Effect of the 3π Resonance on the 2π Resonance*

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It is shown that the 4π intermediate state in π-π scattering plays an important role in determining the position and perhaps even the existence of the π-π resonance. This large effect occurs by virtue of the 3π resonance. Quantitative estimates are given. The effect of this result on the nucleon form factors is briefly discussed and a simple model for these functions is proposed. The possibility of higher energy pion resonances is also considered.

I. INTRODUCTION

It was pointed out by Federbush, Goldberger, and Treiman[1] that an attractive π-π interaction in the $T=J=1$ state improved the qualitative understanding of the Stanford isotopic vector form factor results.[2] Frazer and Fulco,[3] using the Mandelstam representation,[4] were able to achieve a quantitative fit of the data by assuming that a low-energy π-π resonance was present. This resonance was then found, experimentally, to occur at a center-of-mass energy of 5.0–5.3 in units of the pion mass.[5]

Nambu[6] suggested that a meson with quantum numbers $T=0$, $J=1$ would help explain the large spatial extent of the isotopic scalar form factors. It was pointed out by Chew[7] that this particle could also be interpreted as a bound state or resonance of a three-pion system in which each pair of pions was in a relative $T=J=1$ resonant state. An attempt[8] was made to calculate the scalar form factor based on the physical picture of the three-pion system implicit in the above remark. It was shown that in addition to the π-π resonance, an intrinsic three-pion resonance of rather low energy or a bound state was needed in order to get agreement with experiment. No attempt was made to calculate the position of the 3π resonance in terms of the 2π resonance.[9] This resonance was discovered by Maglić, Alvarez, Rosenfeld, and Stevenson[10] and has a mass of 5.6.

We would like to point out here that the logical reverse of the above theoretical picture is very amusing. The most striking fact which one can infer from the form factors is that the 3π resonance should occur at the same energy or perhaps even lower than the 2π resonance. It is this circumstance which will be exploited here in an attempt to show that the 3π resonance can easily produce the 2π resonance through the 4π intermediate state. If this conjecture is true, it means that the 2π and the 3π resonances are very strongly coupled systems and probably owe their separate existence to this mutual coupling. However, it could be that the 3π resonance is more "fundamental" in some mysterious sense and that the 2π resonance is a dynamical consequence.

The graphs which couple the two resonances under consideration do not occur in the strip approximation.[11] Therefore, since we are interested in rough estimates only, a simple model of these resonating systems will be developed and from it a qualitative relation between the resonances will be derived. The essential result, namely that the 3π resonance could drive the 2π resonance, is independent of the approximations used here.

We will first develop a model of the 3π resonance by assuming that there is an *effective* interaction constant g for the six-point graph shown in Fig. 1. The π-π interaction will be neglected, except insofar as it is responsible for the existence of a six-point interaction in the first place. The generalized N over D method[12] will be used to construct unitary scattering amplitudes for the coupled processes, $3\pi\rightarrow 3\pi$, $2\pi\rightarrow 2\pi$, and $2\pi\leftrightarrow 4\pi$. The relevant amplitudes are defined as

$$M_{3'3}=(3!)^{-\frac{1}{2}}\langle^{(-)}k_1'k_2'\,|\,J_{3'}\,|\,k_1k_2k_3{}^{(+)}\rangle \times(2^5\omega_1'\omega_2'\omega_1\omega_2\omega_3)^{\frac{1}{2}},\quad(1.1)$$

$$M_{2'2}=(2!)^{-\frac{1}{2}}\langle k_1'\,|\,J_{2'}\,|\,k_1k_2{}^{(+)}\rangle(2^3\omega_1'\omega_1\omega_2)^{\frac{1}{2}},\quad(1.2)$$

$$M_{2'4}=(2!)^{-\frac{1}{2}}\langle k_1'\,|\,J_{2'}\,|\,k_1k_2k_3k_4{}^{(+)}\rangle(2^5\omega_1'\omega_1\omega_2\omega_3\omega_4)^{\frac{1}{2}},\quad(1.3)$$

$$M_{2'4}=(4!)^{-\frac{1}{2}}\langle^{(-)}k_1'k_2'\,|\,J_4{}^{\dagger}\,|\,k_1k_2k_3{}^{(+)}\rangle \times(2^5\omega_1'\omega_2'\omega_1\omega_2\omega_3)^{\frac{1}{2}},\quad(1.4)$$

and the isotopic labels have been suppressed. One relation that will be needed is found by contracting

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1 P. Federbush, M. Goldberger, and S. Treiman, Phys. Rev. 112, 642 (1958).

