where ν_L is the total meson energy in the lab, κ^2 is one-fourth of the invariant momentum transfer squared, and M is the Λ mass.[7]

According to reference (6), we have approximately

$$-\frac{6}{q^2}\mathrm{Re}f_{P\frac{3}{2}}=\mathrm{Re}\frac{E+M}{2W}\frac{1}{4\pi}\times[A'+\tfrac{1}{2}q^2A''+(W-M)(B'+\tfrac{1}{2}q^2B'')], \quad (3)$$

where the primes denote derivatives with respect to κ^2 evaluated at $\kappa^2=0$, and q is the center-of-mass meson momentum. The total energy and the total nucleon energy in the c.m. system are denoted by W and E, respectively. The dispersion relations are now used to evaluate the right-hand side of Eq. (3), where we keep only s and $p_{\frac{3}{2}}$ terms under the dispersion integrals, and only keep terms to first order in the recoil (i.e., $1/M$).

For our purposes, it is only necessary to record the correction terms due to the coupling of s wave into p wave to first order in recoil. These correction terms are denoted by a bar over the appropriate quantities. We then have that

$$-\frac{6}{q^2}\mathrm{Re}\bar{f}_{P\frac{3}{2}}\approx\frac{1}{2\pi^2M}\int_1^\infty d\nu_L'\frac{\sigma_s q'}{(\nu_L'+\nu_L)^2}\equiv\epsilon(\nu_L), \quad (4)$$

where σ_s is the total s-wave π-Λ cross section. We might note that the correction approaches zero as M becomes large, contrary to the situation in pion-nucleon scattering where the coupling of p wave into s wave goes linearly with the mass. If one estimates the $p_{\frac{3}{2}}$ contribution to the dispersion integrals by the global symmetry solution to the p-wave equations, then one sees that the shift $\Delta\omega$ in the resonance energy (ω_0) is given approximately by

$$\Delta\omega/\omega_0=-(f^2/3)(q_0{}^6/\omega_0)\epsilon(\nu_L{}^0). \quad (5)$$

This gives a decrease in the resonant energy of only 10 Mev or less, even if the maximum value for σ_s ($4\pi/q^2$) is taken.[8] This is then a maximum estimate of the effect within the framework given here.

This simple calculation (which has ignored any possible nonglobal effects in the p wave) argues that if global symmetry is a valid pion-baryon coupling scheme, then the Y_1^* should, at least in part, be a $p_{\frac{3}{2}}$ resonance. Should this later turn out not to be the case, we would then have an argument against global symmetry.

[7] The invariant scattering amplitude T is given by $T=-A+i\gamma\cdot QB$ with $Q=\frac{1}{2}(q_1+q_2)$ where q_1 and q_2 are the initial and final pion four-momenta.

[8] Note that no low- or high-energy cutoff is required.

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Wave Functions*

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A convenient definition of wave functions, in terms of S-matrix quantities, may be obtained from the dispersion relations for hypothetical reactions in which a probe interacts impulsively with a target. The wave functions defined in this way are useful for constructing an impulse approximation for complicated reactions. These wave functions also satisfy a wave equation.

I. INTRODUCTION

DISPERSION theory techniques have been applied with great success to such problems as the analysis of the form factors of "simple" particles (those which do not lead to anomalous thresholds) and the analysis of a transition amplitude $T(a+b\to c+d)$ involving four simple particles.[1] There is no doubt that dispersion theory is also applicable to more complicated problems, in which additional particles are present, either in the final state of a production process, or through the virtual decay of a nonsimple particle. In these more complicated problems, techniques for practical calculation are relatively undeveloped although, partial results have already been obtained by straightforwardly extending the systematic methods used in simpler problems.[2,3] However, in production processes it is possible that the systematic method of writing down a dispersion relation in one or more variables will not turn out to be the most useful one, because there are so many independent variables, and because the singularities, which lie on complicated curves in complex regions, are very difficult to treat exactly. It is certainly clear, that more intuitive or "physical" approaches which would help single out the most

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[1] For an account, see G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, to be published).
[2] R. E. Cutkosky, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), Vol. 10, p. 236.
[3] R. Blankenbecler, Phys. Rev. **122**, 983 (1961).

important variables and suggest useful first approximations are also desirable.

In many problems, experience with classical mechanics and with wave mechanics suggests that some form of impulse approximation will be useful. The topic of this paper, an outgrowth of this observation, is the construction from the theory of S-matrix singularities of an impulse approximation scheme which parallels that of elementary wave mechanics as closely as possible. In order to do this, it is first necessary to define, in terms of S-matrix quantities, "wave functions" analogous to those of the elementary theory. We note that it is implicit in the elementary wave mechanics that wave functions can be measured directly in impulsive experiments.

The approach developed in this paper may be summarized as follows: An S-matrix definition of wave functions is obtained in terms of the scattering from a given "target" of an idealized "probe" which interacts impulsively with it. Such wave functions are then used to define an impulse approximation for the interaction of real probes, using the familiar wave mechanical expressions as a guide. This is done by examining the singularities of the transition amplitude which lie closest to the physical region, and then writing an expression which reproduces the most important singularities exactly, and others approximately. The correction to the impulse approximation can then be estimated from a dispersion relation for the difference between the approximate and exact amplitudes, or, perhaps, from a more complicated form of the impulse approximation which incorporates the most important remaining singularities.

II. CONSTRUCTION OF WAVE FUNCTIONS

A. Definition

We consider an arbitrary initial state i and final state f of the target A. For the probe a we first use two fictitious particles ξ and η; we imagine that the only interaction between ξ, η, and the target is that the ξ is changed into the η in the presence of the target through a weak local interaction described by the operator $\mathcal{O}_a(x)$. The matrix element for the reaction $\xi+i\rightarrow\eta+f$ is

$$M_{aA}(w)=\langle f|\,\mathcal{O}_a(0)\,|i\rangle, \tag{1}$$

where w is the square of the four-momentum transferred between the probe and the target. Since the interaction is an arbitrary local one, we may restrict our attention to a given angular momentum state in the crossed reaction $\xi+\bar{\eta}\rightarrow\bar{\imath}+f$.

A matrix element such as $M_{aA}(w)$ has been used as a "wave function" by Blankenbecler and Cook.[4] However, we do not wish to consider $M_{aA}(w)$ as a wave function of the target A because it still depends on the form of the operator $\mathcal{O}$ which is characteristic of the fictitious probe; rather, we wish to express $M_A(w)$ in terms of a linear combination of wave functions $\Phi_A(w;B,v)$ which represent the target. There must be one component for each physical state accessible through a reaction $\xi+\bar{\eta}\rightarrow B$; the mass of these states is $v^{\frac{1}{2}}$, and B denotes the other quantum numbers. [It is convenient to work with the quantity $\Phi_A(w;B,v)\equiv(v-w)\Psi_A(w;B,v)$ rather than with Ψ_A, which corresponds directly to the ordinary wave function. The factor $(v-w)$ may be identified with the familiar expression (H_0-E).]

