Vertex Function, Coupling Constant, and Composite Particle*†

LU SUN LIU‡
The Enrico Fermi Institute for Nuclear Studies and the Department of Physics, The University of Chicago, Chicago, Illinois

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A vertex function with only three kinds of strongly interacting particles is studied. As one of them becomes unstable, the structure singularities (one of which corresponds to the anomalous threshold) of this vertex function are shown to move out of the unphysical sheet and remain on the physical cut. In writing out its dispersion relation, we have to choose the correct Riemann sheet for its absorptive part. This choice is made by using a new simple method. A physical interpretation of this structure anomaly is given. With the help of this vertex function, we discuss a model of composite particles, stable or unstable. We obtain the trilinear scalar-type coupling constant in terms of the three masses by using an unsubtracted dispersion relation of this vertex function. A method to estimate the lifetime of the unstable particle is proposed. Another condition between the coupling constant and the masses is obtained from consideration of the charge structure of the composite particle. The two independently obtained conditions can be used to determine both the coupling constant and the binding energy simultaneously. As an explicit example, we consider Σ as a bound state of Λ and π through a scalar-type $\Sigma\Lambda\pi$ coupling. The calculated mass of Σ is in excellent agreement with its observed mass and the coupling constant is found to be $g^2/4\pi = 1.4$. A brief discussion on various problems concerning the present work is also given.

I. INTRODUCTION

IN recent years much effort has been devoted to the exploration of the analytic properties of the vertex function on the basis of Lorentz invariance, microscopic causality and the spectral conditions. The problem of investigating the location and the nature of the singularities of this function is highly important. Källèn and Wightman[1] considered the vertex function corresponding to the diagram in Fig. 1 and obtained interesting structure singularities as a function of all three invariant variables z_1, z_2, and z_3. This kind of structure singularities (anomalous thresholds) was also found by Karplus, Sommerfield, and Wichmann,[2] by Nambu,[3] and by Oehme[4] to be closely related with the masses of the external and internal particles, and has been recently clarified by the work of many authors.[5–9] They considered the singularities below the onset of the physical threshold. We may call these singularities *lower* anomalous thresholds.

In this paper we report that these structure singularities may move out of the unphysical sheet and stay on the physical cut when unstable particles are involved. Here the instability refers to strong interactions only. We call these singularities *upper* structure singularities. In order to avoid unnecessary complications, we shall restrict ourselves to three fields only. We give a simple derivation of the dispersion relation of the three-field vertex in the presence of the new anomaly. A physical interpretation of the upper anomaly is also given; it corresponds to the real decay effects of the unstable particle. For comparison we recall that the lower anomaly may be simply interpreted as the quantum mechanical tunnel decay effect of the loosely bound particle.[5,8–10] In this sense the loosely bound particles may be regarded as behaving like unstable particles.[11] Only these particles can give rise to structure singularities. We hope that the present work will lead to a better understanding of this structure anomaly, and will bring out the possibility of treating composite particles in the framework of causal and relativistic field theories.[5] In this formalism a composite particle manifests itself through structure singularities. Although we have studied only the vertex function, it is clear that scattering amplitudes and reaction amplitudes may have similar singularities.

A vertex function consisting of three external fields provides us with an ideal tool for the study of a composite particle with only two constituents. This approach has been undertaken recently by Blankenbecler and Cook.[7] They have pointed out that many features

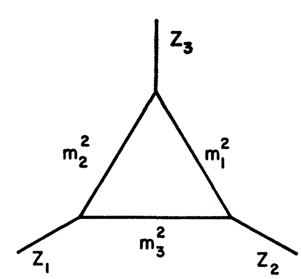

FIG. 1. Feynman diagram for a vertex function.

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‡ Now at the Department of Physics, Northwestern University, Evanston, Illinois.

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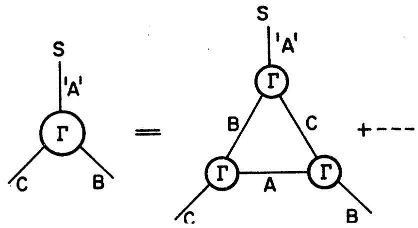

FIG. 2. A dispersion diagram (reduced diagram) for the vertex function $\Gamma(s)$ where particle A is off the energy shell.

of the bound state, e.g., a deuteron, can be obtained by considering the vertex function

$$(4N^0D^0)^{\frac{1}{2}}\langle N|f_p(0)|D\rangle. \tag{1.1}$$

(We follow the notations in Blankenbecler and Cook's paper.) This is much simpler than the customarily introduced amplitude

$$\langle 0|T(\psi_N(N)\psi_p(p)|D\rangle, \tag{1.2}$$

which involves ambiguities concerning the treatment of the relative time dependence. Furthermore, it has the advantage of allowing us to use dispersion theory. In view of their interesting results, such as the relation between anomalous thresholds and the Schrödinger equation and the estimate of the asymptotic (D-S) ratio for the deuteron, we would like to explore the properties of a composite particle, stable or unstable, with the help of the vertex function. We take a simple composite-particle model with only three kinds of strongly interacting particles. This model, describing a self-consistent bound or resonant system of two particles, requires the crossing symmetry which is characteristic of a relativistic field theory. We obtain the trilinear scalar coupling constant in terms of the three masses by using an unsubtracted dispersion relation of the vertex function. Certain properties of this model are revealed through this relation between the coupling constant and masses.

On the other hand, if the composite particle also possesses a charge, this charge must result from the interaction of its constituents (which are coupled to the composite particle through a strong interaction) with the electromagnetic field in the present model. Therefore we can obtain another condition between the coupling constant and the masses through the calculation of the electromagnetic form factor of the composite particle by assuming an unsubtracted dispersion relation. This condition derived from the charge structure is independent of that obtained from the composite structure (the vertex function), as mentioned in the last paragraph. Thus it is possible to determine both the coupling constant and the binding energy of a composite system from the two independent conditions. As an explicit application, we consider the Σ particle as a bound state of Λ and π through the recently speculated odd $\Sigma\Lambda$ parity and scalar-type $\Sigma\Lambda\pi$ coupling.[12] The determined value of the mass of Σ particle is in excellent agreement with its observed value and the coupling constant $g^2/4\pi$ is 1.4.