2 See, for example, R. Hofstadter and R. Herman, Phys. Rev. Letters 6, 293 (1961).

3 W. Frazer and J. Fulco, Phys. Rev. 117, 1609 (1960).

4 S. Mandelstam, Phys. Rev. 112, 1344 (1958).

5 See, for example, A. R. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Letters 6, 628 (1961), where other references are given.

6 Y. Nambu, Phys. Rev. 106, 1366 (1957).

7 G. Chew, Phys. Rev. Letters 4, 142 (1960).

8 R. Blankenbecler and J. Tarski, Phys. Rev. 125, 782 (1962).

9 An attempt to make such a calculation is being made by L. I. Schiff, using potentials and the variational principle. I wish to thank Professor Schiff for discussions concerning his approach to this problem.

10 B. Maglić, L. Alvarez, A. Rosenfeld, and L. Stevenson, Phys. Rev. Letters 7, 178 (1961). I wish to thank Dr. Maglić for informative discussions concerning this experiment.

11 G. Chew and S. Frautschi, Phys. Rev. Letters 5, 580 (1960).

12 R. Blankenbecler, Phys. Rev. 122, 983 (1961).

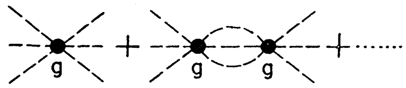


FIG. 1. The 3π interactions.

one of the left pions in $M_{3'3}$ and one of the right pions in the first form for $M_{2'4}$. Then by continuing to negative values of one of the pion energies, we find that

$$[M_{3'3}]=2[M_{2'4}] \tag{1.5}$$

for the connected parts of the matrix elements. Our next task is to work out the unitarity relations satisfied by these functions and to try to calculate them.

II. THE 3π RESONANCE

Since we are interested in discussing only the $T=0$, $J=1$ state of the 3π system, a scalar invariant function will be introduced by defining

$$M_{3'3}=\epsilon_{\alpha'\beta'\gamma'}\epsilon_{\alpha\beta\gamma}\sum_\mu N_\mu(k')N_\mu(k)H_{3'3}(s), \tag{2.1}$$

where

$$N_\mu(k)=-i\epsilon_{\mu\nu\lambda\sigma}k_1{}^\nu k_2{}^\lambda k_3{}^\sigma, \tag{2.2}$$

and $(\alpha\beta\gamma)$ are the isotopic spin labels of the pions. The function $H_{3'3}$ is assumed to depend only on s, the square of the center-of-mass energy of the 3π system. The absorptive part of $H_{3'3}$, $A_{3'3}$, is found by contracting one of the initial pions and after performing the phase space integrals; it turns out to be

$$A_{3'3}(s)=|H_{3'3}(s)|^2R_3(s)\Theta(s-9), \tag{2.3}$$

where $R_3(s)$ is the 3-pion phase space factor. It is given by

$$R_3(s)=\frac{1}{6(2\pi)^3}\int_4^{(s^{\frac{1}{2}}-1)^2}du\, k_3{}^3(s,u)Q^3(u)(s/u)^{\frac{1}{2}}, \tag{2.4}$$

where

$$4sk_3{}^2(s,u)=[s-(u^{\frac{1}{2}}+1)^2][s-(u^{\frac{1}{2}}-1)^2],$$

and $Q^2=\frac{1}{4}u-1$. The extra powers of momentum in the phase-space integral come from the fact, evident from (2.2), that we are dealing with P states.

