Hyperons with and without Doublet Symmetry*

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The object of this study is a system containing a baryon with the properties of the Λ-Σ quartet, in strong interaction with mesons which may be bound to form various isobar states. The model involves two independent coupling constants g and g'. It will be shown that the case $g=g'$ which corresponds to doublet (or global) symmetry is singular in that it is strongly affected by a certain quantum-mechanical resonance.

1. INTRODUCTION

THE hypothesis of "global symmetry" in strange particle physics has appealed to many people.[1] Admittedly, in view of some notorious violations, this symmetry cannot be considered a strict law of nature. The most obvious blemish is the inequality of the baryon masses. If one wants to take the global symmetry at all seriously, the observed facts challenge him also to take deviations from this symmetry seriously whatever their cause may be. This remark applies equally to the "restricted" or "doublet symmetry" which is thought to govern the hyperons of strangeness -1, possibly as a special manifestation of global symmetry. Deviations from the doublet symmetry, such as may generate the Λ-Σ mass difference, are the object of the following investigation.

In particular, we are interested in certain exceptional aspects of the doublet symmetry which do not trivially follow from the symmetry as such. Generally speaking, small deviations from perfect symmetry may have large effects if the interactions in question are strong.

To be more specific, our concern will be the mass spectrum of hyperons with strangeness -1, including the excited states which manifest themselves as resonances in hyperon-pion interactions. It is well known that global symmetry predicts such resonances as counterparts to the $(p,\frac{3}{2},\frac{3}{2})$ resonance in nucleon-pion scattering, and the predictions are supported, to some extent, by recent observations. What new features do we expect as a result of deviations from global symmetry? We propose to show that the symmetric case is quite singular in that it is strongly affected by a certain quantum-mechanical resonance. This mechanism becomes ineffective as soon as the primary hyperon-pion interactions are allowed, even very slightly, to violate the doublet (or global) symmetry. As a consequence, the spectrum changes radically.

The existence of the "nucleon isobar" $(p,\frac{3}{2},\frac{3}{2})$ was first predicted on the basis of the "strong coupling theory,"[2] and one may still take advice from this theory for a qualitative analysis of low isobar states and their properties, provided, of course, he has some confidence in conventional field theory based on a Hamiltonian involving Yukawa interactions. Admittedly, such a theory cannot claim to be quantitative because it is strictly nonrelativistic with regard to the heavy particle (its recoil is entirely neglected, and the pion momentum spectrum is cut off). As we shall see, the strong coupling method affords a very simple picture of the resonance mechanism mentioned above, and we shall argue that, in a qualitative way, the conclusions must be more generally valid.

The real pion is pseudoscalar and interacts with the (static) nucleon in p states. To avoid grave complications which are certainly not essential to our argument, we shall first study a *scalar* meson theory (s-state interactions) in which the strong-coupling solutions can be exhibited in closed form. Afterwards, the pseudoscalar theory ($\sigma\cdot\nabla$ coupling) will be discussed in a less complete manner. As a further simplification, the calculations will be explicitly given only for "low cutoff" (large source radius), but the results will also be stated for the more interesting case of high cutoff. No apology will be needed for ignoring other complicating features like π-π interactions, K-meson couplings, and so forth.

2. SCALAR THEORY. PRELIMINARIES

We assume the heavy particle, fixed at some point, to possess four states labeled $\Lambda, \Sigma_1, \Sigma_2, \Sigma_3$, where Λ is isoscalar and the three other components form an isovector $\mathbf{\Sigma}$. Actually, each such state is a spin doublet, but this doubling can be ignored in the scalar theory. These "bare" particles are supposed to have equal masses. The scalar isovector meson field and its canonical conjugate will be denoted by $\boldsymbol{\psi}(x), \boldsymbol{\pi}(x)$; they are Hermitian and obey the canonical commutation relations. The Hamiltonian is chosen as

$$H=H_0+H_1, \tag{1}$$

$$H_0=\frac{1}{2}\int d^3x\{\boldsymbol{\pi}^2+\boldsymbol{\psi}\cdot(\mu^2-\nabla^2)\boldsymbol{\psi}\}, \tag{2}$$

$$H_1=g(\Lambda^\dagger\mathbf{\Sigma}+\mathbf{\Sigma}^\dagger\Lambda)\cdot\boldsymbol{\psi}_{\mathrm{av}}+g'i(\mathbf{\Sigma}^\dagger\times\mathbf{\Sigma})\cdot\boldsymbol{\psi}_{\mathrm{av}}, \tag{3}$$

where $\boldsymbol{\psi}_{\mathrm{av}}$ is a mean value of $\boldsymbol{\psi}$ at the site of the heavy

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[1] E.g., T. D. Lee and C. N. Yang, Phys. Rev. **122**, 1954 (1961). Many references to earlier papers are given there.
[2] For a survey and references to original work, see G. Wentzel, Revs. Modern Phys. **19**, 1 (1947), Sec. 4. The interpretation of the observed pion-proton scattering cross sections in terms of the $(p,\frac{3}{2},\frac{3}{2})$ resonance was first proposed by K. A. Brueckner, Phys. Rev. **86**, 106 (1952).