Finally, the integral equation of the vertex function, the possibilities of obtaining a potential describing the bound system, and some other physical implications are briefly discussed.

II. UPPER STRUCTURE SINGULARITIES

We discuss a simple model involving only three kinds of scalar particles, A, B, and C, with corresponding masses M_a, M_b, and M_c, interacting strongly through a scalar-type coupling. We may consider one of the three particles as a composite system of the other two. In this section we would like to investigate the nature of structure singularities of a vertex involving unstable particles and how the compositeness manifests itself through these singularities.

The vertex function is defined by

$$\Gamma(s)=(4B^0C^0)^{\frac{1}{2}}\langle B|j_a(0)|C\rangle, \tag{2.1}$$

where $j_a(0)$ is the current operator for the scalar particle A, and

$$s=-(C-B)^2. \tag{2.2}$$

We express $\Gamma(s)$ in the standard way as

$$\Gamma(s)=i(2C^0)^{\frac{1}{2}}\int d^4x\, e^{-iB\cdot x}\langle 0|[j_b(x),j_a(0)]\theta(x_0)|C\rangle,$$

where an equal-time commutator term has been dropped. We see that $\Gamma(s)$ is analytic in the lower-half s plane. Introducing a sum over a complete set of states $|n\rangle$ and carrying out the integrations over x, we find

$$\mathrm{Im}\Gamma(s)=-\pi(2C^0)^{\frac{1}{2}}\sum_n \langle 0|j_a(0)|n\rangle\langle n|j_b(0)|C\rangle \times\delta(n+B-C). \tag{2.3}$$

We restrict ourselves to the consideration of the least massive of the intermediate states, namely, that consisting of one B and one C. In this approximation,

$$\mathrm{Im}\Gamma(s)=-\pi(2C^0)^{\frac{1}{2}}\int\frac{d^3B'd^3C'}{(2\pi)^3}\langle 0|j_a(0)|B'C'\rangle \times\langle B'C'|j_b(0)|C\rangle\delta(B'+C'+B-C). \tag{2.4}$$

This intermediate state generates an integral equation for the vertex function $\Gamma(s)$. For the study of structure singularities we now make use of the lowest order approximation, namely, taking

$$(4B'^0C'^0)^{\frac{1}{2}}\langle 0|j_a(0)|B'C'\rangle=\Gamma_0$$

and the Born term for the amplitude

$$(8B'^0C'^0C^0)^{\frac{1}{2}}\langle B'C'|j_b(0)|C\rangle.$$

We shall discuss the integral equation for $\Gamma(s)$ in Sec.

[12] S. Barshay and M. Schwartz, Phys. Rev. Letters 4, 618 (1960); Y. Nambu and J. J. Sakurai, *ibid.* 6, 377 (1961). These papers contain further references.

VI. For the matrix element $\langle B'C'|j_b(0)|C\rangle$ we take the Born term by retaining only one A state, corresponding to the graph in Fig. 2. The result is

$$F\equiv(8B'^0C'^0C^0)^{\frac{1}{2}}\langle B'C'|j_b(0)|C\rangle$$
$$=\frac{\Gamma_0^2}{(B+C')^2+M_a^2}+\frac{\Gamma_0^2}{(C-B)^2+M_a^2}, \quad (2.5)$$

where $\Gamma_0=\Gamma(M_a^2)$ is the renormalized coupling constant. The first term in Eq. (2.5) describes the structure of the composite particle and the second, which is a renormalization term, may be absorbed into the definition of a vertex. Here, for the purpose of studying the structure singularities, we shall subtract this renormalization term since it will not introduce any such singularities. By introducing relative and center-of-mass coordinates, one can easily perform the integration in Eq. (2.4) which is reduced to an angular integration. The resulting absorptive part is then given by

$$\mathrm{Im}\Gamma(s)=-\frac{\Gamma_0\{[s-(M_c-M_b)^2][s-(M_c+M_b)^2]\}^{\frac{1}{2}}}{16\pi s}\times F_0(s), \quad (2.6)$$

where $F_0(s)$ is the S-wave projection of F in the center-of-mass frame of particles B and C. Explicitly,

$$F_0(s)=\frac{\Gamma_0^2}{4|\mathbf{B}|^2}\ln\frac{\Lambda+2|\mathbf{B}|^2}{\Lambda-2|\mathbf{B}|^2},$$
$$|\mathbf{B}|^2=\frac{[s-(M_c-M_b)^2][s-(M_c+M_b)^2]}{4s}, \quad (2.7)$$
$$\Lambda=\frac{s^2+2s(M_a^2-M_b^2-M_c^2)-(M_c^2-M_b^2)^2}{2s}.$$

Here the S-wave amplitude $F_0(s)$ results from the s-wave coupling among the scalar particles A, B, and C. Since this form of the absorptive part of the vertex evaluated in the lowest order is identical to the result of perturbation theory calculated from the diagram in Fig. 2, we shall use some well-known perturbation results whenever it is convenient. Combining (2.6) and (2.7), we get

$$\mathrm{Im}\Gamma(s)=-\frac{\Gamma_0^3\ln\{s(s-2M_c^2-2M_b^2+M_a^2)/[M_a^2s-(M_c^2-M_b^2)^2]\}}{16\pi\{[s-(M_c-M_b)^2][s-(M_c+M_b)^2]\}^{\frac{1}{2}}}. \quad (2.8)$$

That $\mathrm{Im}\Gamma(s)$ is real leads to the choice of the principal branch for the logarithm in (2.8).

The function $\mathrm{Im}\Gamma(s)$ behaves like $\ln s/s$ for large s, and hence one may assume an unsubtracted dispersion relation

$$\Gamma(s)=-\frac{1}{\pi}\int_{(M_c+M_b)^2}^{\infty}ds'\frac{\mathrm{Im}\Gamma(s')}{s'-s+i\epsilon}. \quad (2.9)$$

The two structure singularities s_1 and s_2 of the vertex function $\Gamma(s)$ are given by the poles of the argument of the logarithm in (2.8) except the one, $s=0$, which always remains on the second Riemann sheet. They are

$$s_1=2M_c^2+2M_b^2-M_a^2,$$
$$s_2=(M_c^2-M_b^2)^2/M_a^2. \quad (2.10)$$

These values are the same as one would obtain from the general formula given in references 1 and 2. Here s_2 has the same mathematical structure as the lower anomalous threshold usually considered in the case of a loosely bound system. The condition[13] for the appearance of structure singularities of the vertex function with the variable z_3 in Fig. 1 on the physical sheet is

$$m_1(z_1-m_2^2-m_3^2)+m_2(z_2-m_1^2-m_3^2)>0. \quad (2.11)$$

[13] See, for example, reference 2.