We require that $\Phi_A(w;B,v)$ satisfy

$$\Phi_A(w;B,w)=F_{BA}(w), \tag{2}$$

where $F_{BA}(w)$ denotes the transition amplitude for the process $A\rightarrow B$, and that

$$M_A(w)=\sum_B\int M_B(v)^\dagger(v-w)^{-1}\boldsymbol{\rho}(B,v)dV\Phi_A(w;B,v), \tag{3}$$

where $\rho(B,v)$, a density of states factor, includes delta-function terms for discrete states. To construct such a Φ, we start with the dispersion relation for $M_A(w)$:

$$M_A(w)=\sum_B\int M_B(v)^\dagger\rho(B,v)(v-w)^{-1}dv\,F_{BA}(v)+\sum_C\int M_C(u)^\dagger(u-w)^{-1}f_{CA}(u)du, \tag{4}$$

(there is no need to treat subtractions explicitly). The first integral in (4) is that associated with all the "normal" thresholds and has the proper form to be identified with (3). The second integral arises from all other branch cuts, that is, from anomalous thresholds. If in the second integral we replace $M_C(u)$ by the right-hand side of (3) we obtain:

$$M_A(w)=\sum_B\int M_B(v)^\dagger\rho(B,v)(v-w)^{-1}F_{BA}(v)dv+\sum_{CB}\int dv\int du(v-u)^{-1}(u-w)^{-1}M_B(v)^\dagger\times\rho(B,v)\Phi_C(u;B,v)f_{CA}(u), \tag{5}$$

which has the form of Eq. (3) with

$$\Phi_A(w;B,v)=F_{BA}(v)+(v-w)\sum_C\int\Phi_C(u;B,v)\times f_{CA}(u)(v-u)^{-1}(u-w)^{-1}du. \tag{6}$$

From Eq. (6) the wave function can be constructed in terms of "mass shell" or "S-matrix" quantities. This fact, and the invariance of Φ in Lorentz transformations, is the result of using a dispersion relation in line (4) rather than some other formula. In Eq. (6) it is obviously necessary to pay very close attention to the

[4] R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 745 (1960).

analytic continuation of the functions to the correct branches. Along a real contour, w and v are to be understood as referring to $w+i\epsilon$ and $v+i\epsilon$, while u is to be replaced by $u-i\epsilon$ everywhere except in f_{CA}.

Note in particular that according to the definition given above, when the target A is such that there are no anomalous thresholds, the w dependence of $\Phi_A(w;B,v)$ is trivial: $\Phi_A(w;B,v)\equiv F_{BA}(v)$. In this case the "wave functions" correspond to those obtained in the non-relativistic theory from zero-range potentials, since the entire w dependence then comes from the factor $(v-w)^{-1}$. This will be discussed in more detail later. On the other hand, the entire effect of the anomalous thresholds is contained in the w dependence of the functions Φ.

B. A Simple Illustration

Here we illustrate the construction of wave functions by applying the prescription of section A to the deuteron. For simplicity we treat only the problem of scalar particles.

We use the notation that N, m, and D are the masses of the nucleon, meson, and deuteron, and $\alpha^2=N^2-\frac{1}{4}D^2$. We consider the matrix element $M(w)$ which corresponds to Fig. 1(a). We denote by μ and C the strength of the two vertices of Fig. 1(b) (in which the nucleon is on the mass shell), and by $\mu(w)$ and $C(w)$ the transition amplitudes for the processes $\xi+\bar{\eta}\to N+\pi$ and $\bar{N}+d\to N+\pi$, respectively. If we include in the dispersion integral for $M(w)$ only the nucleon-plus-one-pion contribution, we have:

$$M(w)=\frac{\mu C}{N^2-w}+\int_{w_0}^{(N+m)^2}\frac{\mu(u)^\dagger f(u)du}{u-w}+\int_{(N+m)^2}^{\infty}\frac{\mu(v)^\dagger\rho(v)C(v)dv}{v-w},\quad(7)$$

where

$$w_0=N^2+m^2D^2/(2N^2)+2m\alpha(1-m^2/4N^2)^{\frac{1}{2}}D/N.\quad(8)$$

The three terms of (7) correspond to Figs. 1(b)–(d).

The function $f(u)$ appearing in (7) is

$$f(u)=(2\pi)^{-2}\int d^4k\, CgN\delta(q_1^2-N^2)\delta(q_2^2-N^2)\delta(q_3^2-m^2)$$

$$=(8\pi)^{-1}CgN(\Delta_+-u)^{-\frac{1}{2}}(\Delta_--u)^{-\frac{1}{2}},\quad(9)$$

where $\Delta_\pm=(D\pm N)^2$, and g is the renormalized pion-nucleon coupling constant. For $\mu(u)$ we have the dispersion relation

$$\mu(u)=\frac{\mu gN}{N^2-u}+\int_{(N+m)^2}^{\infty}\frac{\mu(v)^\dagger\rho(v)F(v)dv}{v-u},\quad(10)$$

where $F(v)$ is the pion-nucleon scattering amplitude. Now we insert (9) and (10) into the second term of (7)

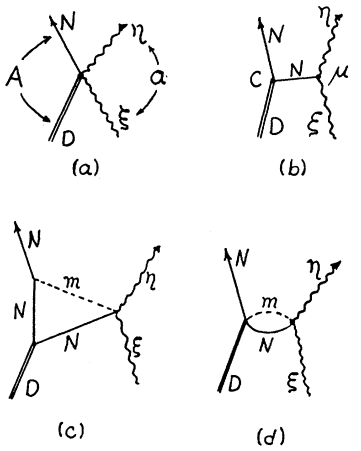


FIG. 1. Some reduced Feynman graphs for the wave function of the deuteron.