A solution for $H_{3'3}(s)$ which satisfies unitarity, and hence sums the graphs illustrated in Fig. 1, is given by

$$H_{3'3}(s)=g(s)D^{-1}(s), \tag{2.5}$$

where

$$D(s)=1-\frac{s}{\pi}\int_9^\infty ds'\frac{g(s')}{s'(s'-s)}R_3(s'). \tag{2.6}$$

The effective coupling constant $g(s)$ contains all the singularities of $H_{33}(s)$ except the physical cut which is exhibited in $D(s)$. If we wish to have a resonance, then $g(s)$ should be large (in some sense) and attractive (positive). In order to introduce convenient parameters to describe the 3π resonance, let us define

$$H_{3'3}{}^{-1}(s)=R_3(s)[\cot\Delta(s)-i], \tag{2.7}$$

$$\cot\Delta(s)=(s-t_S)/\Gamma t_S{}^{\frac{1}{2}}, \tag{2.8}$$

where t_S is the position of the 3π scalar resonance and Γ is its full width, because for $s\sim t_S$, we have

$$\cot\Delta(s)\simeq(s^{\frac{1}{2}}-t_S{}^{\frac{1}{2}})/(\Gamma/2).$$

At resonance, the amplitude becomes

$$H_{33}(t_S)=iR_3{}^{-1}(t_S). \tag{2.9}$$

This shows that the height of the three-particle resonance depends on the inverse of the available phase space. This is in complete analogy with the two-particle case. Let us now turn to π-π scattering where these results on the 3π resonance will be used.

III. THE 2π RESONANCE

There are two distinct sets of graphs which contribute to the scattering amplitude; Fig. 2 illustrates typical "potential" graphs which were studied by Chew and Mandelstam[13] and Fig. 3 shows the particular 4π state which we would like to examine in detail. The potential graphs contribute the negative cut in the scattering amplitude, and they will be approximated in a simple phenomenological manner here. The essential point, as far as the inelastic contribution to π-π scattering is concerned, is that the coupling between the two- and four-pion states is described by the effective coupling constant g which was used in the description of the 3π resonance. In addition, we see that the graph involving the 3π resonance in an intermediate state contributes to the $T=1$ amplitude only; hence there is no difficulty associated with getting large S waves from this mechanism. Now since g is "large," this inelastic state is strongly coupled to the 2π state. We will neglect the four-pion graph in which one or two π-π resonances are excited. These occur at a much higher energy than the graph of interest and can only increase the effect we are calculating anyway.

The unitarity condition for M_{22} is, formally,

$$I_MM_{22}=\pi\Sigma_2M_{22}{}^*M_{22}+\pi\Sigma_4M_{24}{}^*M_{42}. \tag{3.1}$$

Each of these terms will be examined in turn. First, the reduced $T=J=1$ amplitude, H_{22}, is introduced by the relation

$$M_{22}(t)=3\mathbf{Q}\cdot\mathbf{Q}'H_{22}(t)P_1, \tag{3.2}$$

where $Q^2=Q'^2=\frac{1}{4}t-1$ and $P_1=\frac{1}{2}(\delta_{\alpha\alpha'}\delta_{\beta\beta'}-\delta_{\alpha\beta'}\delta_{\beta\alpha'})$. P_1 is the isotopic spin one projection operator.

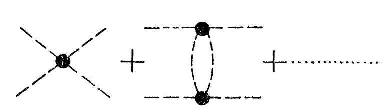


FIG. 2. "Potential" graphs.

[13] G. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

The transition amplitude from the two-pion to the four-pion state is written in the form

$$M_{24}=\epsilon_{\alpha\beta\alpha'}\epsilon_{\beta'\gamma'\delta'}\sum\nolimits_\mu N_\mu N_\mu H_{24}(t,s), \tag{3.3}$$

where $t^{\frac{1}{2}}$ is the center-of-mass energy of the initial and final states and $s^{\frac{1}{2}}$ is the energy of the resonating three-pion group. This amplitude is, of course, the analytic continuation of M_{33} to negative energies of one of the pions. In our previous discussion of M_{33} the t dependence was neglected. This dependence will now be taken into account in such a way as to satisfy unitarity in the t variable.