In our case, this condition means either

$$M_c>M_a+M_b,\quad M_c+M_b>0,$$
and
$$M_a+M_c>M_b, \quad (2.12a)$$
or
$$M_b>M_a+M_c,\quad M_b+M_c>0,$$
and
$$M_a+M_b>M_c. \quad (2.12b)$$

In either case an unstable particle introduces structure singularities on the physical sheet. Since (2.12a) and

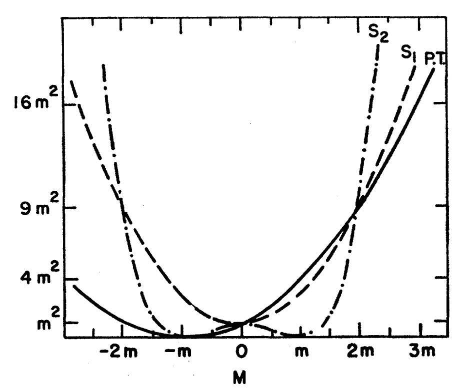

FIG. 3. The structure singularities $s_1=2M^2+m^2$, $s_2=(M^2-m^2)^2/m^2$ and the physical threshold $\mathrm{P.T.}=(M+m)^2$ for the vertex function $\Gamma(s)$ in Fig. 2 with $M_a=M_b=m$ and $M_c=M$ are plotted versus the composite particle mass M.

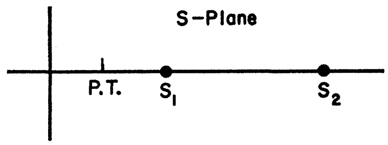

FIG. 4. Structure singularities $s_1=2M_c^2+2M_b^2-M_a^2$, $s_2=(M_c^2-M_b^2)^2/M_a^2$, and the physical threshold P.T.$=(M_c+M_b)^2$ for the vertex function $\Gamma(s)$ in Fig. 2.

(2.12b) are equivalent, we take (2.12a), or M_c as an unstable particle. For simplicity we assume $M_a=M_b$ and plot the structure singularities in (2.10) versus M_c in Fig. 3. In this case the condition for the existence of upper structure singularities is $M_c>2M_a$. We see that in the case of $M_c>0$ the structure singularities remain on the unphysical sheet when $M_c<2M_a$, coincide with the physical threshold when $M_c=2M_a$, and stay on the physical cut when $M_c>2M_a$. When $M_c=0$, the structure singularities always coincide with the physical threshold. Since we are interested in the upper structure singularities, we shall not consider the special case of zero mass particles.

We now turn to the discussion of the dispersion relation of the vertex function $\Gamma(s)$ in the case of unstable particles. The fact that the argument of the logarithm in (2.8) becomes negative in the interval $s_1<s<s_2$ (see Fig. 4) indicates that we have to make at least a choice of the branch of the many-valued logarithm function. Although it seems natural to push these structure singularities off the physical cut by introducing an imaginary quantity to the mass of the unstable particle, the derivation of the dispersion integral is anticipated to be rather complicated, or even incorrect, for one may have chosen a wrong Riemann sheet. Since this direct approach will involve computational difficulties and ambiguities, we try a quite different method.

We express the dispersion integral in the following form

$$\Gamma(s)=\frac{\Gamma_0^3}{16\pi^2}\left\{\int_{(M_c+M_b)^2}^{s_1} ds'\frac{\ln[s'(s'-s_1)/M_a^2(s'-s_2)]+in'\pi}{(s'-s+i\epsilon)\{[s'-(M_c-M_b)^2][s'-(M_c+M_b)^2]\}^{\frac{1}{2}}}\right.$$
$$+\int_{s_1}^{s_2} ds'\frac{\ln[s'(s'-s_1)/M_a^2(s_2-s')]+in\pi}{(s'-s+i\epsilon)\{[s'-(M_c-M_b)^2][s'-(M_c+M_b)^2]\}^{\frac{1}{2}}}$$
$$\left.+\int_{s_2}^{\infty} ds'\frac{\ln[s'(s'-s_1)/M_a^2(s'-s_2)]+in''\pi}{(s'-s+i\epsilon)\{[s'-(M_c-M_b)^2][s'-(M_c+M_b)^2]\}^{\frac{1}{2}}}\right\}, \tag{2.13}$$

$n'=0, \pm2, \pm4$, etc.,

$n=\pm1, \pm3, \pm5$, etc.,

$n''=0, \pm2, \pm4$, etc.,

where the integers n', n, and n'' should be determined. By making use of a normal dispersion integral we shall show that $n'=n''=0$ and $n=-1$.

Let us consider a vertex function

$$\Gamma(t)=(4A^0B^0)^{\frac{1}{2}}\langle A|j_c(0)|B\rangle, \tag{2.14}$$

where $t=-(B-A)^2$. By making the substitution $A\rightarrow C$, $B\rightarrow A$, and $C\rightarrow B$ in the vertex function $\Gamma(s)$, we obtain $\Gamma(t)$. The absorptive part of $\Gamma(t)$ is

$$\mathrm{Im}\Gamma(t)=-\frac{\Gamma_0^3\ln\{t(t-2M_b^2-2M_a^2+M_c^2)/[M_c^2t-(M_b^2-M_a^2)^2]\}}{16\pi\{[t-(M_b-M_a)^2][t-(M_b+M_a)^2]\}^{\frac{1}{2}}}. \tag{2.15}$$

