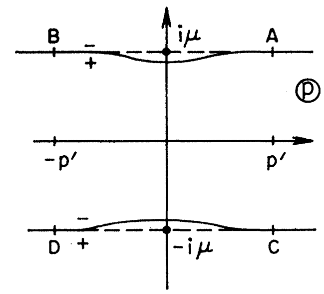

FIG. 7. Cuts of the M_{32}-pole term in the $|\mathbf{p}|$ plane for fixed $|\mathbf{p}'|$.

The corresponding projection operators can be taken from the tables of Ciulli and Fischer.[8]

$$\text{(a)}\quad P_{JL_+l_-}=\frac{1}{(4\pi)^{\frac{1}{2}}}\{P_{L+1}'(\sigma\hat{p})-(\sigma\hat{p}')P_L'\}=P_{L_{+-}},$$

$$\text{(b)}\quad P_{JL_-l_+}=\frac{1}{(4\pi)^{\frac{1}{2}}}\{P_L'(\sigma\hat{p}')-(\sigma\hat{p})P_{L-1}'\}=P_{L_{-+}}.$$

The argument of the Legendre polynomials is $z=z_1=\cos\vartheta_1$. Because

$$(p-p')^2+\mu^2=\mu^2-(p_0-p_0')^2+\mathbf{p}^2+\mathbf{p}'^2-2|\mathbf{p}|\,|\mathbf{p}'|z, \quad (5.21)$$

we obtain the following partial amplitudes, according to the definition (3.13):

$$\begin{aligned} M_{L_{+-}}{}^{32}&=\frac{(4\pi)^{\frac{1}{2}}}{4m}\left\{\frac{\epsilon}{p'}Q_L(\Lambda)-\frac{\epsilon^{-1}}{p}Q_{L+1}(\Lambda)\right\},\\ \Lambda&=\frac{\mu^2-(p_0-p_0')^2+p^2+p'^2}{2pp'},\\ M_{L_{-+}}{}^{32}&=\frac{(4\pi)^{\frac{1}{2}}}{4m}\left\{\frac{\epsilon}{p'}Q_L(\Lambda)-\frac{\epsilon^{-1}}{p}Q_{L-1}(\Lambda)\right\},\\ p&=|\mathbf{p}|,\\ p'&=|\mathbf{p}'|. \end{aligned} \quad (5.22)$$

The Q_L are Legendre functions of the second kind. The amplitudes for isospin eigenstates can be obtained from (5.22) by multiplication with the corresponding factor, taken from (5.18).

Q_L has branch points at $\Lambda=\pm1$, associated with

$$F(p,p')=\ln\frac{(p+p')^2+\mu^2-(p_0-p_0')^2}{(p-p')^2+\mu^2-(p_0-p_0')^2}.$$

The branch points are determined by

$$(p\pm p')^2=-\mu^2(1-\mu^2/4m^2)+pp'\mu^2/4m^2. \quad (5.23)$$

If we have p_0' and s as variables of the partial amplitudes, we are interested in the singularities of $F(p(s),p')$ for a fixed physical value of p'. There are two cuts in the p plane (Fig. 7), which may be so chosen that

$$(p\pm x)^2=-\mu^2(1-\mu^2/4m^2)+px\mu^2/4m^2,\quad -p'\leq x\leq p'. \quad (5.24)$$

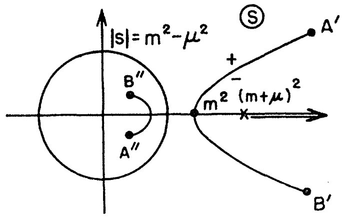

FIG. 8. Cuts of the M_{32}-pole term in the s plane for fixed $|\mathbf{p}'|$.