The integrals in Eq. (3.1) can be evaluated and lead to the result for the absorptive part of H_{22}

$$A_{22}(t)=\Theta(t-4)H_{22}{}^*R_2(t)H_{22}+\Theta(t-16)\int_9^{(t^{\frac{1}{2}}-1)^2} ds\, H_{24}{}^*R_4(t,s)H_{42}. \tag{3.4}$$

where

$$R_2(t)=Q^3(t)/8\pi t^{\frac{1}{2}}, \tag{3.5}$$

$$R_4(t,s)=2t^{\frac{1}{2}}P_4{}^3(t,s)R_3(s)/9(2\pi)^2, \tag{3.6}$$

and

$$4tP_4{}^2(t,s)=[t-(s^{\frac{1}{2}}+1)^2][t-(s^{\frac{1}{2}}-1)^2].$$

The unitarity condition for M_{24} is also easily evaluated;

$$A_{24}(t,s)=\Theta(t-4)H_{22}{}^*R_2(t)H_{24}(t,s)+\Theta(t-16)\int_9^{(t^{\frac{1}{2}}-1)^2} ds'\, H_{24}{}^*(t,s')R_4(t,s')H_{44}(t,s',s). \tag{3.7}$$

In order to solve these coupled nonlinear equations, one introduces[12] the linear equations

$$H_{22}(t)D_{22}(t)+\int_9^\infty ds\, H_{24}(t,s)D_{42}(t,s)=N_{22}(t), \tag{3.8}$$

$$H_{22}(t)D_{24}(t,s)+H_{24}(t,s)=N_{24}(t,s), \tag{3.9}$$

$$H_{42}(t,s')D_{24}(t,s)+H_{44}(t,s',s)=0, \tag{3.10}$$

where the N's do not have the physical cuts in t, and the D's are defined as

$$D_{22}(t)=1-Q^2(t)\int_4^\infty \frac{dt'}{\pi}\frac{R_2(t')}{Q^2(t')(t'-t)}N_{22}(t'), \tag{3.11}$$

$$D_{24}(t,s)=-Q^2(t)\int_4^\infty \frac{dt'}{\pi}\frac{R_2(t')}{Q^2(t')(t'-t)}N_{24}(t',s), \tag{3.12}$$

and

$$D_{42}(t,s)=-Q^2(t)\int_{(s^{\frac{1}{2}}+1)^2}^\infty \frac{dt'}{\pi}\frac{R_4(t',s)}{Q^2(t')(t'-t)}N_{42}(t',s). \tag{3.13}$$

By assuming that $N_{44}=0$, we have neglected the possibility of an intrinsic four-particle resonance occurring in H_{44}. This could be easily added if necessary. In general, the effect of an attractive (repulsive) 4π

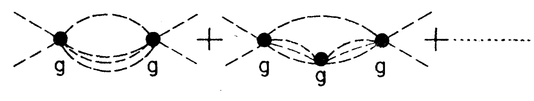


FIG. 3. The 4π contributions.

interaction is to increase (decrease) the effect of the inelastic channels on $H_{22}(t)$. One can now easily check that the solutions to (3.8) and (3.9) satisfy the unitarity conditions (3.4) and (3.7).

The explicit solution for $H_{22}(t)$ is

$$H_{22}(t)\left[D_{22}(t)-\int_9^\infty ds\, D_{24}(t,s)D_{42}(t,s)\right]=\left[N_{22}(t)-\int_9^\infty ds\, N_{24}(t,s)D_{42}(t,s)\right].$$

The dynamical problem consists in determining the N's. We will be motivated here by the twin desires of simplicity and convergence. Therefore, out of ignorance we will choose as our model

$$N_{22}(t)=2a(4+t_0)(t+t_0)^{-1}t^{-\frac{1}{2}}, \tag{3.14}$$

and

$$N_{24}(t,s)=N_{42}(t,s)=(2a)^{-1}N_{22}(t)H_{33}(s), \tag{3.15}$$

where a is the scattering length. Following the work of Chew and Mandelstam,[13] we expect the parameter t_0, which characterizes the left-hand cut, to be approximately

$$t_0\simeq 15. \tag{3.16}$$

Our result does not depend critically upon this choice.