particle, involving a (real) source function $u(x)$:

$$\psi_{av}=\int d^3x\, u\psi\left[\int d^3x\, u\right]^{-1}. \tag{4}$$

g and g' are independent real coupling parameters (pure numbers if $\hbar=1$) which we may choose $\geqslant 0$ without loss of generality. The special case $g'=g$ is well known to correspond to the doublet (or "restricted") symmetry. Indeed, if one transforms to states

$$2^{-\frac{1}{2}}(\Sigma_1\pm i\Sigma_2) \quad \text{and} \quad 2^{-\frac{1}{2}}(\Lambda\mp\Sigma_3)$$

(forming two doublets usually called Y, Z), the 4×4 interaction matrix [see Eq. (10) below] splits, if $g'=g$, into two 2×2 matrices and thus reduces in essence to the nucleon-meson interaction.

The case of "low cutoff" is obtained by assuming

$$\int d^3x\, u \exp(ik\cdot x)\approx 0 \text{ except for } |k|\ll\mu. \tag{5}$$

Of course, this implies that the source radius a is large compared with the mesonic Compton wavelength: $a\mu\gg 1$. As mentioned in the introduction, this assumption will be made merely for the purpose of short-cutting unessential parts of the argument. How to deal with the case $a\mu\lesssim 1$ is well known in principle.

It is the virtue of the low cutoff to make the problem of how to split the meson field into "bound" and "free" parts a trivial one: the former has just the space dependence of the source function $u(x)$, while the latter is orthogonal to it, the coupling between the two parts (due to the ∇^2 term in H_0) being negligible on account of (5). For our purposes, we can then omit the free field right away. This amounts to substituting in (2):

$$\psi\rightarrow u(x)\mathbf{q}, \quad \pi\rightarrow u(x)\mathbf{p},$$

where $[p_\rho,q_\sigma]=-i\delta_{\rho\sigma}$, provided $u(x)$ is normalized according to

$$\int d^3x\, u^2=1. \tag{6}$$

Then, H_0 becomes simply

$$H_0=\tfrac{1}{2}(\mathbf{p}^2+\mu^2\mathbf{q}^2), \tag{7}$$

and ψ_{av} in H_1 differs from $\mathbf{q}$ only by a factor:

$$\psi_{av}=\mathbf{q}\left[\int d^3x\, u\right]^{-1}. \tag{8}$$

Thus, our problem reduces to a wave-mechanical one in three dimensions (charge space). However, H_1 is a 4×4 matrix in terms of the bare particle states. Using the definitions

$$\gamma=g\left[\int d^3x\, u\right]^{-1}, \quad \alpha=g'/g=\gamma'/\gamma, \tag{9}$$

we can write out the matrix (3) as follows:

$$H_1=\gamma\begin{Vmatrix} 0 & q_1 & q_2 & q_3 \\ q_1 & 0 & i\alpha q_3 & -i\alpha q_2 \\ q_2 & -i\alpha q_3 & 0 & i\alpha q_1 \\ q_3 & i\alpha q_2 & -i\alpha q_1 & 0 \end{Vmatrix}, \tag{10}$$

while H_0, of course, carries the unit matrix.

It is typical of the strong coupling method that one starts with diagonalizing the "large" interaction term, through a unitary transformation U:

$$U^\dagger H_1 U=\text{diagonal},$$

leaving H_0 to be transformed at a later stage. Denoting an eigenvector of H_1 (i.e., a column of U) by X_ρ, $\rho=0, 1, 2, 3$, we have the equations:

$$\begin{aligned} q_1X_1+q_2X_2+q_3X_3&=\lambda X_0,\\ q_1X_0+i\alpha(q_3X_2-q_2X_3)&=\lambda X_1 \end{aligned} \tag{11}$$

and so on (cyclic permutations). The solutions are of two kinds:

Type I.

$$X_0\neq 0, \quad X_\rho=X_0q_\rho/\lambda \quad (\rho\neq 0), \tag{12}$$

$$\lambda^2=q_1^2+q_2^2+q_3^2\equiv r^2. \tag{13}$$

Note that this solution is entirely independent of α.