Clearly, $\Gamma(t)$ is different from $\Gamma(s)$ both in mathematical structure and in physical meaning, but should be identical to $\Gamma(s)$ at one point, namely

$$\Gamma(t=M_c^2)=\Gamma(s=M_a^2)=\Gamma_0. \tag{2.16}$$

By either applying the condition in (2.11) or explicitly considering the poles of the logarithm in $\mathrm{Im}\Gamma(t)$, one can easily show that $\Gamma(t)$ has no structure singularities on the physical sheet if the condition in (2.12a) holds true. Thus, the vertex $\Gamma(t)$ satisfies a normal dispersion relation

$$\Gamma(t)=-\frac{1}{\pi}\int_{(M_a+M_b)^2}^{\infty} dt'\frac{\mathrm{Im}\Gamma(t')}{t'-t+i\epsilon}. \tag{2.17}$$

The continuity condition $\Gamma(t=M_c^2)=\Gamma(s=M_a^2)=\Gamma_0$ in (2.16) allows us to determine the correct branch of the logarithm in $\Gamma(s)$.

$$\Gamma(t=M_c^2)=\frac{\Gamma_0^3}{16\pi^2}\left\{P\int_{(M_a+M_b)^2}^{\infty}dt'\frac{\ln\{t'(t'-2M_b^2-2M_a^2+M_c^2)/[M_c^2t'-(M_b^2-M_a^2)^2]\}}{(t'-M_c^2)\{[t'-(M_b-M_a)^2][t'-(M_b+M_a)^2]\}^{\frac{1}{2}}}\right.$$
$$\left.-i\pi\frac{\ln\{2M_c^2(M_c^2-M_b^2-M_a^2)/[M_c^4-(M_b^2-M_a^2)^2]\}}{\{[t'-(M_b-M_a)^2][t'-(M_b+M_a)^2]\}^{\frac{1}{2}}}\right\}. \quad (2.18)$$

On the other hand, after carrying out the integrations in (2.13), we have

$$\Gamma(s=M_a^2)=\frac{\Gamma_0^3}{16\pi^2}\left\{P\int_{(M_c+M_b)^2}^{\infty}ds'\frac{\text{Principal branch of }\ln[s'(s'-s_1)/M_a^2(s'-s_2)]}{(s'-M_a^2)\{[s'-(M_c-M_b)^2][s'-(M_c+M_b)^2]\}^{\frac{1}{2}}}\right.$$
$$+in'\pi\frac{\ln[(M_c^2+M_b^2-M_a^2)/2M_cM_b]}{\{[M_c^2-(M_b-M_a)^2][M_c^2-(M_b+M_a)^2]\}^{\frac{1}{2}}}+in\pi\frac{\ln\{2M_c^2(M_c^2-M_b^2-M_a^2)/[M_c^4-(M_b^2-M_a^2)^2]\}}{\{[M_c^2-(M_b-M_a)^2][M_c^2-(M_b+M_a)^2]\}^{\frac{1}{2}}}$$
$$+in''\pi$$
$$\left.\times\frac{\ln[2M_b^2(M_c^2-M_b^2+M_a^2)/(M_c^2-M_b^2-M_a^2)\{M_c^2+M_b^2-M_a^2-[M_c^2-(M_b-M_a)^2]^{\frac{1}{2}}[M_c^2-(M_b+M_a)^2]^{\frac{1}{2}}\}]}{\{[M_c^2-(M_b-M_a)^2][M_c^2-(M_b+M_a)^2]\}^{\frac{1}{2}}}\right\}$$
$$(2.19)$$

A comparison between (2.18) and (2.19) gives

$$n'=n''=0,\quad n=-1. \quad (2.20)$$

This completes the dispersion relation in (2.13).

We see that in the unstable particle case the absorptive part of the vertex function becomes complex. This is, however, not an unusual feature since unstable particles have a decay lifetime, which corresponds to an imaginary mass.

We would like to summarize our method for solving the difficulties associated with the upper anomaly. Let us consider the vertex in Fig. 1. It may have structure singularities when z_3 is its variable, but it may not have them when z_1 or z_2 is its variable. When z_1, z_2, and z_3 are simultaneously on their corresponding mass shells, the three vertex functions with variables z_1, z_2, and z_3, respectively, should equal the same renormalized coupling constant. This matching helps us solve the problem of upper structure singularities. We hope that this simple method may be found useful and applicable to other cases.

III. PHYSICAL INTERPRETATION

In this section the physical meaning of the upper anomaly will be discussed. We shall see that in the upper anomalous region the internal particles propagate freely, i.e., they are real particles. Thus the upper anomaly corresponds to the decay effect of the unstable particle.

Consider the diagram in Fig. 2. Let $\mathbf{q}$ be the momentum of the internal particle A. (See Fig. 2.) Kinematic calculation in the c.m. frame of particles C and B gives

$$\begin{aligned}C_0&=(s+M_c^2-M_b^2)/2\sqrt{s},\\ B_0&=(s-M_c^2+M_b^2)/2\sqrt{s},\\ |\mathbf{C}|^2=|\mathbf{B}|^2&=[s-(M_c-M_b)^2]\\ &\quad\times[s-(M_c+M_b)^2]/4s.\end{aligned} \quad (3.1)$$

The magnitude of $\mathbf{q}$ is given through the equation of energy conservation

$$(\mathbf{q}^2+M_a^2)^{\frac{1}{2}}+B_0=C_0,$$

or

$$\mathbf{q}^2=[(M_c^2-M_b^2)^2-M_a^2s]/s. \quad (3.2)$$

If the internal particle A is a real one, its momentum should be within the physically possible limits

$$|\mathbf{q}|_{\text{minimum}}^2\leqslant\mathbf{q}^2\leqslant|\mathbf{q}|_{\text{maximum}}^2,$$

where

$$|\mathbf{q}|_{\text{minimum}}^2=0\quad\text{and}\quad|\mathbf{q}|_{\text{maximum}}^2=4|\mathbf{B}|^2,$$

or

$$2M_c^2+2M_b^2-M_a^2\leqslant s\leqslant(M_c^2-M_b^2)^2/M_a^2. \quad (3.3)$$

The above relation shows that s is exactly in the upper anomalous region. For any s in this region we can always find a c.m. scattering angle with which the internal particle propagates freely. This consideration makes it plausible that the upper anomaly should be interpreted as the decay effect of the unstable particle.