In order to normalize the function H_{24} with the function H_{33}, we have required [see Eq. (1.5)]

$$H_{24}(t=4,\, s)=\tfrac{1}{2}H_{33}(s). \tag{3.17}$$

Following the results of Ref. 8, the complex singularities have been included in our model only insofar as they affect the phase of the production amplitudes. The evaluation of $H_{22}(t)$ is now straightforward and the result is

$$H_{22}(t)=\frac{N_{22}(t)}{a}\{[a+Q^2I_4(t)]^{-1}-Q^2I_2(t)\}^{-1}, \tag{3.18}$$

where

$$I_2(t)=\int_4^\infty \frac{dt'}{\pi}\frac{R_2(t')}{Q^2(t')(t'-t)}\frac{N_{22}(t')}{a}, \tag{3.19}$$

$$I_4(t)=(4+t_0)^2\int_{16}^\infty \frac{dt'}{\pi}\frac{J_4(t')}{Q^2(t')(t'-t)(t'+t_0)t'^{\frac{1}{2}}}, \tag{3.20}$$

and

$$J_4(t)=\int_9^{(t^{\frac{1}{2}}-1)^2} ds\, R_4(t,s)|H_{33}(s)|^2. \tag{3.21}$$

A few general remarks about this solution are probably in order. If we let the coupling of the inelastic

channel get very large, $b\to\infty$, then the inelastic cut disappears from the solution (3.16) and one gets the same result as taking the strong coupling limit, $a\to\infty$. It is amusing that in either limit the amplitude becomes independent of both a and b.

The integral $I_4(t)$ is positive for t less than 16 and hence has the effect of making the potential more attractive when compared to the purely elastic case. In fact, if $Q^2I_4(t)$ is a rapidly increasing function of t, then the inelastic channel can yield a resonance.[14] It is this behavior that we would like to explore in a more quantitative fashion. We will show that $Q^2I_4(t)$ increases linearly for reasonable values of t and is quite large in the neighborhood of the 2π resonance.

The four-particle phase space tends to keep the inelastic contribution to $A_{22}(t)$ small until the energy reaches a rather large value. However, the fact that there is a 3π resonance changes this qualitative result drastically. The physical point here is the fact that when the 3π resonance can be excited, it tries to change the four-particle phase space into a two-particle phase space. This, in turn, rises much more rapidly as a function of the energy. Thus the inelastic contribution can be quite sizable and energy-dependent. Since the integrand, and therefore the integral $I_4(t)$, rises rapidly when the 3π resonance at t_S can be excited, one would expect that the 2π resonance at t_V should occur in the neighborhood of

$$t_V^{\frac{1}{2}}\lesssim t_S^{\frac{1}{2}}+1. \tag{3.22}$$

IV. NUMERICAL RESULTS

In order to achieve a more quantitative feeling for the effects of the 4π state without undue calculations, the 3π resonance will be replaced by a delta function:

$$\begin{aligned}|H_{33}(s)|^2&=\Gamma^2t_S[R_3(s)]^{-2}[(s-t_S)^2+\Gamma^2t_S]^{-1}\\&=\Gamma t_S^{\frac{1}{2}}[R_3(t_S)]^{-2}\pi\delta(s-t_S).\end{aligned} \tag{4.1}$$

These expressions have been adjusted to have approximately the same area. Using this result in Eq. (3.21), we find

$$J_4(t)=\frac{\Gamma t_S^{\frac{1}{2}}}{9(2\pi)^2R_3(t_S)}t^{\frac{1}{2}}P_4{}^3(t,t_S). \tag{4.2}$$

It is expected that the nonresonant corrections to the delta-function approximation should be down by a factor of $\Gamma/t_S^{\frac{1}{2}}$ and also should contain a slowly increasing 4-particle phase space.