and identify $\Phi(w)$ as the coefficient of $\mu/(N^2-w)$ and $\Phi(w;v)$ as the coefficient of $\mu(v)\rho(v)/(v-w)$ in the expression thereby obtained. In terms of the function $\varphi(w)$ defined by

$$\varphi(w)=\tfrac{1}{2}\int_{w_0}^{(N+m)^2}\frac{N^2du}{(u-w)(\Delta_+-u)^{\frac{1}{2}}(u-\Delta_-)^{\frac{1}{2}}}$$

$$=\frac{N^2}{(\Delta_+-w)^{\frac{1}{2}}(w-\Delta_-)^{\frac{1}{2}}}\left\{\tan^{-1}\left[\frac{(\Delta_--w)(\Delta_+-w_0)}{(\Delta_+-w)(w_0-\Delta_-)}\right]^{\frac{1}{2}}-\tan^{-1}\left[\frac{(\Delta_--w)[\Delta_+-(N+m)^2]}{(\Delta_+-w)[(N+m)^2-\Delta_-]}\right]^{\frac{1}{2}}\right\},\quad(11)$$

we obtain:

$$\Phi(w)=C+\frac{Cg^2}{4\pi}[\varphi(w)-\varphi(N^2)]\quad(12a)$$

and

$$\Phi(w;v)=C(v)+\frac{CgF(v)}{4\pi N}[\varphi(w)-\varphi(v)].\quad(12b)$$

It will be seen later (Sec. IV) that the fact that we have considered a target in which the initial and final states are both discrete has lead to considerable simplification.

III. IMPULSE APPROXIMATION

A. Formulation

We now consider the interaction with the target A of an actual probe, which does not consist of fictitious particles. We denote such a probe by A', to emphasize that it must now be treated in the same way as A; in particular, it is represented by a wave function $\Phi_{A'}(w;B,v)$. If the states B include a discrete one, the exact transition amplitude for $i+i'\to f+f'$ has a pole with residue $F_{A'B}F_{AB}$; our impulse approximation should be constructed in a way which reproduces such poles exactly. Moreover, the wave function Φ_A which has been defined here should enter into the impulse

approximation in the same way that the wave function of a target which is described by nonrelativistic wave mechanics enters into the conventional Born approximation for rearrangement collisions. These requirements lead us to define the impulse approximation to the matrix element for the reaction $i+i' \rightarrow f+f'$ to be

$$I_{A'A}(w)=\sum_B \int dv\, \Phi_{A'}(w; B,v)\rho(B,v) \times (v-w)^{-1}\Phi_A(w; B,v), \quad (13)$$

(we might also call this a Born approximation). In many cases, we might wish to add exchange terms to (13). If the integral over v does not converge, then (13) is to be regarded as a formal equation, valid after subtraction.

In Eq. (13) we sum over a restricted set of states B with definite quantum numbers. For example, in deuteron breakup reactions, we might include the discrete nucleon state and all others (e.g., meson-nucleon scattering states) with $J=T=\frac{1}{2}$ and even parity. It is necessary to include only states with a low angular momentum, because the sum over all angular momentum would not converge throughout the physical region. Moreover, intervening states B which have a large angular momentum would give to $I_{A'A}$ a polynomial energy dependence of high order. The impulse approximation implies an interaction localized in time, which requires for consistency a polynomial of small degree.

The peripheral collision model of high-energy interactions, as usually formulated,[5,6] corresponds to Eq. (13) with the wave functions replaced by transition amplitudes. The wave-function formalism, by providing an unambiguous way of extrapolating transition amplitudes off the mass shell, will extend the range of validity of this model. A similar process is the "quasi-elastic" scattering of protons from protons bound in a nucleus.[7] Here Φ_A corresponds to the wave function of a proton in the nucleus, and $\Phi_{A'}$ to the wave function of a proton in a p-p scattering state. The w dependence of $\Phi_{A'}$ is presumably negligible compared with that of Φ_A in this problem; then (13) reduces to the formula usually applied. Another example is the scattering of particles from a deuteron; the cross sections for inelastic scattering of K mesons,[8] π mesons,[9] and electrons[10] from deuterons have been analyzed according to the impulse approximation, with the deuteron being represented by a phenomenological wave function. In these analyses, the improvement of the impulse approximation over the "pole" approximation is quite striking.

It is obvious that in an experiment in which the impulse approximation is a good one, the "wave function" describes all the properties of a target which are necessary to an interpretation of the experiment, and conversely, all the properties of the target which may be inferred from an analysis of such experiments with any probes. This suggests that the idea of wave functions may be useful in the semi-phenomenological analysis of experiments even when they have not first been calculated theoretically.

B. Analysis of Residual Singularities

In order to see what is omitted by the impulse approximation, we look at the dispersion relation (with respect to w) for the exact amplitude, and compare it with that obtained from (13). As an illustration, let us consider the amplitude $F(s,t,u)$ for the reaction $a+b \rightarrow c+d$ between spinless particles, where we suppose at first that there are only normal thresholds. We denote by $F_l(t)$ the partial wave amplitudes defined with respect to the t channel, and similarily denote by $F_l'(u)$ partial amplitudes for the u channel. We may construct a $\varphi_l(t)$ so that $[\varphi_l(t)]=[F_l(t)]$ along the right-hand cut, but $\varphi_l(t)$ has no left-hand discontinuity; similarily we construct a $\varphi_l'(u)$. If we now take Eq. (13) and identify w with the momentum transfer variable t, we see that the contribution of the lth angular momentum to $I_{A'A}(t)$ is $g_l\varphi_l(t)$ where the factor g_l is a kinematical factor which is a polynomial of degree l in s. This is true because $I_{A'A}{}^{(l)}$ has no left-hand cut, and because its right-hand discontinuity is also $[F_l(t)]$ in this case where there are no anomalous thresholds. It is clear that if we add an exchange term we have

$$I(t,u)=I_{A'A}(t)+I_{B'B}(u)=\sum_{l=0}^{L} g_l\varphi_l(t)+\sum_{l=0}^{L'} g_l'\varphi_l'(u). \quad (14)$$

In other words, $I(t,u)$ differs from $F(s,t,u)$ in two ways: the t and u branch cuts are approximated by the first few terms of the partial wave expansions, and the s cuts are completely absent. It is, of course, obvious that the "impulse" or "Born" approximation is not a unitary one.

We may reintroduce the s thresholds approximately by subtracting from (14) the contribution to the first L'' partial waves in the s channel, and adding the exact amplitudes $g_l''F_l''(s)$ in their place. The amplitudes $F_l''(s)$ are determined, in the usual manner, from partial wave dispersion relations in which the right-hand discontinuities are obtained from unitarity, and the left-hand discontinuities from the crossed reactions.

[5] F. Salzman and G. Salzman, Phys. Rev. **120**, 599 (1960); Phys. Rev. Letters **5**, 377 (1960).

[6] S. D. Drell, Revs. Modern Phys. **33**, 458 (1961). This paper gives a general discussion and many other references. See also references 12 and 16.