A study of the condition in (2.11) shows that only loosely bound and unstable particles can produce structure singularities. The lower anomaly is interpreted as the virtual decay effect of the loosely bound particle due to the quantum-mechanical tunnel phenomenon.[5,8–10] This phenomenon means that the wave

function of the loosely bound particle spreads out well beyond the range of the binding force, and hence its constituent particles may be able to escape from each other. In this situation the loosely bound particle behaves like an unstable one. Thus both the lower and the upper anomaly which are of the same mathematical nature have the same physical meaning, namely, the virtual and the real decay effects of the composite particles.

In view of the appearance of such structure singularities and their physical meaning, we believe that it would probably be possible to define a local field operator for a composite particle in terms of the field operators of its constituents.[14] Its composite nature will manifest itself in this formalism through structure singularities.

IV. COUPLING CONSTANT AND COMPOSITE PARTICLE

We assume the validity of the unsubtracted dispersion relation in (2.9). After substituting (2.8) into (2.9) and putting $s=M_a^2$, we obtain an equation for Γ_0 which has two solutions. One is the trivial solution, $\Gamma_0=0$. The other solution gives the dependence of Γ_0 on the three masses as

$$\frac{16\pi^2}{\Gamma_0^2}=\int_{(M_c+M_b)^2}^{\infty} ds\frac{\ln\{s(s-2M_c^2-2M_b^2+M_a^2)/[M_a^2 s-(M_c^2-M_b^2)^2]\}}{(s-M_a^2+i\epsilon)\{[s-(M_c-M_b)^2][s-(M_c+M_b)^2]\}^{\frac{1}{2}}}. \tag{4.1}$$

In the presence of upper structure singularities, Eq. (4.1) should be treated properly as in Sec. II. In (4.1) we have retained only the structure term. The renormalization term corresponds to the self-energy of particle A and hence can be absorbed into the definition of $\Gamma(s)$. We would like to give some justification to this procedure. The composite particle amplitude can also be defined by

$$\Gamma(u)=(4A^0C^0)^{\frac{1}{2}}\langle A\,|\,j_b(0)\,|\,C\rangle, \tag{4.2}$$

where

$$u=-(C-A)^2.$$

In the lowest approximation $\mathrm{Im}\Gamma(u)$ likewise consists of two terms, one being the structure term and the other the renormalization term. We note that the structure terms in all the Γ's give an identical contribution to the coupling constant, but the renormalization terms in these vertex functions, corresponding to self-energies of different particles, are quite different. Since in principle the three differently defined vertex functions should give the same coupling constant, it is advisable to subtract the renormalization term in the lowest approximation.

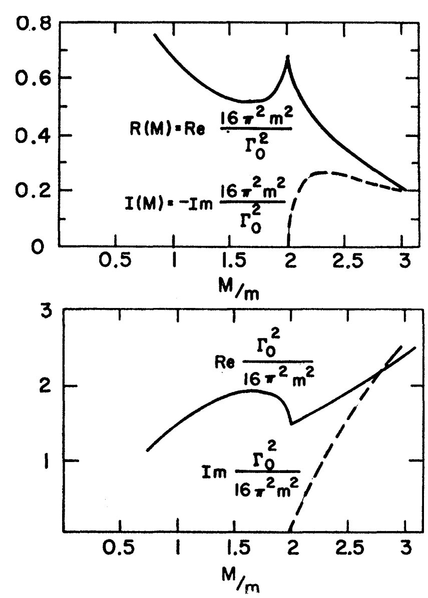

FIG. 5. Scalar-type M-m-m coupling constant as a function of the composite particle mass M which has two constituents, each of mass m.

The dependence of the coupling constant on the composite particle mass is obtained for a simple case in which $M_a=M_b=m$ and $M_c=M$, and the result is shown in Fig. 5. For M vanishingly small, the calculation of coupling constant from Eq. (4.1) is not very clear and reliable, since it involves a principal value integral which appears to be divergent. We recall that in this case the structure singularities coincide with the physical threshold (see Fig. 3). Thus for clarity we do not include the rather special case $M=0$ in our calculation of the coupling constant.

From Fig. 5 we can see some features of the present model. There appears to be a clear distinction between the stable (bound) and the unstable (resonant) cases. We start with the former. For this case the coupling constant is real and has an upper limit. It is also to be noted that the possible values of the coupling constant are confined in a narrow region. For a small coupling constant only one bound state exists. As we increase the value of the coupling constant, two bound states appear (two mass values corresponding to the same coupling constant), one being tightly bound and the other loosely bound. When its value is further increased, these two bound states approach nearer and nearer to each other and finally combine into one at its upper limit. Beyond this limit the bound particle state cannot exist but a resonant state may appear. In order to understand these features we reexamine the vertex function in Fig. 2. The composite particle C appears

[14] For references, see reference 5.

not only as an external line but also as an internal one which is responsible for the final-state interaction of its two constituents. Thus the system considered is a self-consistent one in the sense that there is no need to introduce new fields for the interaction between the constituent particles. If we change the external composite particle mass, the internal particle mass changes accordingly. Since their contributions to the coupling constant are in opposite directions, we would expect the aforementioned behavior.

In the unstable case the upper anomaly appears. By applying the treatment in Sec. II [see Eq. (2.19)], we obtain the following relation on the coupling constant

$$16\pi^2m^2/(\Gamma_0(M))^2=R(M)-iI(M), \quad \text{for} \quad M\geqslant 2m,$$

where

$$\begin{aligned} R(M)&=m^2\int_{(M+m)^2}^{\infty} ds\frac{\text{Principal branch of }\ln\{s(s-2M^2-m^2)/[m^2s-(M^2-m^2)^2]\}}{(s-m^2)\{[s-(M-m)^2][s-(M+m)^2]\}^{\frac{1}{2}}},\\ I(M)&=\pi m^2\frac{\ln[2(M^2-2m^2)/M^2]}{M(M^2-4m^2)^{\frac{1}{2}}}. \end{aligned} \tag{4.3}$$