Next we transform to the p^2 plane: $p=(p^2)^{\frac{1}{2}}$. Because $F(p)=-F(-p)$, the cuts AB and CD are mapped onto different sheets of $p=(p^2)^{\frac{1}{2}}$. Therefore, there is a kinematical cut $0\leq p^2<\infty$. Finally, we go to the s plane with $p^2=p^2(s)$, (4.3), and obtain the mapping of AB, its reflection at the circle $|s|=(m^2-\mu^2)$ and the kinematical cut $(m+\mu)^2\leq s<\infty$ (Fig. 8). The point $p^2=-\mu^2(1-\mu^2/4m^2)$ corresponds to $s=m^2$. The cut $A'B'$ and its reflection at the circle are approximately given by the solution of

$$p^2(s)=x^2-\mu^2+2i\mu x,\quad -p'\leq x\leq p',\tag{5.25}$$

for s. It is a deformed parabola.

The other possibility is that we have s and w as variables. The singularities of $F(p(s),p(s,w))$ for a fixed physical value of w are more difficult to determine, because we have to solve (5.23) for s with $p=p(s)$ (4.3) and $p'=p'(s,w)$ (4.10). We shall not do so, because we do not need the position of the singularities at all. It is clear that there may be also a kinematical cut, associated with $p(s,w)$: $(m+\sqrt{w})^2\leq s<\infty$.

Now it is obvious that we have for partial amplitudes with $l_r=0$:

$$\begin{aligned}M_{Ll}^{32}(-p,p')&=(-1)^lM_{Ll}^{32}(p,p'),\\ M_{Ll}^{32}(p,-p')&=(-1)^LM_{Ll}^{32}(p,p').\end{aligned}\tag{5.26}$$

We may then avoid the kinematical cuts by dividing by $p^lp'^L$.

Finally, let us make a remark about the choice of quantum numbers for the $\pi-\pi-N$ state. It is likely that the $(\pi N)-\pi$ coupling quantum numbers s, v, j_r, l_r, l_c, J, M are "better" than the $(\pi-\pi)-N$ coupling: s, w, l_r, l_c, L, J, M we have used. [v is the c.m. energy of one of the mesons and the nucleon, e.g., $v=-(p'+q_1')^2$; l_r and l_c are the orbital angular momenta of the $(\pi-N)$ relative motion and the relative motion of the left π and the $(\pi-N)$ center of mass. The nucleon spin is coupled with l_r to angular momentum j_r.] From the corresponding projection operators, constructed analogously to (3.10) and (3.11), it is seen, however, that it would be very difficult to project out partial waves from (5.1) in these coordinates. So, we could not make a reasonable assumption about dynamical singularities, but have otherwise the advantage that it is very easy in these coordinates to associate a $(\pi-N)$ phase shift with the three-particle state in the sense of (5.15).

VI. INTEGRAL EQUATIONS

We shall now set up the integral equations for the partial amplitudes in the single-meson exchange approximation, i.e., we do consider the dynamical singularities of M_{22} as given by the Mandelstam representation, approximate the negative singularities of $M_{32}(M_{23})$ by the simplest one-meson exchange diagram (Fig. 4) and neglect negative singularities of M_{33}. Consequently, the two mesons in a three-particle state have no relative angular momentum $(l_r=0)$. According to the Pauli principle (Sec. III) only even values of T can appear: $T=0$ $(I=\frac{1}{2})$, $T=2$ $(I=\frac{3}{2})$. Then we have exactly four amplitudes M_{22}, M_{23}, M_{32}, M_{33} for each combination IJl, which are to be determined from the linear unitarity conditions (4.19):

$$\begin{aligned}&J=l+\tfrac{1}{2}=L-\tfrac{1}{2},\quad M_{l_+}{}^{22}, M_{L_-l_+}{}^{23}, M_{L_-l_+}{}^{32}, M_{L_-L_-}{}^{33};\\ &J=l-\tfrac{1}{2}=L+\tfrac{1}{2},\quad M_{l_-}{}^{22}, M_{L_+l_-}{}^{23}, M_{L_+l_-}{}^{32}, M_{L_+L_+}{}^{33}\end{aligned}\tag{6.1}$$

(l and L are the orbital angular momentum quantum numbers of two- and three-particle states).