[7] P. Hillman, H. Tyrén, and Th. A. J. Maris, Phys. Rev. Letters **5**, 107 (1960); B. Gottschalk and K. Strauch, *ibid.* **120**, 1005 (1960); H. Tyrén, P. Hillman, and Th. A. J. Maris, Nuclear Phys. **7**, 1, 10 (1958).

[8] W. Chinowsky *et al.*, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), Vol. 10, p. 451.

[9] M. Derrick, J. Fetkovich, T. Fields, G. Pewitt, and G. Yodh (private communication).

[10] L. Durand, Phys. Rev. Letters **6**, 631 (1961); Phys. Rev. **123**, 1393 (1961).

Within the approximation of truncating the partial-wave expansions at L and L', the "potential" branch cut which appears in the s channel is given exactly by the "Born approximation" (14). The approximation (14) does not correspond exactly, therefore, to the Born approximation to scattering from a potential of the Yukawa type, but it reduces to it when the potential is weak and includes some corrections when the potential is strong. After the first L'' partial waves are corrected for unitarity, the approximation is identical to that introduced by Cini and Fubini,[11] and is therefore not new. The essential point is that when all the thresholds are normal, the formalism introduced here reduces to one which has been shown to lead to useful calculational techniques.

Next we examine the case in which there are anomalous thresholds which occur at points $t=$constant or $u=$constant. We consider again the partial wave amplitudes $F_l(t)$ and $F_{l'}(u)$, and construct an amplitude $\varphi_l(t)$ in which the left-hand cut of $F_l(t)$ is absent. The branch cuts starting from the normal thresholds in t have discontinuities $[\varphi_l(t)]=[F_l(t)]$ as before, but the discontinuity $[F_l(t)]$ associated with an anomalous region is approximated. These anomalous discontinuities are associated with reduced graphs such as in Fig. 2 in which there is always one vertex α to which two of the external lines are attached. We omit the contribution of any of the so-called "super-anomalous" graphs which are more open and do not lead to singularities of $F(s,t,u)$ at points $t=$constant. The discontinuity $[F_l(t)]$ associated with Fig. 2 is proportional to the partial wave amplitude $F_l(t;\alpha)$ associated with the vertex α; we approximate this amplitude by the same prescription used in relating $\varphi_l(t)$ to $F_l(t)$. That is, in the dispersion integral for $F_l(t;\alpha)$ we omit the "left-hand" branch cuts (and those associated with super-anomalous graphs), and approximate the discontinuities associated with other non-normal branch cuts by repeating systematically this procedure. It is not hard to see that this procedure of systematically neglecting left-hand branch cuts in successive dispersion integrals is equivalent to using in place of $F(s,t,u)$, the impulse approximation (13), in which Φ_A and $\Phi_{A'}$ have been replaced by the iterative solution of the integral equation (6).

If we write a dispersion integral for the entire discontinuity function $[F_l(t)]$ associated with Fig. 2, not just for the factor $F_l(t;\alpha)$, we find the left-hand discontinuities of $F_l(t;\alpha)$ correspond to discontinuities $[[F_l(t)]]$ which are given by more open reduced graphs, in which all four external lines are attached to separate vertices. Cini and Fubini have pointed out that their approximation to the Mandelstam representation consists in ignoring the singularities on the

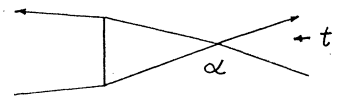


FIG. 2. An anomalous threshold graph.

Landau curves derived from such graphs. In this more general case we are ignoring the same kind of Landau singularity but are required to explicitly discard those which appear as super-anomalous or as complex singularities as well as some of those on hidden Riemann sheets. In the impulse approximation (13), the s thresholds are, of course, also absent.

This discussion of the singularities reproduced by the impulse approximation shows that when the initial and final states of the target and probe are all discrete states, the impulse approximation (13) could have been derived from a standard (but rather intricate) analysis of partial-wave dispersion relations. The advantage of introducing the concept of wave functions is that it provides a familiar physical interpretation for the approximation and enables us to bring forth physical arguments for its validity which also apply to continuum states.

IV. CONTINUUM STATES

Although the formalism introduced in Sec. II can be applied even when the initial or final states of the target A are continuum states, there are some new features which are sufficiently complicated to require special treatment. These features are associated with the fact that there is some arbitrariness in specifying the w dependence of the matrix element $M_{aA}(w)$ in Eq. (1) when i and f are not discrete,[12] as we can see by considering the simplest case, in which i is a discrete state of mass M_0 and four-momentum P_0, and f is a continuum state in which two particles of masses M_1 and M_2 have four-momenta P_1 and P_2. Then we may write $M_{aA}=M_{aA}(w,s,t_i)$, where $s=(P_1+P_2)^2$ and $t_i=(P_i-P_0)^2$; these variables are not independent, since

$$s+t_1+t_2=w+M_0^2+M_1^2+M_2^2. \tag{15}$$

If w varies, s, t_1, and t_2 can not all remain constant, and if, for instance, s and t_1 are fixed and t_2 is determined from (15), then a threshold of M_{aA} at $t_2=t_2'$ will give a w threshold at $w'=s+t_1+t_2'-\sum M^2$ in the dispersion relation (4). Such a threshold must be treated as an anomalous one in the construction of the wave function $\Phi_A(w;B,v)$, even though it may correspond to a normal t_2 threshold.

The most natural way to choose the variables and define the w dependence of the matrix element is to consider a final state f which has a definite angular momentum J. These partial wave matrix elements may be obtained by projection:

$$M_{aA}(w,s,J)=\int \mathcal{Y}_J(\theta,\varphi)^\dagger M_{aA}(w,s,t_i)d\Omega,$$

[11] M. Cini and S. Fubini, Ann. Phys. **3**, 352 (1960); *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), Vol. 10, p. 310.

[12] See the discussion by P. V. Landshoff and S. B. Treiman Nuovo cimento **19**, 1249 (1961).

where $\mathcal{Y}_J(\theta,\varphi)$ is a spin orbital function with total angular momentum J. We then must consider, in addition to the w singularities of the original amplitude, the additional singularities which arise from coincidence of the end points $\cos\theta=\pm1$ of the angular integration with the t_i singularities. This procedure may be generalized to matrix elements involving more initial or final particles.