Here the coupling constant becomes complex and extends to a wide range (see Fig. 5). Just at the instability point $M=2m$, $\mathrm{Re}(\Gamma_0^2/16\pi^2m^2)$ has a cusp and $\mathrm{Im}(\Gamma_0^2/16\pi^2m^2)$ rises from zero with an infinite slope. These sudden changes characterize the difference between stable and unstable particles. Here we have kept the mass of the unstable particle real and obtained a complex coupling constant. However, an unstable particle may be regarded as having a complex mass, of which the real part is its mass and the imaginary part is related to its lifetime. We propose the following method to evaluate the lifetime. By replacing the mass M of the unstable particle by $M-\frac{1}{2}i\gamma$ in (4.3) and requiring the coupling constant to be real, we can in principle obtain a condition for γ and hence get the lifetime and the coupling constant. In order to illustrate this procedure, we assume that $16\pi^2m^2/[\Gamma_0(M-\frac{1}{2}i\gamma)]^2$ can be approximated by its Taylor's expansion to just the first power of γ. This is valid in the region of $M=2.3m$ to $2.6m$, where the functions $R(M)$ and $I(M)$ are slowly varying. Thus the vanishing of the imaginary part of $16\pi^2m^2/[\Gamma_0(M-\frac{1}{2}i\gamma)]^2$ yields

$$\gamma(M)\cong-\frac{2I(M)}{dR(M)/dM}, \tag{4.4}$$

$$\frac{\Gamma_0^2}{16\pi^2m^2}\cong\left[R(M)+I(M)\frac{dI(M)/dM}{dR(M)/dM}\right]^{-1}. \tag{4.5}$$

The lifetime τ of the unstable particle is

$$\tau(M)=\frac{1}{\gamma(M)}\cong-\frac{dR(M)/dM}{2I(M)}. \tag{4.6}$$

Since $dR(M)/dM$ is negative (see Fig. 5) and $I(M)$ is positive, τ is positive. We note that here the positiveness of the lifetime is directly related to the logarithm branch, $n=-1$ in (2.20), in the upper anomalous region.

As a comparison we write down the lifetime calculated from the lowest order perturbation theory

$$\tau(M)=16\pi M^2/(M^2-4m^2)^{\frac{1}{2}}\Gamma_0^2. \tag{4.7}$$

For example, we find in the case of $M=2.4m$, $\Gamma_0^2/16\pi^2m^2\cong2.4$ from (4.5) and $\tau\cong0.68/m$ from (4.6), which is 3.2×10^{-24} sec for an unstable particle of a mass of 336 Mev with two constituent particles, each of mass of 140 Mev. Substitution of the calculated value of Γ_0^2 from (4.5) into (4.7) yields $\tau\cong0.58/m$. The two values of the lifetime agree reasonably.

We would like to stress again that this simple composite particle model describes a self-consistent bound or resonant system of two particles. The existence of this self-consistent system is due to the crossing symmetry which is a characteristic of the causal, relativistic field theory. This can be seen from (2.5) which is obtained from crossing symmetry. Some composite particles, e.g., a deuteron as a bound state of a proton and a neutron, cannot be described by this model if the conservation laws in strong interaction; namely, conservation of baryon number, strangeness, etc., are assumed to be true. In other words, these particles cannot exist in a self-consistent bound or resonant state. In the case of a deuteron we usually introduce pion field to maintain the $N-N$ bound system. It is reasonable to say that the conservation laws destroy the self-consistency of a bound or resonant system and require the existence of other fields to maintain the system.

V. COUPLING CONSTANT FROM CHARGE STRUCTURE AND PHYSICAL EXAMPLE

As mentioned in the first section, in the case of a charged composite particle the present model indicates that this charge results from the strong interaction with its constituents, which in turn interact with the electromagnetic field. Therefore, from the calculation of the electromagnetic form factor, we obtain another condition between the coupling constant and the masses, which is independent of the one obtained in Sec. IV. This charge structure calculation corresponds

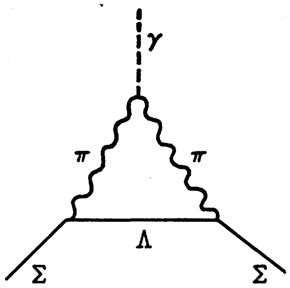

FIG. 6. Σ form factor from the contribution of a scalar-type $\Sigma\Lambda\pi$ coupling.

to the normalization of the wave function of the composite system in the Schrödinger theory. Thus, in principle, both the coupling constant and the binding energy of a composite system can be determined from these two methods.

For clarity, we illustrate this procedure through a physical example. We assume Σ particle to be a bound state of Λ and π, and all three are coupled together by a scalar interaction as proposed by several authors.[12]

Let Γ_0 be the scalar-type coupling constant with Σ and Λ considered as scalar particles, and g the same coupling constant with Σ and Λ considered as spinors. We relate them nonrelativistically in the following equation:

$$\frac{g^2}{4\pi}=\frac{\Gamma_0^2}{16\pi M_\Sigma M_\Lambda}. \tag{5.1}$$

In the following we shall evaluate the form factor of Σ to obtain a relation between $g^2/4\pi$ and the masses of Σ, Λ, and π by assuming an unsubtracted dispersion relation. We proceed in the usual way.[15] The Σ form factor F is discussed from the pair annihilation point of view. Thus

$$J_\mu=\left(\frac{p^0\bar{p}^0}{M_\Sigma^2}\right)^{\frac{1}{2}}\langle 0|\,j_\mu(0)\,|\bar{p}p \text{ in}\rangle$$
$$=-\bar{v}(\bar{p})F(-(p+\bar{p})^2)i\gamma_\mu u(p), \tag{5.2}$$

where $j_\mu(0)$ is the photon current. The absorptive part of J_μ is given by

$$\operatorname{Im}J_\mu=-\pi(p^0/M_\Sigma)^{\frac{1}{2}}\sum_s \bar{v}(\bar{p})\langle 0|\,j_\mu\,|s\rangle\langle s|\,f\,|p\rangle \times\delta(p_s-\bar{p}-p)$$
$$=-\bar{v}(\bar{p})\operatorname{Im}Fi\gamma_\mu u(p), \tag{5.3}$$

where f is the Σ current operator. If we restrict ourselves to the lowest mass state, i.e., that of two pions, the absorptive part is