Let us make contact with the usual phase-shift notation of the two-particle scattering amplitudes $f_{l_\pm}(s)=\exp(i\delta_{l_\pm})\sin\delta_{l_\pm}/p$. For the amplitudes $M_{l_\pm}(s)$ defined by (3.13) we have the elastic unitarity condition

$$\operatorname{Im}M_{l_\pm}=(mp/16\pi^2W)M_{l_\pm}{}^*M_{l_\pm};\quad W=\sqrt{s}.\tag{6.2}$$

Consequently,

$$f_{l_\pm}=(m/16\pi^2W)M_{l_\pm}.\tag{6.3}$$

We then define

$$f_{ik}=(m/16\pi^2W)M_{ik}.\tag{6.4}$$

The four amplitudes f_{ik}, belonging to a special set I, J, l, satisfy the unitarity conditions

$$\begin{aligned}&\operatorname{Im}f_{22}(s)\\ &\quad=p(s)f_{22}(s-i\epsilon)f_{22}(s+i\epsilon)\\ &\qquad+\int_{4\mu^2}^{\infty}dw'\rho_\pi(w',s)p(s,w')f_{23}(s-i\epsilon,w'-i\epsilon)\\ &\qquad\qquad\times f_{32}(s+i\epsilon,w'+i\epsilon),\\ &\operatorname{Im}f_{23}(s,w)\\ &\quad=p(s)f_{22}(s-i\epsilon)f_{23}(s+i\epsilon,w)\\ &\qquad+\int_{4\mu^2}^{\infty}dw'\,\rho_\pi(w',s)p(s,w')f_{23}(s-i\epsilon,w'-i\epsilon)\\ &\qquad\qquad\times f_{33}(s+i\epsilon,w'+i\epsilon,w),\\ &\operatorname{Im}f_{33}(s,w,\bar{w})\\ &\quad=p(s)f_{32}(s-i\epsilon,w)f_{33}(s+i\epsilon,\bar{w})\\ &\qquad+\int_{4\mu^2}^{\infty}dw'\rho_\pi(w's)p(s,w')f_{33}(s-i\epsilon,w,w'-i\epsilon)\\ &\qquad\qquad\times f_{33}(s+i\epsilon,w'+i\epsilon,\bar{w}),\end{aligned}\tag{6.5}$$

where

$$\rho_\pi(w,s)=\frac{1}{4(2\pi)^3}\frac{q(w)}{\sqrt{w}}\Theta(w-(m+\sqrt{s})^2)\tag{6.6}$$

is the density of two-meson states in phase space. Equation (6.5) can be obtained from (4.1) by performing the partial wave projection and observing (6.4).

Next we have to choose amplitudes which do not have any kinematical singularities. The reason is that no safe basis exists for approximating the corresponding residues or discontinuities. As has been discussed by several authors,[3,10] the discontinuities are neither bounded by unitarity nor given in terms of coupling constants. The only thing one could do is to prescribe the discontinuities of certain graphs across the kinematical cuts. But even if the dynamical singularities can be approximated by such graphs, there is no reason to believe that this is also good for kinematical singularities.

As is well known, the amplitudes $f_{l_\pm}$ can be expressed in terms of the partial-wave projections of the invariant functions A, B [Eq. (2.13)] by

$$f_{l_\pm}=\frac{E+m}{2W}\left\{\frac{A_l+(W-m)B_l}{4\pi}\right\}+\frac{E-m}{2W}\left\{\frac{-A_{l\pm1}+(W+m)B_{l\pm1}}{4\pi}\right\},\quad(6.7)$$

where

$$A_l(s)=\int_{-1}^{+1}\frac{dz}{2}A(s,t(s,z))P_l(z),$$

and analogously for B_l. $t(s,z)=-2p^2(s)(1-z)$ is the momentum transfer expressed by s and the scattering angle z. $(E+m)$ and $(E-m)$ can be written as functions of the total energy $W=\sqrt{s}$:

$$E\pm m=[(W\pm m)^2-\mu^2]/2W.\quad(6.8)$$

In order to avoid kinematical singularities, we have to work in the W plane and to multiply the $f_{l_\pm}$ by W^2, to drop a possible pole at the origin.[3] The elastic unitarity condition can also be used on the left-hand physical cut in the W plane ($W\leq -m-\mu$) observing the "reflection principle" of MacDowell[12]:

$$f_{l_\pm}(-W)=-f_{(l+1)_-}(W).\quad(6.9)$$

A similar consideration can be made for the partial wave projections of M_{32} and M_{23}. Observing that we have assumed $l_r=0$, we are reduced to two independent vectors. According to (2.14), we may, therefore, write in analogy to the two-particle case:

$$M_{32}=\gamma_5[A-\tfrac{1}{2}i\gamma(q+q')B]\quad(6.10)$$

where

$$q'=q_1'+q_2'.$$

[12] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).

Using the spinors (3.3), we obtain

$$\bar{u}(p')M_{32}u(p)=\frac{[(E'+m)(E-m)]^{\frac12}}{2m}(A-WB)(\boldsymbol{\sigma}\cdot\hat{p})-\frac{[(E+m)(E'-m)]^{\frac12}}{2m}(A+WB)(\boldsymbol{\sigma}\cdot\hat{p}').\quad(6.11)$$

The partial amplitudes can then be projected out with (5.20):

$$M_{L_\pm l_\mp}{}^{32}=\frac{[(E'+m)(E-m)]^{\frac12}}{2m}(A_L-WB_L)(\boldsymbol{\sigma}\cdot\hat{p})-\frac{[(E+m)(E'-m)]^{\frac12}}{2m}\times(A_{L\pm1}+WB_{L\pm1})(\boldsymbol{\sigma}\cdot\hat{p}'),\quad(6.12)$$

where

$$A_L(s,w)=\int_{-1}^{+1}\frac{dz}{2}A(s,w,t(s,w,z))P_L(z),\quad(6.13)$$

and $t(s,w,z)$ is the nucleon's momentum transfer:

$$t(s,w,z)=2m^2+2p(s)p(s,w)z-2[m^2+p^2(s)]^{\frac12}\times[m^2+p^2(s,w)]^{\frac12}.\quad(6.14)$$

$p(s)$ and $p(s,w)$ are given by (4.10) and (4.3). Combining Eqs. (6.12)–(6.14) and (6.4), we have the reflection principle,

$$f_{L_+l_-}(-W,\,w)=-f_{(L+1)_-(l-1)_+}(W,w).\quad(6.15)$$

There is no difficulty to show the same for M^{33}, which is essentially a two-particle scattering amplitude in our approximation. So we have

$$f_{L_+L_+}(-W,\,w)=-f_{(L+1)_-(L+1)_-}(-W,\,w).\quad(6.16)$$

The reflection principles (6.9), (6.15), (6.16) then enable us to use the unitarity conditions (6.5) also on the left-hand physical cut in the W plane. Working in the W plane then and multiplying by W^2 removes the kinematical singularities from f_{22} and analogously from f_{33}. But we are still left with kinematical singularities in $M_{L_\pm l_\mp}{}^{32}$, which arise from the $p(s)$ and $p(s,w)$ functions. To see what happens, if we cross the corresponding cuts, we write (6.12) in the form:

$$M_{L_\pm l_\mp}{}^{32}=\left(\frac{E'+m}{E+m}\right)^{\frac12}p\frac{(A_L-WB_L)}{2m}(\boldsymbol{\sigma}\cdot\hat{p})-\left(\frac{E+m}{E'+m}\right)^{\frac12}\times p'\frac{(A_{L\pm1}+WB_{L\pm1})}{2m}(\boldsymbol{\sigma}\cdot\hat{p}').\quad(6.16)$$