We shall discuss in some detail the case of spinless particles, in order to examine the location of the additional singularities, and to see their influence in the simplest situation. Let us define θ_i to be the angle, in the barycentric frame of reference, between $\mathbf{p}_0$ and $\mathbf{p}_i$; obviously, $\theta_1=\pi-\theta_2$. Then we have

$$t_1=-(2s)^{-1}[s^2-s(M_0^2+M_1^2+M_2^2+w)+(M_1^2-M_2^2)(M_0^2-w)-\cos\theta_1 A(s,M_1^2,M_2^2)A(s,M_0^2,w)], \quad (16)$$

where

$$A(\alpha,\beta,\gamma)^2=\alpha^2+\beta^2+\gamma^2-2\alpha\beta-2\alpha\gamma-2\beta\gamma. \quad (17)$$

If $M_{aA}(w,s,t_i)$ has a singularity at $t_1=T$, then the locations of the two induced w singularities are obtained from the solution of $T=t_1(\cos\theta_1=\pm1)$, which are

$$2M_1^2w_\pm=-[M_1^4-2M_1^2(s+T+M_0^2+M_1^2)+(s-M_1^2)(M_0^2-T)]\pm A(s,M_1^2,M_2^2)A(M_1^2,M_0^2,T). \quad (18)$$

In the most important case, where T is smaller than the threshold for producing the two particles of mass M_0 and M_1, w_+ and w_- are complex conjugates. There may also be anomalous singularities at $t=T(w)$; then (18) is an implicit equation for $w_\pm$.

The partial wave matrix element may also have kinematical singularities when the relation between t_i and $\cos\theta_i$ is singular, that is, at the values of w for which $A(s,M_0^2,w)=0$. These kinematical singularities may be eliminated by factoring from the matrix element corresponding to orbital angular momentum l, the lth power of the barycentric momentum. The four-momenta which in the barycentric frame reduce to the relative momenta are

$$q=(2s)^{-1}[s(P_1-P_2)-(P_1+P_2)(M_1^2-M_2^2)] \quad (19)$$

and

$$k=(2s)^{-1}[s(P_0-P_a)-(P_1+P_a)(M_0^2-w)], \quad (20)$$

where $P_a=P_1+P_2-P_0$ and $P_a^2=w$. The squares of the relative momenta are given by

$$-4sq^2=A(s,M_1^2,M_2^2)^2, \quad -4sk^2=A(s,M_0^2,w)^2. \quad (21)$$

Let us denote by $\mathcal{Y}_{lm}(q)$ the spherical harmonic multiplied by q^l. When all the particles have spin zero, the matrix element may be written in the form

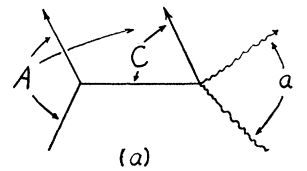
$$M_{aA}(w,s,t_i)=\sum_{lm}\mathcal{Y}_{lm}(k)^\dagger\mathcal{Y}_{lm}(q)M_{aA}(w,s,l), \quad (22)$$

where $M_{aA}(w,s,l)$ is an invariant function which is free of these kinematical singularities at $k^2=0$. It should be noted that when the state f is a discrete one of spin l, the proper treatment of the spin is equivalent to factoring out a term like $\mathcal{Y}_{lm}(k)^\dagger$. This factor must, of course, be supplied again when the impulse approximation is written down.

The physical interpretation of the factor k^l is that it gives the effect of the centrifugal barrier on the wave function. Salzman and Salzman[5] suggested that such a factor should be included when the excitation of the $(\frac{3}{2},\frac{3}{2})$ resonance in peripheral collisions is treated. Their arguments were based on the static nucleon model, but the result is of course much more general. It is interesting that experimental studies of the excitation of this resonance agree better with "pololological" calculations if this factor is omitted.[13–15] From this it should not be concluded that it is wrong to correct for centrifugal barriers, but that the pion-nucleon interaction has such a long range that the w dependence of the rest of the wave function compensates approximately for that of the factor k.

The interactions which have the longest range contribute to so many partial waves when s is large that it is necessary to sum over all partial waves in order to obtain useful results. It is possible to do this, within the formalism described here, because these peripheral interactions have an especially simple form. Let us first look at a simple example in which $M_{aA}(w,s,t_i)$ has a pole term $gM_{aC}(w)(t_1-T)^{-1}$ [see Fig. 3(a)]. Let $t_1=a+b\cos\theta$ (cf. 16), so that

$$M_{aA}(w,s,l)=\frac{C(l)}{b^l}\int_{-1}^{1}\frac{P_l(\cos\theta)d(\cos\theta)\,gM_{aC}(w)}{t-T}=\frac{C(l)}{b^{l+1}}\int_{a-b}^{a+b}\frac{P_l((t-a)/b)dt\,gM_{aC}(w)}{t-T}, \quad (23)$$



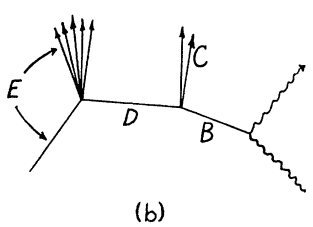


FIG. 3. Graphs for peripheral interactions.

[13] J. Iizuka and A. Klein, Phys. Rev. 123, 669 (1961).
[14] R. Serber (private communication).
[15] N. C. Hien, thesis, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, 1961 (unpublished).

where $C(l)$ is a normalizing constant. Then $M_{aA}(w,s,l)$ has branch points at $w_\pm$ which are joined by a cut for which

$$[M_{aA}(w,s,l)]=\frac{2\pi iC}{b^{l+1}}P_l\left(\frac{T-a}{b}\right)gM_{aC}(w). \quad (24)$$

Let us now suppose that $M_{aC}(w)$ satisfies a dispersion relation with only normal thresholds:

$$M_{aC}(w)=\sum_B \int dw' \frac{M_{aB}(w')^\dagger\rho(B,w')F_{BC}(w')}{w'-w}. \quad (25)$$

Then, according to Eq. (6),

$$\Phi_A(w;s,l;B,w')=\frac{C}{b'^{l+1}}\int_{a'-b'}^{a'+b'}\frac{P_l((t-a')/b')dt\,gF_{BC}(w')}{t-T}$$
$$+C\int_{w_-}^{w_+}\frac{dw''}{b''^{l+1}}P_l\left(\frac{T-a''}{b''}\right)$$
$$\times gF_{BC}(w')\frac{w'-w}{(w''-w)(w'-w'')}, \quad (26)$$

where $a'=a(w')$, etc. This expression can be simplified, by using the fact that

$$\int_{w_-}^{w_+}\frac{dw''P_l((T-a'')/b'')}{b''^{l+1}(w''-w)}$$
$$=\frac{1}{b^{l+1}}\int_{a-b}^{a+b}\frac{dt\,P_l((t-a)/b)}{t-T}, \quad (27)$$

to

$$\Phi_A(w;s,l;B,w')$$
$$=\frac{C(l)}{b^{l+1}}\int_{a-b}^{a+b}\frac{dt}{t-T}P_l\left(\frac{t-a}{b}\right)gF_{BC}(w'). \quad (28)$$

We are now able to sum over all angular momenta, and obtain the simple result

$$\Phi_A(w;s,t_i;B,w')=\sum_l C(l)b^lP_l(\cos\theta)\Phi_A(w;s,l;B,w')$$
$$=gF_{BC}(w')[t(w,\theta)-T]^{-1}, \quad (29)$$

from which it is clear that the w dependence associated with the cut from w_- to w_+ is merely a reminder to use the correct kinematical relation between t and w.

