Effect of Quantum Fluctuation on Vertical Oscillations of an Electron Moving in a Magnetic Field*†

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In this work two extreme cases of vertical motion of a radiating electron in a magnetic field are studied by means of Dirac's relativistic quantum theory. First, free motion of the electron along the magnetic field is considered (continuous spectrum). In the second case the motion of the electron along the field is limited by a potential well with infinite walls (discrete spectrum). The analysis of these cases establishes a connection between quantum fluctuations and the classical radiation damping force.

1. FREE MOTION OF AN ELECTRON ALONG THE FIELD

To investigate the motion of a relativistic electron in a magnetic field directed along the z axis,

$$A_x=-\tfrac{1}{2}yH,\quad A_y=\tfrac{1}{2}xH,\quad A_z=0, \tag{1}$$

it is necessary to use wave functions which are solutions to the Dirac equation. In cylindrical coordinates these solutions have the following form[1]:

$$\psi=\left[\frac{\gamma}{(1+k_0/K)2\pi L}\right]^{\frac{1}{2}}\begin{bmatrix} 0 \\ i\left(1+\dfrac{k_0}{K}\right)I_{n,s}(\rho) \\ \dfrac{(4\gamma n)^{\frac{1}{2}}}{K}I_{n-1,s}(\rho)e^{-i\varphi} \\ -\dfrac{ik}{K}I_{n,s}(\rho)\end{bmatrix}e^{i(l\varphi+kz-cKt)}. \tag{2}$$

Here the electron energy is $E=c\hbar K$, where

$$K=(k_0^2+k^2+4\gamma n)^{\frac{1}{2}}; \tag{3}$$

$m_0=\hbar k_0/c$ is the electron mass; $\rho=\gamma r^2$; $\gamma=eH/2c\hbar$; $k=(2\pi/L)m$ is the momentum along the field; and $n=0, 1, 2\cdots$, $l=n-s$, $s=0, 1, 2\cdots$, $m=0, \pm1, \pm2\cdots$ are the principal, orbital, radial, and magnetic quantum numbers, respectively. The function $I_{n,s}(\rho)$ is related to the Laguerre polynomials,

$$Q_s^l=(-1)^s\sum_{j=0}^{j=s}(-1)^j\frac{s!(s+l)!\rho^{s-j}}{j!(s-j)!(s+l-j)!},$$

as follows:

$$I_{n,s}(\rho)=\frac{1}{(n!s!)^{\frac{1}{2}}}e^{-\frac{1}{2}\rho}\rho^{\frac{1}{2}(n-s)}Q_s^{n-s}(\rho). \tag{4}$$

The wave function (2) describes a state of the electron with definite direction of spin. Since, in this problem, we are not interested in spin effects, the second solution will not be taken into consideration.

In the transition of the electron from the initial state n, s, k, to final state n', s', k', a photon is emitted, with a frequency determined by the following expression:

$$\omega=c\kappa=c(K-K')=c(k_0^2+k^2+4\gamma n)^{\frac{1}{2}}-c(k_0^2+k'^2+4\gamma n)^{\frac{1}{2}}. \tag{5}$$

The intensity of radiation is then determined by

$$W_{nn',ss',kk'}=\int_0^{\pi}c\hbar\kappa w\sin\theta d\theta, \tag{6}$$

where the transition probability w, related to the matrix elements

$$\bar{\alpha}_\mu=\int\psi^\dagger_{n',s',k'}\exp(-i\boldsymbol{\kappa}\cdot\mathbf{r})\alpha_\mu\psi_{n,s,k}d^3x, \tag{7}$$

is given, including terms of order ν/n, by the following expression:

$$w=\frac{e^2}{\hbar}\frac{\kappa}{1+\partial K'/\partial\kappa}\times[|\bar{\alpha}_1|^2+|\bar{\alpha}_2|^2\cos^2\theta-(\bar{\alpha}_3\bar{\alpha}_2^*+\bar{\alpha}_2\bar{\alpha}_3^*)\sin\theta\cos\theta], \tag{8}$$

where $\nu=n-n'$ is the number of the radiated harmonic. Calculation of the matrix elements gives

$$\begin{gathered}|\bar{\alpha}_1|^2=\frac{\gamma n}{K^2}|A_{kk'}|^2I_{s,s'}^2(x)[I_{n-1,n'}(x)-I_{n,n'-1}(x)]^2,\\ |\bar{\alpha}_2|^2=\frac{\gamma n}{K^2}|A_{kk'}|^2I_{s,s'}^2(x)[I_{n-1,n'}(x)+I_{n,n'-1}(x)]^2,\\ \bar{\alpha}_3\bar{\alpha}_2^*+\bar{\alpha}_2\bar{\alpha}_3^*=-(\nu\beta/2n)\cos\theta|\bar{\alpha}_2|^2.\end{gathered} \tag{9}$$