Observing (6.13) and (6.14), we see that the coefficients

of the roots are multiplied by:

$$\text{for } M_{L_+l_-}{}^{32}\begin{cases}(-1)^L \text{ crossing the } p' \text{ cut } W^2>(m+\sqrt{w})^2,\\ (-1)^{L+1}=(-1)^l \text{ crossing the } p \text{ cut } W^2>(m+\mu)^2,\end{cases}$$

$$\text{for } M_{L_-l_+}{}^{32}\begin{cases}(-1)^L \text{ crossing the } p' \text{ cut } W^2>(m+\sqrt{w})^2,\\ (-1)^{L-1}=(-1)^l \text{ crossing the } p \text{ cut } W^2>(m+\mu)^2.\end{cases}$$

Removing also the roots, we may then define

$$\begin{aligned} h_{L_+l_-}{}^{32}(W,w)&=\frac{W}{p^lp'^L}\left(\frac{E+m}{E'+m}\right)^{\frac{1}{2}}f_{L_+l_-}{}^{32}(W,w),\\ h_{L_-l_+}{}^{32}(W,w)&=\frac{W}{p^lp'^L}\left(\frac{E'+m}{E+m}\right)^{\frac{1}{2}}f_{L_-l_+}{}^{32}(W,w),\end{aligned}\tag{6.17}$$

as amplitudes without kinematical singularities. Equation (6.17) also means that we have "subtracted" just the correct "threshold" behavior at $p\to 0$ and $p'\to 0$, observing the reflection principle (6.15) at left-hand "threshold" in the W plane. We may check at the perturbation amplitudes (5.24) that in fact no singularities are introduced by the definition (6.17). Looking at the unitarity conditions (6.5) we are now forced to define h^{22} and h^{32} as follows:

$$\begin{aligned} h_{l_+}{}^{22}(W)&=\frac{W}{p^{2l}(E+m)}f_{l_+}{}^{22}(W);\\ h_{l_-}{}^{22}(W)&=\frac{W}{p^{2l}}(E+m)f_{l_-}{}^{22}(W),\end{aligned}\tag{6.18}$$

$$\begin{aligned} h_{L_+L_+}{}^{33}(W,w,\bar{w})&=\frac{W}{p'^L(w)p'^L(\bar{w})[E'(w)+m]^{\frac{1}{2}}[E'(\bar{w})+m]^{\frac{1}{2}}}\times f_{L_+L_+}{}^{33}(W,w,\bar{w}),\\ h_{L_-L_-}{}^{33}(W,w,\bar{w})&=\frac{W}{p'^L(w)p'^L(\bar{w})}[E'(w)+m]^{\frac{1}{2}}\times[E'(\bar{w})+m]^{\frac{1}{2}}f_{L_-L_-}{}^{33}(W,w,\bar{w}),\end{aligned}\tag{6.19}$$

where automatically also the correct threshold behavior has been "subtracted," in agreement with the reflection principles (6.9) and (6.16). The amplitudes (6.18) are those considered by Frazer and Fulco.[10]

Let us now specialize to the $J=l+\frac{1}{2}=L-\frac{1}{2}$ case. We may then drop the indices and write down the linear unitarity conditions (4.19) for the four coupled amplitudes h_{ik}. The D functions are given by