The result (29) can be immediately generalized to a matrix element which is given by a peripheral approximation:

$$M_{aA}(w;s,t_i)=\sum_{BD}\int dw'dt'\,M_{aB}(w')^\dagger\rho(B,w')(w'-w)^{-1}$$
$$\times F_{BCD}(w',t')\rho(D,t')(t'-t)^{-1}\Phi_E(t;D,t'). \quad (30)$$

The structure of this approximation and the identification of the states $B-E$ is given symbolically in Fig. 3(b). Note in particular that we have used not the impulse approximation as previously defined, but a cruder one, for the vertex (BCD) in which two states are considered to be "off the mass shell." In order to treat properly the anomalous thresholds associated with such a vertex, we need to generalize the wave functions so far defined. However, the effect of such additional structure of the vertex (BCD) on the wave function Φ_A can be calculated, for a given partial wave, by the method already outlined. Moreover, the approximation (30) should be sufficient for many applications, provided that the contribution of pole terms to F_{BCD} is absent or is subtracted.[16] From (30) we obtain, by repeating the steps leading from Eq. (23) to Eq. (29):

$$\Phi_A(w;s,t_i;B,w')=\sum_D\int dt'\,F_{BCD}(w',t')$$
$$\times\rho(D,t')(t'-t)^{-1}\Phi_E(t;D,t'). \quad (31)$$

To the right-hand side of (31) we can also add "exchange terms" corresponding to different pairings of the external lines. If the impulse approximation (13) for a many-particle reaction is formulated with approximate wave functions of the type of (31) it is immaterial whether states such as "C" are supposed to be a part of the final state of the target or of the probe, because they automatically enter symmetrically into the result.

When many particles are produced in a peripheral collision, the particularly important graphs have a "chain-of-poles" structure.[17] The effect of such graphs is correctly represented by the approximation (31), because we may use this approximation for Φ_E, etc. Note that at a fixed θ, $dt/dw>0$ [cf. Eq. (16)] when the variables are in the physical region; the inclusion of the "chain-of-poles" contribution to the wave function therefore has the qualitative effect of decreasing the matrix element for peripheral collisions from that of the one-pole approximation. The decrease is especially important when it is *not* the extreme edge of the interaction region which contributes, so the average range of interaction is thereby effectively increased.[18]

V. CONSTRUCTION OF A WAVE EQUATION

Wave functions not only provide the description of states in wave mechanics, but are also essential to the expression, through the Schrödinger equation, of the evolution of states. The wave functions defined here do not have this dual role; the dynamics of the target

[16] A convenient further restriction to impose on the states B, C, and D is to require that the vertex (BCD) contain no more than 4 external lines. This enables one to avoid the problem of anomalous thresholds in the physical region of F_{BCD}.

[17] V. B. Berestetsky and I. Ya. Pomeranchyk, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1961), Vol. 10, p. 333.

[18] The author is very grateful to Professor M. Ruderman for an interesting discussion of these effects from another point of view.

is supposed to be determined from the location and strength of the singularities of the S matrix $S_{fi}\equiv F_{BA}(v)$ of the target particles alone, while the wave function was introduced as a convenient shorthand for another set of S-matrix elements. It is indeed possible to construct a Schrödinger-like equation which our wave functions satisfy, but this equation is a mere transcription of the given structure of the wave function. Nevertheless, the existence of a Schrödinger-like equation is not coincidental, because our wave functions do contain the dynamical rules in a disguised form, inasmuch as their analytic structure is governed by the same reduced graphs which determine the dynamical singularities of the matrix S_{fi}. It is quite likely that this equation will be useful for calculating neither wave functions nor scattering amplitudes; still, the possibility of constructing an equation which resembles closely the non-relativistic Schrödinger equation helps to justify the name "wave functions" and the transference of our intuition. Moreover, this possibility is essential for justifying the use in the impulse approximation of the same non-relativistic wave functions that have been derived from phenomenological studies of the dynamical properties of nuclear systems.

We shall follow the prescription of Blankenbecler and Cook,[4] which is based on the observation that $h_l(p^2)$, defined by

$$p^l P_l(\cos\theta_p)h_l(p^2)=\int\frac{d^3q}{(2\pi)^3}\frac{P_l(\cos\theta_q)}{[(\mathbf{p}-\mathbf{q})^2+a^2][q^2+b^2]}, \quad (32)$$

has a branch point at $p^2=-(a+b)^2$; the discontinuity is obtained from the integral

$$\begin{aligned}2\pi p^l P_l(\cos\theta_p)[h_l(p^2)]&=\int d^3q\,\delta((p-q)^2+a^2)\delta(q^2+b^2)q^l P_l(\cos\theta_q)\\&=\int_0^{2\pi}d\phi(4p)^{-1}b^l P_l(\cos\theta_q)\\&=2\pi(4p)^{-1}b^l P_l(\cos\theta_p)P_l(Z),\end{aligned} \quad (33)$$

where $2bpZ=(p^2+b^2-a^2)$. Now, the Schrödinger equation for a discrete state may be put in the form

$$\mathcal{Y}_{lm}(\mathbf{p})\varphi(p^2)=\int\frac{d^3k}{(2\pi)^3}\int_{\mu^2}^{\infty}\frac{g(\sigma^2)d\sigma^2}{\sigma^2+(\mathbf{p}-\mathbf{q})^2}\frac{\mathcal{Y}_{lm}(q)}{q^2+\alpha^2}\varphi(q^2), \quad (34)$$

if the potential is a superposition of Yukawa ones, so from the remarks above it follows that

$$\varphi(p^2)=\int_{(\mu+\alpha)^2}^{\infty}\frac{f(\sigma^2)d\sigma^2}{\sigma^2+p^2}. \quad (35)$$

(the eigenvalue condition is the convergence of this integral) and $f(\sigma^2)$ is obtained as the iterative solution of a Volterra integral equation.[19,20] This iteration can be inverted, and $g(\sigma^2)$ calculated if $f(\sigma^2)$ is known—in this way, any function with a spectral representation could be represented by an equivalent "Schrödinger equation." For details, the papers cited may be consulted. This procedure is obviously rather artificial, but by requiring that the same equation serve for continuum states and that the single quantum exchange contribution to the potential be exactly of the Yukawa type the arbitrariness is lessened.