$I_{n,n'}$ is determined by relation (4), $x=(\kappa^2\sin^2\theta)/4\gamma$, and the coefficient

$$A_{kk'}=\frac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\exp[i(k-k')z-i\kappa z\cos\theta]dz. \tag{10}$$

It is readily seen that the square modulus of this

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[1] A. A. Sokolov and I. M. Ternov, J. Exptl. Theoret. Phys. (U.S.S.R.) 25, 698 (1953).

coefficient has the meaning of the Kronecker δ symbol:[2]

$$|A_{kk'}|^2=\delta_{k',k-\kappa\cos\theta}{}^2 \tag{11}$$

which describes the law of conservation of momentum along field. Taking this into consideration, we find the frequency of radiation

$$\kappa=\left(\frac{4\gamma n}{\beta^2-\beta_1{}^2}\right)^{\frac{1}{2}}\frac{1-\beta_1\cos\theta}{\sin^2\theta}\times\left[1-\left(1-\frac{\nu}{n}\frac{[\beta^2-\beta_1{}^2]\sin^2\theta}{[1-\beta_1\cos\theta]^2}\right)^{\frac{1}{2}}\right]. \tag{12}$$

Here $c\beta_1=c(k/K)\ll c$ is the z component of the electron velocity $c\beta=c[(K^2-k_0{}^2)^{\frac{1}{2}}/K]\cong c$. As a result of the approximation in which the matrix elements were replaced by Bessel functions of orders $\frac{1}{3}$ and $\frac{2}{3}$ with imaginary argument, the transition probability becomes

$$w=\frac{1}{3\pi^2}\frac{e^2}{\hbar}\frac{\nu}{R}\frac{(1-\beta_1{}^2)^{\frac{1}{2}}}{(1-\beta_1\cos\theta)^2}\left\{\frac{(1-\beta_1{}^2)^2\sin^2\theta}{(1-\beta_1\cos\theta)^2}\epsilon^2\mathcal{K}_{2/3}{}^2(\tfrac{1}{3}\nu\epsilon^{\frac{3}{2}})+\left(\frac{\cos\theta-\beta_1}{\sin\theta}\right)^2\epsilon\mathcal{K}_{\frac{1}{3}}{}^2(\tfrac{1}{3}\nu\epsilon^{\frac{3}{2}})\right\}|A_{kk'}|^2 I_{s,s'}{}^2, \tag{13}$$

where

$$\epsilon=\left[1-\frac{\beta^2\sin^2\theta(1-\beta_1{}^2)}{(1-\beta_1\cos\theta)^2}\right]\left(1+\frac{\nu}{2n}\right),\quad I_{s,s'}=I_{s,s'}(x).$$

In previous work[1] using this method, an influence of quantum effects on the intensity of radiation and radial betatron oscillation was established. Here we want to investigate the motion along the z axis. Let us now multiply the change of the momentum along the field during the transition, $\Delta k=k'-k=-\kappa\cos\theta$, by the probability w and sum over all possible transitions:

$$\frac{d\langle k\rangle_{\text{av}}}{dt}=\sum_{n',s',k'}\int_0^\pi(k'-k)w\sin\theta d\theta=-\langle k\rangle_{\text{av}}\frac{W}{E}, \tag{14}$$

where

$$W=\tfrac{2}{3}(e^2c/R^2)(E/m_0c^2)^4.$$

Thus, the electron experiences a radiation force which causes an attenuation in agreement with the classical theory of motion.[3,4] It is worth mentioning that the attenuation automatically follows from the quantum theory [see Eq. (14)].