$$\begin{aligned} D_{22}(W)&=1-(W-m)\left\{\int_{m+\mu}^{\infty}+\int_{-\infty}^{-(m+\mu)}\frac{dW'}{\pi}\frac{p^{2l+1}(W')}{W'}\frac{(E+m)N_{22}(W')}{(W'-W)(W'-m)}\right\},\\ D_{23}(W,w)&=-(W-m)\left\{\int_{m+\mu}^{\infty}+\int_{-\infty}^{-(m+\mu)}\frac{dW'}{\pi}\frac{p^{2l+1}(W')}{W'}\frac{(E+m)N_{23}(W',w)}{(W'-W)(W'-m)}\right\},\\ D_{32}(W,w)&=-(W-m)\left\{\int_{m+\sqrt{w}}^{\infty}+\int_{-\infty}^{-(m+\sqrt{w})}\frac{dW'}{\pi}\rho_\pi(W',w)\frac{p^{2L+1}(W',w)}{W'(E'+m)}\right\}\frac{N_{32}(W',w)}{(W'-W)(W'-m)},\\ D_{33}(W,w,\bar{w})&=\delta(w-\bar{w})-(W-m)\left\{\int_{m+\sqrt{w}}^{\infty}+\int_{-\infty}^{-(m+\sqrt{w})}\frac{dW'}{\pi}\rho_\pi(W',w)\frac{p^{2L+1}(W',w)}{W'(E'+m)}\frac{N_{33}(W',w,\bar{w})}{(W'-W)(W'-m)}\right\}.\end{aligned}\tag{6.20}$$

We have made a subtraction at $W=m$ as usual. Next we introduce (6.20) into the definition of the N_{ik}, (4.16), and obtain coupled integral equations for the N_{ik}:

$$\begin{aligned} N_{22}(W)={}&G_{22}(W)+\int_{m+\mu}^{\infty}+\int_{-\infty}^{-(m+\mu)}\frac{dW'}{\pi}K_{22}(W',W)\frac{p^{2l+1}(W')}{W'}(E+m)N_{22}(W')\\ &+\int_{4\mu^2}^{\infty}dw'\left\{\int_{m+\sqrt{w'}}^{\infty}+\int_{-\infty}^{-(m+\sqrt{w'})}\right\}\frac{dW'}{\pi}\rho_\pi(W',w')K_{23}(W',W|w)\frac{p^{2L+1}(W',w')}{W'(E'+m)}N_{32}(W',w'),\\ N_{23}(W,w)={}&G_{23}(W,w)+\int_{m+\mu}^{\infty}+\int_{-\infty}^{-(m+\mu)}\frac{dW'}{\pi}K_{22}(W',W)\frac{p^{2l+1}(W')}{W'}(E+m)N_{23}(W',w)\\ &+\int_{4\mu^2}^{\infty}dw'\left\{\int_{m+\sqrt{w'}}^{\infty}+\int_{-\infty}^{-(m+\sqrt{w'})}\right\}\frac{dW'}{\pi}\rho_\pi(w',W')K_{23}(W',W|w')\frac{p^{2L+1}(W',w')}{W'(E'+m)}N_{33}(W',w',w),\\ N_{32}(W,w)={}&G_{32}(W,w)+\int_{m+\mu}^{\infty}+\int_{-\infty}^{-(m+\mu)}\frac{dW'}{\pi}K_{32}(W',W|w)\frac{p^{2l+1}(W')}{W'}(E+m)N_{22}(W'),\\ N_{33}(W,w,\bar{w})={}&\int_{m+\mu}^{\infty}+\int_{-\infty}^{-(m+\mu)}\frac{dW'}{\pi}K_{32}(W'W|w)\frac{p^{2l+1}(W')}{W'}(E+m)N_{23}(W',\bar{w}),\end{aligned}\tag{6.21}$$

where

$$K_{22}(W',W)=\frac{G_{22}(W')(W'-m)-G_{22}(W)(W-m)}{(W'-W)(W'-m)},$$
$$K_{23}(W',W|w)=\frac{G_{23}(W',w)(W'-m)-G_{23}(W,w)(W-m)}{(W'-W)(W'-m)}. \tag{6.22}$$

The integral equations can be derived from (4.13), transforming into the $W=\sqrt{s}$ plane and observing the subtraction at $W=m$.