For continuum states, the starting point is the equation

$$\mathcal{Y}_{lm}(\mathbf{p})[\varphi(p^2)-\varphi_0(p^2)]=\int\frac{d^3k}{(2\pi)^3}\int_{\mu^2}^{\infty}\frac{g(\sigma^2)d\sigma^2}{\sigma^2+(\mathbf{p}-\mathbf{q})^2}\frac{\mathcal{Y}_{lm}(\mathbf{q})\varphi(q^2)}{q^2-p_0^2-i\epsilon}, \quad (36)$$

where $\varphi_0(p^2)$ is the Born approximation. In addition to the branch points at $p^2=-(n\mu-ip_0)^2$, which are the analogs of the discrete state ones, there are additional branch points at $p^2=-(n\mu\pm ip_0)^2$ arising from the presence of the inhomogeneous term, which correspond to those discussed in Sec. IV. The approach we are following here differs from that of Martin,[20] because we are interested in the solutions satisfying outgoing wave boundary conditions, which correspond directly to the wave functions defined in the previous sections.

The differences between our wave functions and the vertex function studied by Blankenbecler and Cook imposes some minor changes in their procedure. First, the wave function has many components, and it is necessary to write an equation in which their amplitudes are coupled. We shall examine here the case in which there is only one discrete component, for which we write a Schrödinger equation; the effect of the continuum components will be incorporated into the boundary condition. Secondly, we see from Eq. (12) that the one-meson contribution to $\Phi(w)$ is given by a branch cut from w_0 to $(N+m)^2$, whereas the branch cut in the non-relativistic theory extends to infinity. The effect of the branch point at $w=(N+m)^2$ could be represented by an interaction whose range, in the case of the deuteron, is $\approx(Nm)^{-\frac{1}{2}}$, but is strongly energy dependent and even complex for continuum states. This energy dependence could be safely ignored within the range of energies considered by Charap and Fubini,[21] but that range is not broad enough to be useful. It should be noted that this complication is also present in the work of Blankenbecler and Cook, but was not pointed out by them because it is associated with normal rather than anomalous thresholds, and because they examined only a single energy. We propose to

[19] G. C. Wick (private communication); S. H. Vosko, J. Math. Phys. **1**, 505 (1960).
[20] A. Martin, Nuovo cimento **14**, 403 (1959); **15**, 99 (1960).
[21] J. Charap and S. Fubini, Nuovo cimento **14**, 540 (1959).

treat this effect also by modifying the boundary condition.

A wave equation in which the one-quantum exchange corresponds to a Yukawa potential is obtained if $\Phi(w)$ is not itself identified as a nonrelativistic amplitude, but rather, is related to one in an indirect but simple way. We shall outline here in an example how this may be done. Two spinless particles whose masses are M and M' interact through a scalar quantum of mass M; we consider an S state of energy $\sqrt{s}\equiv M_0$. The particle which has the mass M is the "spectator." We denote by $\Phi(w)$ the component of the wave function corresponding to the discrete state M'. The one-quantum discontinuity of $\Phi(w)$ is shown by Eq. (9) to be proportional to $[(M_0+M)^2-w]^{-\frac{1}{2}}[w-(M_0-M)^2]^{-\frac{1}{2}}$. A comparison with Eq. (33) shows that p^2 can not be linearly related to w. If we were to choose p^2 to be the square of the barycentric momentum [cf. Eq. (21)] this discontinuity would have the correct form, but then there would be a kinematical singularity at $p^2=-M^2$ because w would not be a single-valued function of p^2. The only simple possibility is to define

$$\mathbf{p}^2=4M^2[(M_0-M)^2-w]/[(M_0+M)^2-w], \quad (37)$$

[or, $\mathbf{p}=4MM_0\mathbf{k}/\{(M_0+M)^2-w\}$, where $\mathbf{k}$ is the relative momentum]; then, if we also define

$$\chi(\mathbf{p}^2)=\frac{(M_0+M)^2-w}{(M_0+M)^2-M'^2}\Phi(w), \quad (38)$$

$[\chi(p^2)]$ is proportional to $\mathbf{p}^{-1}$ and the only kinematical singularity introduced is a pole which is easily subtracted.

The complicated relation (37) represents correctly the differences between relativistic and nonrelativistic kinematics, as is seen from the following argument. Consider the Lorentz frame in which the initial compound system and the final spectator particle have velocities $-\mathbf{v}$ and $+\mathbf{v}$, respectively. If the spectator particle were initially at rest in the barycentric frame, the momentum $\boldsymbol{\Delta}$ transferred to the spectator would be related to $\mathbf{v}$ by $\mathbf{v}=\frac{1}{2}\boldsymbol{\Delta}(M^2+\frac{1}{4}\Delta^2)^{-\frac{1}{2}}$. This momentum $\boldsymbol{\Delta}$ would be a reasonable variable to use if it were not for the FitzGerald contraction of the compound system and the spectator in this frame. In fact, if a is a length characteristic of the size of the particles when they are at rest, we might expect that the wavelength $\lambda=\boldsymbol{\Delta}^{-1}$ should be of the order of $a(1-v^2)^{\frac{1}{2}}$ in order to probe the structure of the system [these arguments do not apply to the structure associated with normal thresholds]. Consequently, the equivalent nonrelativistic momentum $\mathbf{p}$ should be one for which the ratio of $\mathbf{p}^{-1}$ to the size of an uncontracted object is equal to the ratio of $\boldsymbol{\Delta}^{-1}$ to the actual contracted dimensions, or

$$\mathbf{p}=\boldsymbol{\Delta}(1-v^2)^{\frac{1}{2}}=M\boldsymbol{\Delta}(M^2+\tfrac{1}{4}\Delta^2)^{-\frac{1}{2}}. \quad (39)$$

To see the equivalence between (37) and (39), refer to the diagram in Fig. 4. The relation between $\mathbf{p}$ and the four-momenta P and P' is that the three-momentum $\mathbf{p}$ is obtained by a stereographic projection from the sphere $P^2=M^2$ onto the three-flat tangent to this sphere and perpendicular to $P_0=P+P'$.