The function $I_4(t)$ will now be evaluated. The position of the 3π resonance will be chosen as[10]

$$t_S=32. \tag{4.3}$$

This choice leads to a value of the three-particle phase space of approximately

$$R_3(32)=150/6(2\pi)^3. \tag{4.4}$$

In order to simplify the presentation of numerical results, we will introduce the quantity $J(t)$ as

$$I_4(t)=\frac{8\Gamma t_S^{\frac{1}{2}}(4+t_0)^2}{9(2\pi)^2R_3(t_S)}J(t), \tag{4.5}$$

where

$$J(t)=\int_{(t_S^{\frac{1}{2}}+1)^2}^{\infty}dt'\ j(t')(t'-t)^{-1}, \tag{4.6}$$

and

$$j(t')=P_4{}^3(t',t_S)(t'-4)^{-1}(t'+t_0)^{-1}. \tag{4.7}$$

A plot of $j(t)$ is given in Fig. 4 as the solid line. The function will be evaluated analytically by using the approximation

$$j(t)\simeq\frac{(t-42)^{\frac{3}{2}}}{16.6(t-4)(t+t_0)},$$

which is plotted in Fig. 4 as the dashed line. This approximation slightly underestimates the value of $J(t)$. The integral in $J(t)$ can now be carried out readily:

$$J(t)=\frac{\pi}{16.6(t_0+4)}\left\{\frac{t_0+42}{t_0+t}[(t_0+42)^{\frac{1}{2}}-(42-t)^{\frac{1}{2}}]+\frac{38}{4-t}[(38)^{\frac{1}{2}}-(42-t)^{\frac{1}{2}}]\right\}. \tag{4.8}$$

A simple approximation for this function, valid for t in the range between 15 and 40, is

$$Q^2(t)J(t)\simeq(t-6)/290. \tag{4.9}$$

In the model under consideration, it is obvious from Eq. (3.18) that the inelastic channel simply gives, in effect, an energy-dependent scattering length when compared to the purely elastic case. Rather than making an attempt to calculate the detailed position of a π-π resonance, which could hardly have any significance at the present stage of approximation, let us simply estimate the effect of the inelastic on the effective scattering length $A(t)$, where

$$A(t)=N_{22}(4)+Q^2(t)I_4(t). \tag{4.10}$$

$A(t)$ is defined more generally by the relation

$$H_{22}(t)=\frac{N_{22}(t)}{N_{22}(4)}\left\{\frac{1}{A(t)}+\frac{D_{22}(t)-1}{N_{22}(4)}\right\}^{-1}. \tag{4.11}$$

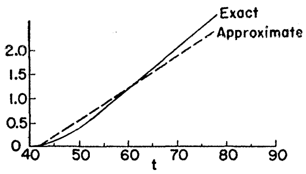
Collecting our previous numerical results, we see that

$$A(t)\simeq a+31.5\Gamma(t-6)/22. \tag{4.12}$$

Since the 3π resonance is of the order of 20 Mev wide,

[14] To this author's knowledge, this connection was first shown by W. Frazer and J. Ball in the partial wave case (to be published) and independently by M. Goldberger and the author for the complete amplitude (to be published). The method used in the present paper seems to have the calculational advantage that the inelastic contribution cannot exceed the unitarity limit.

FIG. 4. Plots of $j(t)\times 10^3$.



we see that at the π-π resonance position,

$$A(28)\simeq a+4.5. \tag{4.13}$$

This is a very large correction to the elastic scattering length and could easily cause a resonance.[15] It may be that our simple model has overestimated the effect of the 4π state, because of its sensitivity to the high-energy behavior of the amplitudes.