The G_{ik} functions contain the approximations made for the dynamical singularities of the scattering amplitudes. If their location is known, they may be chosen by "polology" arguments. In our approximation G_{23} and G_{32} are given by the single-meson exchange terms. We have for instance for the $J=l+\frac{1}{2}=L-\frac{1}{2}$, $I=\frac{3}{2}$ amplitude (5.20), (5.21), and (6.17):

$$^{\frac{3}{2}}G_{L_-l_+}{}^{32}=-i\frac{g\lambda(20)^{\frac{1}{2}}}{4(4\pi)^{\frac{1}{2}}}\left\{\frac{\epsilon^2Q_L(\Lambda)}{p^lp'^{L+1}}-\frac{Q_{L-1}(\Lambda)}{p^{l+1}p'^L}\right\}. \tag{6.23}$$

For fixed w and $W\to\pm\infty$, G_{32} then behaves as $W^{-2L}\ln(\Lambda\to 1)$, where ln denotes the logarithmic singularity of Q_L for $\Lambda\to 1$. That $\Lambda\to 1$, follows from Eqs. (4.10) and (4.3). G_{23} is the same function, but has a $(+i)$. The simplest approximation for G_{22} is, of course, the single-nucleon term (Fig. 9)[3]:

$$^{\frac{3}{2}}G_{l_+}{}^{22}=\frac{g^2}{2p^{2l+2}}\left\{(W-m)Q_l(\Delta)+\frac{E-m}{E+m}(W+m)Q_{l+1}(\Delta)\right\}, \tag{6.24}$$

where

$$\Delta=1+(m^2+2\mu^2-W^2)/2p^2(W)$$

and $l\geq 1$. For $l=0$, we have additionally the $W^2=m^2$ pole term from the crossed diagram of Fig. 9. $G_{l_+}{}^{22}$ behaves as $W^{-(2L-1)}\ln(\Delta\to 1)$ for $W\to\pm\infty$. It can easily be checked that all integrals in an iteration solution of (6.21) exist. So, there is no principal difficulty to calculate the N_{ik} functions in the physical region and then the D_{ik} according to (6.20). Finally, we have to solve the linear unitarity conditions (4.19) for the amplitudes h_{ik}. It may be mentioned that by "subtracting" the threshold behavior of the amplitudes, the convergence in the integrals, defining the N and D functions, is not affected. The advantage is only that the correct threshold behavior is guaranteed. We emphasize that it is more favorable in this case to set up integral equations for the N functions rather than for the D functions. Integrals over contours in the complex plane are then avoided (Sec. IV). It is, therefore, not absolutely necessary to determine the location of the perturbation function singularities, which serve as approximations for the discontinuities across the dynamical cuts.

As a first approximation, one may put

$$N_{22}=G_{22},\quad N_{23}=G_{23},\quad N_{32}=G_{32},\quad N_{33}=0. \tag{6.25}$$

G_{32} and G_{23} shall be given by (6.23). For N_{22} an "elastic" N function may be used, which has been calculated taking only the elastic unitarity condition into account. For instance, in the $J=\frac{3}{2}$, $I=\frac{3}{2}$ case we can take the N function, calculated by Frautschi and Walecka.[3] Then (6.25) may serve as a basis for the treatment of inelastic effects. The D functions can then be obtained from (6.20). Because $D_{33}=\delta(w-\bar{w})$, the linear unitarity conditions have the simple form:

$$N_{22}=h_{22}D_{22}+\int_{4\mu^2}^{\infty}dw'h_{23}D_{32},$$
$$N_{23}=h_{22}D_{23}+h_{23}. \tag{6.26}$$

Consequently,

$$h_{22}=\left(N_{22}-\int_{4\mu^2}^{\infty}dw'N_{23}D_{32}\right)\Big/\left(D_{22}-\int_{4\mu^2}^{\infty}dw'D_{23}D_{32}\right). \tag{6.27}$$

Contact with experiment can then be made by compari-

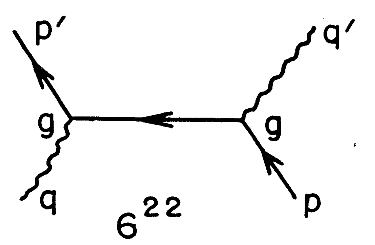

FIG. 9. Lowest order approximation for G_{22}.