By analogy with (14) we obtain for the square of the momentum,

$$\frac{d\langle k^2\rangle_{\text{av}}}{dt}=-2\langle k^2\rangle_{\text{av}}\frac{W}{E}+\frac{13}{24\sqrt{3}}\frac{1}{(1-\beta_1{}^2)^{\frac{1}{2}}}\frac{e^2}{\hbar R^3}\left(\frac{E}{m_0c^2}\right)^5. \tag{15}$$

To derive the last equation, use was made of the relation

$$k'^2-k^2=2k(k'-k)+(k'-k)^2,$$

according to which the quadratic fluctuation of the momentum component along the field is characterized by the following quantity, that grows adiabatically with time:

$$\frac{\langle(\Delta k)^2\rangle_{\text{av}}}{\Delta t}=\sum_{n',s',k'}\int_0^\pi(k'-k)^2w\sin\theta d\theta=\frac{13}{24\sqrt{3}}\frac{1}{(1-\beta_1{}^2)^{\frac{1}{2}}}\frac{e^2}{\hbar R^3}\left(\frac{E}{m_0c^2}\right)^5. \tag{16}$$

Here $\langle(\Delta k)^2\rangle_{\text{av}}=\langle(k'-k)^2\rangle_{\text{av}}$ is the average square fluctuation.

Analogous quantities are met with in a number of problems of statistical physics, for instance in the theory of the Brownian motion.

Since the function (16) is present in Eq. (15), the quantity $\langle k^2\rangle_{\text{av}}$ becomes limited by a nonzero lowest value.

2. LIMITED MOTION ALONG THE FIELD

Assume that the electron is in a potential well with infinite walls and let the width of the well be finite and equal to $2L$. The solutions of the Dirac equation satisfying the boundary conditions have the following form:

$$\psi=\left[\frac{\gamma}{(1+k_0/K)2\pi L}\right]^{\frac{1}{2}}\times\begin{pmatrix}0\\ -\left(1+\frac{k_0}{K}\right)\sin\left[\frac{\pi m}{2L}(z-L)\right]I_{n,s}(\rho)\\ \frac{i(4\gamma n)^{\frac{1}{2}}}{K}\sin\left[\frac{\pi m}{2L}(z-L)\right]e^{-i\varphi}I_{n-1,s}(\rho)\\ -\frac{i\pi m}{2LK}\cos\left[\frac{\pi m}{2L}(z-L)\right]I_{n,s}(\rho)\end{pmatrix}e^{il\varphi-icKt}. \tag{17}$$

Keeping only classical terms in the calculation of the Dirac matrix elements and neglecting $\beta_1{}^2$, we obtain

$$\begin{gathered}|\bar{\alpha}_1|^2=(\gamma n/K^2)|A_{mm'}|^2I_{s,s'}{}^2(I_{n-1,n'}-I_{n,n'-1})^2,\\ |\bar{\alpha}_2|^2=(\gamma n/K^2)|A_{mm'}|^2I_{s,s'}{}^2(I_{n-1,n'}+I_{n,n'-1})^2,\\ \bar{\alpha}_3\bar{\alpha}_2{}^*+\bar{\alpha}_2\bar{\alpha}_3{}^*=0,\end{gathered} \tag{18}$$

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[2] This follows from the relation $\sum_{k'}(k')^\lambda|A_{kk'}|^2=(k-\kappa\cos\theta)^\lambda$, where $\lambda=0, 1, 2, 3\cdots$.

[3] A. A. Sokolov, N. P. Klepikov, and I. M. Ternov, J. Exptl. Theoret. Phys. (U.S.S.R.) **24**, 249 (1953).

[4] Assuming $\beta_1=0$, $\nu/n=0$, and multiplying (13) by $c\hbar\kappa$, we obtain the formula for synchrotron radiation in the classical approximation. [See D. D. Ivanenko and A. A. Sokolov, Doklady Akad. Nauk. U.S.S.R. **59**, 1551 (1948); J. Schwinger, Phys. Rev. **75**, 1912 (1949).]

and

$$A_{mm'}=\frac{1}{L}\int_{-L}^{L}\sin\left[\frac{\pi m}{2L}(z-L)\right]\times\sin\left[\frac{\pi m'}{2L}(z-L)\right]e^{-i\kappa z\cos\theta}dz. \quad (19)$$