$$\operatorname{Im}J_\mu=-\pi(p^0/M_\Sigma)^{\frac{1}{2}}\bar{v}(\bar{p})\sum_{ij}\int\frac{d^3qd^3k}{(2\pi)^3}\langle 0|\,j_\mu\,|q_ik_j\rangle \times\langle q_ik_j|\,f\,|p\rangle\delta(q+k-\bar{p}-p), \tag{5.4}$$

where the indices i and j are pion isospin labels. From the invariance consideration the first matrix element in the integrand, i.e., pion form factor, may be written as

$$(4q^0k^0)^{\frac{1}{2}}\langle 0|\,j_\mu\,|q_ik_j \text{ out}\rangle=i\frac{e}{\sqrt{2}}\epsilon_{3ij}(q-k)_\mu M^*((q+k)^2),$$

where $M^*(0)=1$. Throughout the calculation we shall adopt the point pion approximation of setting $M^*((q+k)^2)=M^*(0)=1$. The second matrix element describes Σ pair annihilation into two pions and is approximated by a Λ pole (see Fig. 6) in the Born term. The result is

$$\operatorname{Im}J_\mu=\frac{\pi e}{4}\int\frac{d^3qd^3k}{(2\pi)^3q^0k^0}(q-k)_\mu\bar{v}(\bar{p})\frac{2M_\Lambda g^2}{(p-q)^2+M_\Lambda^2}T_3 \times u(p)\delta(q+k-\bar{p}-p)$$
$$=-\bar{v}(\bar{p})\operatorname{Im}F_v(-(p+\bar{p})^2)T_3 i\gamma_\mu u(p), \tag{5.5}$$

where $F(-(p+\bar{p})^2)=F_v(-(p+\bar{p})^2)T_3$ and T_3 is the third component of the Σ isospin. Here F_v is the Σ charge form factor. We reduce (5.5) further by going over to the center-of-mass and relative coordinates. The result is

$$\operatorname{Im}F_v(x)=-\frac{eg^2M_\Sigma M_\Lambda(x-4M_\pi^2)^{\frac{1}{2}}}{2\pi x^{\frac{1}{2}}(x-4M_\Sigma^2)}\left\{\frac{x-2M_\Sigma^2+2M_\Lambda^2-2M_\pi^2}{2(x-4M_\pi^2)^{\frac{1}{2}}(x-4M_\Sigma^2)^{\frac{1}{2}}}\times\ln\frac{x-2(M_\Sigma^2-M_\Lambda^2+M_\pi^2)+(x-4M_\pi^2)^{\frac{1}{2}}(x-4M_\Sigma^2)^{\frac{1}{2}}}{x-2(M_\Sigma^2-M_\Lambda^2+M_\pi^2)-(x-4M_\pi^2)^{\frac{1}{2}}(x-4M_\Sigma^2)^{\frac{1}{2}}}-1\right\}, \tag{5.6}$$

where $x=-(p+\bar{p})^2$. We assume that the Σ form factor satisfies an unsubtracted dispersion relation of the form

$$F_v(x)=\frac{1}{\pi}\int_a^\infty dx'\frac{\operatorname{Im}F_v(x')}{x'-x+i\epsilon}, \tag{5.7}$$

where $a=4M_\pi^2$, the physical threshold, in the case of $M_\Sigma^2\leqslant M_\Lambda^2+M_\pi^2$, and

$$a=(1/M_\pi^2)[4M_\pi^2M_\Sigma^2-(M_\Sigma^2-M_\Lambda^2+M_\pi^2)^2],$$

[15] G. F. Chew, R. Karplus, S. Gasiorowicz, and F. Zachariasen, Phys. Rev. **110**, 265 (1958); P. Federbush, M. L. Goldberger, and S. B. Treiman, *ibid.* **112**, 642 (1958).

the lower anomalous threshold, in the case of $M_\Sigma^2 > M_\Lambda^2 + M_\pi^2$. In the latter case the logarithm term of $\mathrm{Im}F_v(x)$ in (5.6) starts from the lower anomalous threshold, while the "1" term always starts from the physical threshold, $x=4M_\pi^2$.

If we assume that only the scalar-type $\Sigma\Lambda\pi$ coupling contributes to the charge structure of Σ, which is regarded as a bound state of Λ and π, we can set $F_v(0)=e$ to obtain

$$\frac{4\pi}{g^2}=\frac{2M_\Sigma M_\Lambda}{\pi}\int_a^\infty dx\frac{(x-4M_\pi^2)^{\frac{1}{2}}}{x^{\frac{3}{2}}(x-4M_\Sigma^2)}\left\{\frac{x-2M_\Sigma^2+2M_\Lambda^2-2M_\pi^2}{2(x-4M_\pi^2)^{\frac{1}{2}}(x-4M_\Sigma^2)^{\frac{1}{2}}}\right.$$
$$\left.\times\ln\frac{x-2M_\Sigma^2+2M_\Lambda^2-2M_\pi^2+(x-4M_\pi^2)^{\frac{1}{2}}(x-4M_\Sigma^2)^{\frac{1}{2}}}{x-2M_\Sigma^2+2M_\Lambda^2-2M_\pi^2-(x-4M_\pi^2)^{\frac{1}{2}}(x-4M_\Sigma^2)^{\frac{1}{2}}}-1\right\}. \quad (5.8)$$

By putting the known mass values of Λ ($M_\Lambda=8.1M_\pi$) and π into (5.8), we calculate the coupling constant as a function of Σ mass, which is shown by the dashed curve in Fig. 7.

On the other hand, we replace the mass of A, B, and C particles by the corresponding mass of Λ, π, and Σ particles in (4.1) and obtain another independent Σ mass behavior of the coupling constant as shown by the solid curve in Fig. 7. The intersection of these two curves gives the coupling constant and the mass of the bound state Σ of Λ and π as follows

$$g^2/4\pi=1.4,\qquad M_\Sigma=8.65M_\pi. \quad (5.9)$$

The above-determined value of the Σ mass is in excellent agreement with the experimentally measured value. We see that the present model works in the case of treating the Σ hyperon as a self-consistent bound system of Λ hyperon and pion through a scalar-type $\Sigma\Lambda\pi$ coupling. We remark that the scalar-type $K\Sigma N$ coupling system cannot be applied in this way owing to the fact that the conservation of baryon number and strangeness forbids the use of this model.