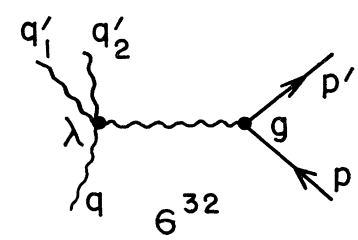

FIG. 10. Lowest order approximation for G_{32}.

son with the effective-range formula:

$$\frac{4}{3\mu^2}\frac{p^3}{W-m}\cot\delta_{33}=\frac{4}{3\mu^2}\frac{W}{(E+m)(W-m)}\operatorname{Re}\frac{1}{h_{22}}$$
$$=\frac{1}{f^2}\frac{(W_r-W)}{W_r-m}, \tag{6.28}$$

where W_r is the position of the 33 resonance.

ACKNOWLEDGMENTS

I am greatly indebted to Dr. M. L. Goldberger for several discussions and reading the manuscript. I further wish to thank Dr. R. Blankenbecler for helpful conversations and a preprint of his work.

The hospitality of the Physics Department of Princeton University is gratefully acknowledged.

PHYSICAL REVIEW VOLUME 125, NUMBER 2 JANUARY 15, 1962

Conservation of Hypercharge*

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(Received May 17, 1961)

It is shown that the law of conservation of hypercharge follows from a certain permutation symmetry of the Lagrangian provided that trilinear interactions are assumed. The permutation involves "primitive" isospin doublet fields, which are linear combinations of fields corresponding to the observed isospin doublets N, Ξ, K, K^G. No permutation symmetry for the observed particles is implied. If the assumption of trilinear interactions is not made, a multiplicative law of hypercharge conservation "modulo 4" is obtained. An argument is presented to show that this is inconsistent with experiment.

I. INTRODUCTION

According to the present description of elementary particles, the strong interactions obey several conservation laws, and hence are invariant under certain "internal" transformations. These are the 3-dimensional isospin group, and two phase groups, which may be taken to be the baryon gauge group and the hypercharge gauge group. Since the known particles satisfy the Gell-Mann–Nishijima relation,

$$Q=I_3+\tfrac{1}{2}U, \tag{1}$$

the conservation of electric charge follows automatically from these for such particles.[1] It is of some interest to see whether the existence of such invariances may be derived from a different assumption, which may be applicable beyond the strong interactions. In this note, we show that a multiplicative law of hypercharge conservation, of the type considered by D'Espagnat and Prentki, and by Racah,[2] is implied by invariance of the strong interactions under a certain permutation of the fields involved. The permutation symmetry is similar to the one studied previously in connection with the muon-electron system,[3,4] and the argument given below is a simple extension of previous arguments to the strong interactions.

It is known that if one restricts interactions to trilinear ones, then the multiplicative law of hypercharge conservation implies the full hypercharge gauge group, which is equivalent to an additive hypercharge conservation law. Such an additive law is required by experiment, for reasons we indicate at the end of this note, and therefore it is necessary to impose some additional restrictions such as trilinear couplings in order to obtain the hypercharge gauge group. We will therefore consider only such couplings in this paper. Such restrictions have been considered before in other connections.

II. DERIVATION OF HYPERCHARGE CONSERVATION

We assume that the strong interactions involve the following groups of particles.

(1) Two baryon isotopic doublets B_1, B_2:

$$B_1=\begin{pmatrix}B_1{}^1\\B_1{}^2\end{pmatrix},\quad B_2=\begin{pmatrix}B_2{}^1\\B_2{}^2\end{pmatrix}. \tag{2}$$

We group these into a quadruplet B by taking

$$B=\begin{pmatrix}B_1\\B_2\end{pmatrix},$$

a doublet in a new space which we call hypercharge space.

(2) A meson isotopic doublet M, and a doublet M^G,

* Supported by the Atomic Energy Commission.
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[1] This need not be the case for particles not obeying Eq. (1). In particular, it is possible to construct interactions for such particles which conserve hypercharge, isospin, and baryon number, but not electric charge.

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