For the transition probability and the frequency of radiation we have the following expressions:

$$w=\frac{e^2}{\hbar}\frac{\nu}{R}\frac{1}{3\pi^2}(\epsilon^2\sin^2\theta\mathcal{K}_{2/3}{}^2+\epsilon\cot^2\theta\mathcal{K}_{\frac{1}{3}}{}^2)|A_{mm'}|^2I_{s,s'}{}^2,$$
$$\kappa=\nu\beta/R,\quad \mathcal{K}_{2/3}=\mathcal{K}_{2/3}(\tfrac{1}{3}\nu\epsilon^{\frac{3}{2}}),\quad \epsilon=(1-\beta^2\sin^2\theta). \quad (20)$$

Since in this case we will have, instead of Eq. (11), the following:

$$|A_{mm'}|^2=\tfrac{1}{2}\{\delta_{m'^2,[m+(2L/\pi)\kappa\cos\theta]^2}+\delta_{m'^2,[m-(2L/\pi)\kappa\cos\theta]^2}\}, \quad (21)$$

then for the square of the momentum we obtain the formula

$$\frac{d\langle k^2\rangle_{\mathrm{av}}}{dt}=\sum_{n',s',m'}\int_0^{\pi}\frac{\pi^2}{4L^2}(m'^2-m^2)w\sin\theta d\theta=\frac{13}{24\sqrt{3}}\frac{e^2}{\hbar R^3}\left(\frac{E}{m_0c^2}\right)^5, \quad (22)$$

which differs from the previous case (continuous spectrum) in that, instead of attenuation due to radiation, we will have weak quantum excitations which practically provide a stabilization of the oscillation since classical attenuation must disappear.

Let us now make some remarks on the motion of the electron in a focussing magnetic field of cyclic accelerator.

It is known that if an electron is slightly deflected from the median plane $z=0$, it performs a harmonic oscillation with the amplitude damped according to the following expression[5]:

$$b=b_0\left(\frac{H(0)}{H(t)}\right)^{\frac{1}{2}}\exp\left(-\frac{1}{2}\int_0^t\frac{W}{E}dt\right). \quad (23)$$

According to quantum mechanics the motion of such an electron can be considered as a motion of the center of a wave packet [see Eqs. (14–16) for the free motion] when the results of quantum theory must coincide with the classical.

Due to the influence of the quantum fluctuations a spread of the wave packet should take place and one must begin using quantum mechanics to describe the vertical oscillations of the electron [see formula (22)] as a kind of "macroatom."[6,7] Thus, the classical attenuation and quantum excitations should represent by themselves two extreme cases of different manifestations of the influence of the emitted photons on the oscillation of the electron in a synchrotron.

According to other theories[8] the influence of the classical radiation damping force and quantum excitation should take place independently of each other. Moreover, according to reference 5 the quantum excitation practically should not influence the vertical oscillation to any extent. Due to the radiation attenuation the amplitude of the latter must approach zero. Experimental investigations[9] show that the amplitude of the vertical oscillations approaches a limit different from zero. It should be noted that we were unable as yet to obtain the theoretical criterion of the transition from the classical motion with quantum fluctuations gradually coming into play [see Eq. (14)], to the other limiting case—that of the quantum motion [see Eq. (22)] when there is no classical attenuation at all.

[5] K. W. Robinson, Phys. Rev. **111**, 373 (1958); A. A. Kolomenskii, A. N. Lebedev, *Proceedings of the* CERN *Conference on High-Energy Accelerators, Geneva, 1956* (European Organization of Nuclear Research, Geneva, 1956), Vol. 1, p. 447.

[6] A. A. Sokolov, D. D. Ivanenko, and I. M. Ternov, Doklady Akad. Nauk. (U.S.S.R.) **111**, 334 (1956).

[7] A. A. Sokolov and I. M. Ternov, Doklady Akad. Nauk. (U.S.S.R.) **117**, 967 (1957).

[8] M. Sands, Phys. Rev. **97**, 470 (1955); Nuovo cimento **15**, 599 (1960).

[9] F. A. Korolev, O. F. Kulikov, and A. G. Ershov, Nuovo cimento **18**, 4257 (1960). The influence of quantum fluctuations on the motion of electrons in a synchrotron for the first time was observed experimentally by Sands.[8]