If the 3π resonance were as low in energy as 20, then the effective scattering length would be

$$A(t)\simeq a+800\Gamma(t-5)/23. \tag{4.14}$$

The increase in the effect is due mainly to the fact that the three-particle phase space factor $R_3(20)$ is smaller than $R_3(32)$ by a factor of 20. This increases the height and effect of the resonance for the same width.

V. CONCLUSIONS

It has been shown that because of the 3π resonance, the 4π intermediate state has a large effect on π-π scattering. There can be little doubt that this is a very crude estimate of the effect of the 4π state. It is also clear from the preceding calculation that this inelastic contribution has a very important role in determining the position and perhaps even the existence of the rather high energy π-π resonance.

The 4π state, by virtue of the 3π resonance, should also play an important role in all pion processes; for example, the isotopic vector form factors which are discussed in the Appendix, pion-nucleon scattering, and photoproduction, to mention but a few.

The 2π and 3π resonances should also drive π-π resonances at higher energies and in higher partial waves via the inelastic mechanism. In particular, one might expect a resonance in the $T=0$ or 2 amplitude in the S or (preferably) D wave at an energy of approximately $2t_v^{\frac{1}{2}}\cong 10.6$ due to the excitation of two-vector resonances in the four-pion intermediate state. If there actually is a $T=2$ D-wave resonance at 10.6, then one might expect a lower energy S-wave resonance in this state. The argument for this is based on the family theorem proved in the second reference of footnote 14. The $T=1$ amplitude is also fed by this intermediate state but its main structure should come when the four pions can pair up to form a vector resonance and a $T=0$ "virtual" state of $t\sim 6$. This should yield some structure at roughly 7.6 meson masses. [This resonance is either P or (preferably) F wave.] The six-pion intermediate state should also yield a contribution to the $T=0$ amplitude when two scalar resonances are excited at $2t_S^{\frac{1}{2}}\cong 11.2$. These resonances, if they occur, will also show up in the $4\pi\to 4\pi$ scattering amplitude as is evident from Eqs. (3.8) to (3.10).

Note added in proof. The discovery of a narrow $T=0$ 3_π resonance at $t_s\simeq 16$ by A. Pevsner, *et al.*, Phys. Rev. Letters (to be published), should remove any doubt that inelastic effects play a major role in producing the π-π resonance. This value of the resonance makes Eq. (3.22) almost an equality and will change the scalar form factor model presented in the Appendix in an obvious way. The change in the vector form factor is a lowering of acceptable values of T_V.

APPENDIX. THE NUCLEON FORM FACTORS

If one assumes that the 3π resonance plays an important role in the π-π problem, then it should also yield an important contribution to the isotopic vector nucleon form factors. Let us first discuss the scalar form factors.

The 3π contribution to the form factors seems to have two distinct "resonances." The lowest mass contribution is the very narrow 3π resonance at 32. In addition, the matrix elements are enhanced when the three intermediate pions can resonate by pairs via the $T=J=1$ resonance.[8] If all three pairs are excited to resonance, an energy of

$$t=3(t_V-1)\cong 81$$

is required. However, if only two pairs are excited (or even one pair) the matrix elements are also enhanced, although to a lesser degree, and this can occur for a much smaller value of t than that given above. Another way of describing this enhancement is that the inelastic state composed of a 2π resonance and an extra pion might be expected to drive the 3π resonance in analogy to the mechanism described in the text which drives the 2π resonance.[16] This effect was, in fact, demonstrated in Ref. 8. Thus, using a simple pole approximation,[17] we expect that the scalar form factors can be expressed as

$$F^S(t)=a^S(1+t/32)^{-1}+b^S(1+t/T_S)^{-1} +(1-a^S-b^S), \tag{A1}$$

where T_S should lie in the range 50–80. The constant term reflects the contribution of the higher mass states and the nonresonant background.

The vector form factors present a coupled channel problem and let us discuss it briefly. As we have seen,

[15] It is interesting to note that the results of Ref. 5 for the π-π cross section tend to lie below the elastic unitarity limit. If this circumstance remains as the experiments are improved, it may mean that inelastic effects are beginning to play a role even at the resonance position.