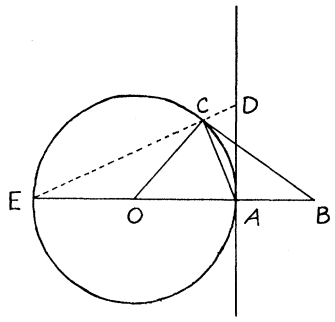


FIG. 4. A Euclidian metric was used in drawing this figure, so the geometry would be simple. The radius of the circle is M, the length of OB is M_0, and the length of BC is $\sqrt{w}$. The segment AC corresponds to the momentum $\boldsymbol{\Delta}$, and it is not hard to show that AD corresponds to $\mathbf{p}$.

Blankenbecler and Cook pointed out that if either of the states i or f is "nonelementary" we should expect $M_{aA}(w)\to 0$ when $w\to\infty$, and that this condition determines the energy of discrete states. It presumably requires that $\mathcal{P}_a(x)$ correspond to a renormalizable interaction. We are therefore led to consider the "boundary condition"

$$\sum_B \int dv\, M_{aB}(B,v)^\dagger \Lambda_A(B,v)=0, \quad (40)$$

where

$$\Lambda_A(B,v)=\lim(w\to\infty)\Phi_A(w;B,v). \quad (41)$$

For the example considered in this section, we write $\Phi(w=\infty)=\Lambda$. The value of Λ determines the effect the continuum components have on the form of $\Phi(w)$. In practice, these continuum components might be determined in terms of the discrete one in some approximation, and Λ then calculated from (40), but in this discussion we shall treat Λ as a phenomenological parameter. It depends on s, and satisfies a dispersion relation.

Since $\Lambda\neq 0$, $\chi(\mathbf{p}^2)$ has a pole at $\mathbf{p}^2=4M^2$; we therefore write

$$\chi(\mathbf{p}^2)=\Lambda\left[\frac{4M^2-\mathbf{p}_0^2}{4M^2-\mathbf{p}^2}-1\right]+\chi'(\mathbf{p}^2), \quad (42)$$

where $\mathbf{p}_0^2=\mathbf{p}^2(w=M'^2)$. Now $\chi'(\mathbf{p}^2)$ does not have a pole at $\mathbf{p}^2=4M^2$, but $\chi'(\infty)=\Lambda$. The one-quantum branch cut of χ' stops at $\mathbf{p}^2=\mathbf{p}_1^2=\mathbf{p}^2((N+m)^2)$, the two-quantum cut stops at $\mathbf{p}_2^2=\mathbf{p}^2((N+2m)^2)$, etc., so we extend these cuts by writing

$$\varphi(\mathbf{p}^2)=\chi'(\mathbf{p}^2)+(\mathbf{p}_0^2-\mathbf{p}^2)\int_{p_1^2}^{\infty}\frac{\Delta f(\sigma^2)d\sigma^2}{(\sigma^2+\mathbf{p}^2)(\sigma^2+\mathbf{p}_0^2)}, \quad (43)$$

where $\Delta f(\sigma^2)$ is the sum of the extrapolated one-quantum, two-quantum, etc., spectral functions. Then $\varphi(\mathbf{p}^2)$ satisfies a wave equation of the type desired and it is normalized so that $\varphi(\mathbf{p}_0^2)=\Phi(M'^2)$. The boundary condition at the origin of coordinate space is not that

the radial wave function vanish; instead it must approach a value proportional to

$$\varphi(\infty)=\lambda=\Lambda-\int_{p_1^2}^{\infty}(\sigma^2+\mathbf{p}_0^2)^{-1}\Delta f(\sigma^2)d\sigma^2. \quad (44)$$

Although the wave functions $\Phi(w)$ and $\varphi(\mathbf{p}^2)$ are not quite the same, the differences are very small when the velocities which enter are really nonrelativistic, and can be neglected in most ordinary nuclear physics problems. Another relativistic effect, that shows up in continuum states, is that the inhomogeneous term $\varphi_0(\mathbf{p}^2)$ which must be chosen to reproduce the extra singularities of the continuum wave functions (Sec. IV) is no longer exactly equal to the Born approximation obtained from the potential.

VI. DISCUSSION

This paper has been devoted to the development of the concept of wave functions, within S-matrix theory, for those contexts in which the wave function enters linearily. We have been concerned with the problem of recognizing, in the dispersion theory formulation of quantum field theory, features which can be understood most easily when they are pictured wave mechanically. The inverse approach to the relations between quantum field theory (as treated by dispersion relations) and the familiar nonrelativistic wave mechanics has been more used in the past; many persons have illustrated dispersion relations by showing that wave mechanics could be recast in a dispersion-theoretic mold.[22] However, there have also been many attempts to rewrite the dispersive equations for the determination of amplitudes in a form which resembles the Schrödinger equation.[4,21,23,24] Except for reference 3, these other studies have been devoted to the elastic scattering amplitude. The potentials introduced via the elastic scattering problem might be useful *ad hoc* technical devices, but their general significance remains unclear.

Application of the concept of wave functions to circumstances in which they appear bilinearly in the elementary theory would be a generalization of great practical importance. We have in mind here in particular such problems as the calculation of the form factors of the deuteron, the impulse approximation to elastic scattering from a deuteron, and the correction within the impulse approximation of the approximation (13) for inelastic scattering through inclusion of interactions in the final state. It may be noted that the formalism introduced here could be applied directly to the form factors of the deuteron—the states i and f would be identified as deuterons, and the wave function would have components representing two mesons, three mesons, etc.—but this would not be useful. What is desired is a way of expressing the properties of the deuteron in terms of assumed properties of individual nucleons as directly as possible. It can be shown that for the problems mentioned here it is possible to write expressions in terms of the type of wave function considered in Sec. II B which resemble the classical expressions, and are quite good approximations, in that they reproduce correctly many of the leading singularities and reduce to the proper limit in nonrelativistic situations. But it is not worthwhile to discuss this in detail here, because the approximations are limited by the exigencies of relativistic kinematics, which ensure that these problems can be treated systematically only if in the intermediate state all the constituents of the deuteron are allowed to stray from the mass shell.

ACKNOWLEDGMENTS

The author has discussed the material of this paper while it was in a formative state with many persons, and wishes especially to thank Professor L. Wolfenstein, Professor G. C. Wick, Professor M. Ruderman, and Professor M. Gell-Mann for their interest and remarks.

[22] A good example is the paper by R. Blankenbecler, M. L. Goldberger, M. N. Khuri, and S. B. Treiman, Ann. Phys. **10**, 62 (1960).

[23] R. E. Cutkosky, Revs. Modern Phys. **33**, 448 (1961).

[24] G. F. Chew and S. C. Frautschi, Phys. Rev. **124**, 264 (1961).