VI. DISCUSSION

The absorptive part in Eq. (2.4) of the vertex function $\Gamma(s)$ considered as a composite particle amplitude has been expressed as a folding of another vertex function $I=(4B'^0C'^0)^{\frac{1}{2}}\langle 0|j_a(0)|B'C'\rangle$ and the amplitude $(8B'^0C'^0C^0)^{\frac{1}{2}}\langle B'C'|j_b(0)|C\rangle$. If we assume that the analytic continuation can be established

$$I=(4B'^0C'^0)^{\frac{1}{2}}\langle 0|j_a(0)|B'C'\rangle=\Gamma(s)=(4B^0C^0)^{\frac{1}{2}}\langle B|j_a(0)|C\rangle,$$

with

$$C=C' \quad\text{and}\quad B=-B', \quad (6.1)$$

Eq. (2.4) becomes

$$\mathrm{Im}\Gamma(s)=-\pi(2C^0)^{\frac{1}{2}}\int\frac{d^3B'd^3C'}{(2\pi)^3(4B'^0C'^0)^{\frac{1}{2}}}\Gamma(-(C'+B')^2)\times\langle B'C'|j_b(0)|C\rangle\delta(B'+C'+B-C). \quad (6.2)$$

One can in principle solve the integral equation for $\Gamma(s)$. We note that amplitude in Eq. (6.2) cannot be directly related to the usual scattering amplitude without going into the negative energy region. We are then confronted with the problem of analytic continuation and maintaining unitary condition in the negative region. The renormalization term introduced by the amplitude $(8B'^0C'^0C^0)^{\frac{1}{2}}\langle B'C'|j_b(0)|C\rangle$ will enter into the integral equation and thus complicate the problem further. Here the question is how this renormalization effect can be treated in a proper and rigorous way.

We remark that a composite particle can also be represented by the vertex function

$$F(s)=(4A^0B^0)^{\frac{1}{2}}\langle 0|j_c(0)|AB\ \mathrm{in}\rangle, \quad (6.3)$$

where $j_c(0)$ is the current operator for the composite particle C, and $s=-(A+B)^2$. If the composite particle C has total angular momentum J, then $j_c(0)$ selects only that part of the state $|A,B\ \mathrm{in}\rangle$ with angular momentum J in the amplitude $F(s)$. In the following we consider the case with $J=0$. In a standard way,

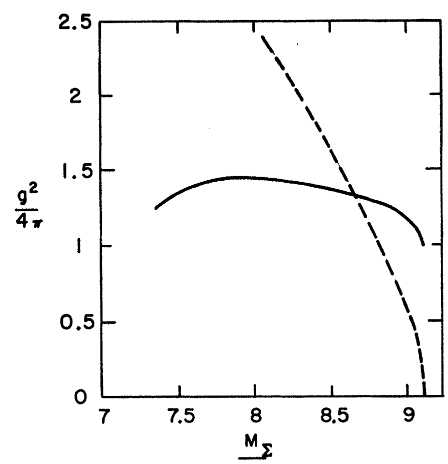

Fig. 7

FIG. 7. Scalar-type $\Sigma\Lambda\pi$ coupling constant as a function of the mass of Σ particle which is considered as a bound state of Λ and π. The solid curve was calculated from the Σ-Λ-π vertex, the dashed curve from the Σ form factor diagram in Fig. 6. The intersection of these two independent curves gives both the coupling constant and the Σ mass.

the absorptive part of $F(s)$ is given by

$$\mathrm{Im}F(s)=\pi(2B^0)^{\frac{1}{2}}\int\frac{d^3A'd^3B'}{(2\pi)^3}\langle 0|j_c(0)|A'B'\rangle$$
$$\times\langle A'B'|j_a(0)|B\rangle\delta(A'+B'-A-B). \quad (6.4)$$

The first matrix element in the integrand leads us back to the vertex function F; the second is the scattering amplitude for the process $A+B\rightarrow A+B$. In the lowest approximation the scattering amplitude with a particle C is its pole

$$(8A'^0B'^0B^0)^{\frac{1}{2}}\langle A'B'|j_a(0)|B\rangle=\frac{\Gamma_0^2}{(A+B)^2+M_c^2}$$
$$+\frac{\Gamma_0^2}{(A-B')^2+M_c^2}. \quad (6.5)$$

After inserting the above scattering amplitude into Eq. (6.4), we see that the first term is recognized as a renormalization effect and the second term contains the structure of the composite particle. This situation is similar to the previous case when we considered the vertex function Γ. Here $F(s)$ represents a self-consistent bound or resonant system of two particles in a scattering state. This also requires the crossing symmetry. If the absorptive part of $F(s)$ starts from the physical threshold $(M_a+M_b)^2$, the integral equation is a standard one and its solution can be immediately written down.[16] However, in our problem of composite particles, this may not be so. In the bound state case it may start from M_c^2, or it may have a pole at M_c^2, and hence the analytic continuation from the physical region is necessary.

There is an essential difference between $\Gamma(s)$ and $F(s)$. In $\Gamma(s)$, one of the two constituents of the composite particle is off the energy shell; in $F(s)$ the composite particle itself is off the energy shell. It would be interesting to compare both $\Gamma(s)$ and $F(s)$ for the description of a composite particle. For the purpose of studying unstable particles, it seems more convenient to consider $F(s)$ since it is originated from the scattering of two particles.

[16] R. Omnès, Nuovo cimento **8**, 316 (1958).

Blankenbecler and Cook have shown that a dispersion calculation of the D-N-N vertex gives the same results as the Schrödinger equation, and it is possible to define a potential describing the bound-state properties in field theory. In the present model the potential describing the self-consistent bound system can be calculated. If we assume a two-pion bound or resonant state B, we may get some information about the π-π potential which produces the state B.

The upper anomaly discussed in Secs. II and III would also appear in a scattering or reaction process involving unstable particles or narrow resonances. It corresponds to a resonance scattering since the intermediate particles behave like real ones. If a resonant state exists in the low-energy region, scattering with this resonant state may occur, and hence resonances in the higher energy region corresponding to the upper anomaly will probably appear. This consideration reveals some hope that resonances frequently occurring in the higher energy region may be closely connected with the resonances in the low-energy region.

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