[16] I am indebted to M. Gell-Mann and F. Zachariasen for this remark.

[17] This type of approximation has been discussed most recently by S. Bergia, A. Stanghellini, S. Fubini, and C. Villi, Phys. Rev. Letters 6, 367 (1961).

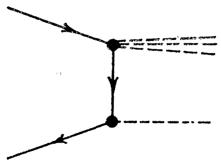


FIG. 5. Nucleon annihilation into 4 pions.

the effective threshold for the production of the 3π resonance and a pion is $(t_S^{\frac{1}{2}}+1)^2$, and the absorptive part from the 4π state should rise rather rapidly above this point. In order to achieve a more quantitative feeling for this effect, and to make sure we are not counting things twice, the coupled channel problem will be solved as follows. The matrix D is introduced as[12] [see also (3.8–3.13)]

$$D(t;s',s)=\begin{bmatrix} 1 & D_{12}(t) & D_{14}(t,s) \\ D_{21}(t) & D_{22}(t) & D_{24}(t,s) \\ D_{41}(t,s') & D_{42}(t,s') & \delta(s'-s) \end{bmatrix}, \quad \text{(A2)}$$

where the subscript one refers to the nucleon pair channel. The solution for the form factors are, formally,

$$F(t)=fD^{-1}(t),$$

where F and f are row matrices. If the effect of nucleon pairs on the 2π and 4π channels are neglected, then the form factors become

$$F_4(t,s)=f_4(s)-F_2(t)D_{24}(t,s), \quad \text{(A3)}$$

$$d(t)F_2(t)=f_2-\int_9^\infty ds\, f_4(s)D_{42}(t,s), \quad \text{(A4)}$$

$$d(t)F_1(t)=f_1d(t)-\left[f_2-\int_9^\infty ds\, f_4(s)D_{42}(t,s)\right] \times\left[D_{21}(t)-\int_9^\infty ds\, D_{24}D_{41}\right] -d(t)\int_9^\infty ds\, f_4(s)D_{41}(t,s), \quad \text{(A5)}$$

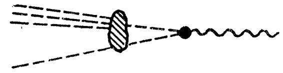


FIG. 6. Electroproduction of 4 pions.

where

$$d(t)=D_{22}(t)-\int_9^\infty ds\, D_{24}(t,s)D_{42}(t,s). \quad \text{(A6)}$$

This function has been discussed in the text and leads to the π-π resonance. In order to satisfy unitarity and reality, the function $f_4(s)$ must be of the form

$$f_4(s)=h_4(s)D^{-1}(s), \quad \text{(A7)}$$

where $h_4(s)$ is real for $9<s<\infty$, and $D(s)$ is given in the text, Eq. (2.6).

In order to complete this discussion we must insert into these formulas suitable approximations for $h_4(s)$, $D_{41}(t,s)$, and $D_{21}(t)$. The latter function can be taken from the results of Ref. 3. In the one-nucleon exchange approximation, illustrated in Fig. 5, the function $D_{41}(t,s)$ is simply related to the process $N+\bar{N}\rightarrow 3\pi$, which also enters the scalar nucleon form factor problem. If we were to make the approximation that the 4π state is coupled to the photon only through the 2π state, as indicated in Fig. 6, then $h_4(s)$ would vanish and the solutions take on a particularly simple form.

Rather than attempting to carry out this calculation in detail, let us approximate the various contributions by simple poles. We expect from the form of (A6) that the 2π and 4π contributions to the vector nucleon form factors should be given approximately by

$$F^V(t)=a^V(1+t/28)^{-1}+b^V(1+t/T_V)^{-1} +(1-a^V-b^V), \quad \text{(A8)}$$

where T_V is expected to lie in the range 45–75. It should be mentioned that the $K-\bar{K}$ contribution has a threshold at $t\sim 50$, therefore the b terms in F^V and F^S also will represent the contribution of this state.