Type II.

$$\begin{aligned} X_0=0, &\quad \mathbf{q}\cdot\mathbf{X}=0,\\ \lambda\mathbf{X}&=-i\alpha\mathbf{q}\times\mathbf{X}. \end{aligned} \tag{14}$$

Iteration of this equation gives

$$\lambda^2=\alpha^2r^2. \tag{15}$$

Hence, the eigenvalues of H_1 are

$$\pm\gamma r, \quad \pm\gamma' r,$$

where $\gamma'=\alpha\gamma$, $\gamma'/\gamma=g'/g$. Ignoring for the moment the U transformation still to be applied to the "kinetic energy" $\frac{1}{2}\mathbf{p}^2$ in (7), we may say that our problem has reduced to that of a particle moving in either one of the four central potentials:

$$V_\pm^{\mathrm{I}}(r)=\tfrac{1}{2}\mu^2r^2\pm\gamma r, \quad V_\pm^{\mathrm{II}}(r)=\tfrac{1}{2}\mu^2r^2\pm\gamma' r \tag{16}$$

($r\geqslant 0$). In the case of the repulsive potentials V_+^{I}, V_+^{II}, all eigenvalues of H are positive and far above the "valley energies" (minimum values of V_-^{I}, V_-^{II}) whose magnitudes are considered "large" in the strong coupling case. Of the two valleys, the deeper one will give the lowest (vibrational-rotational) states, and these lowest states are the only ones we are interested in. So we shall have to distinguish two "normal" cases:

I: $g>g'$, $V(r)=V_-^{\mathrm{I}}(r)$;

II: $g<g'$, $V(r)=V_-^{\mathrm{II}}(r)$;

and the "exceptional" case:

III: $g\approx g'$, where the two valleys (almost) coincide.

In the latter case, as we shall see, the term $U^\dagger\frac{1}{2}\mathbf{p}^2U$

couples the two kinds of states in a resonance-like fashion. Note that $g=g'$ corresponds to the doublet symmetry.

The almost classical nature of this picture should lead one to believe that any consistent field-theoretical treatment of the strong coupling case must somehow reproduce the main features emerging from our approximation.

3. SCALAR THEORY, CONTINUED

As usual, we can write

$$\tfrac{1}{2}\mathbf{p}^2=-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\frac{1}{2r^2}\mathbf{L}^2, \tag{17}$$

where $\mathbf{L}=\mathbf{q}\times\mathbf{p}$ is the iso-angular momentum of the bound meson field. The radial oscillations across the valley can be separated off; they are of no immediate interest because their excitation energy ($\approx\mu$) is still moderately large (though much smaller than the valley depths). What we plan to analyze in detail is the *rotational* fine structure which carries the small factor

$$1/2r^2\approx\mu^4/2\gamma^2 \quad\text{or}\quad \mu^4/2\gamma'^2 \tag{18}$$

(practically constant inside the valley).

We then have to examine the matrix $U^\dagger\mathbf{L}^2U$, where U can now be confined to the two columns corresponding to the eigenvectors (12) and (14), with $\lambda<0$ (attractive potentials only). Labeling these eigenvectors by superscripts K ($K=\mathrm{I}$ or II), the resulting 2×2 matrix is

$$(K|U^\dagger\mathbf{L}^2U|K')=\textstyle\sum_\rho X_\rho{}^{K*}\mathbf{L}^2X_\rho{}^{K'}, \tag{19}$$

assuming normalization according to

$$\textstyle\sum_\rho X_\rho{}^{K*}X_\rho{}^{K'}=\delta_{KK'}. \tag{20}$$

The diagonal element $K=K'=\mathrm{I}$ is easy to calculate. To satisfy (12) and (20) we take

$$X_0{}^{\mathrm{I}}=-2^{-\frac{1}{2}},\quad X_\rho{}^{\mathrm{I}}=2^{-\frac{1}{2}}q_\rho/r\quad(\rho\neq0), \tag{21}$$

and from the formula

$$\mathbf{q}\cdot[\mathbf{L}^2,\mathbf{q}]=2r^2,$$

we conclude immediately:

$$(\mathrm{I}|U^\dagger\mathbf{L}^2U|\mathrm{I})=\mathbf{L}^2+1=l(l+1)+1, \tag{22}$$

where $l=0,1,2,\cdots$. It follows that in case I ($g>g'$, and $g-g'$ not too small, see below) the low-lying states form an ordinary rotational spectrum. Of course, l is the isotopic spin of a state and $L_3=m$ its charge. The bare particle does not contribute to the isospin because it is, in case I, an isoscalar mixture ($\Lambda-\mathbf{\Sigma}\cdot\mathbf{q}/r$).

Next, in order to construct the eigenvector $X_\rho{}^{\mathrm{II}}$ obeying (14) with $\lambda=-\alpha r$, viz.,

$$X_0{}^{\mathrm{II}}=0,\quad \mathbf{q}\times\mathbf{X}^{\mathrm{II}}=-ir\mathbf{X}^{\mathrm{II}}, \tag{23}$$

we introduce a (real) orthogonal transformation matrix $S_{\rho\sigma}$ depending on three Euler angles Θ,Φ,Ψ, where Θ,Φ are the polar coordinates of the vector $\mathbf{q}$:

$$q_\rho=rS_{\rho3}, \tag{24}$$

and Ψ is, for the time being, a constant parameter, kept arbitrary:

$$\begin{aligned}
&S_{11}+iS_{12}=e^{i\Psi}(-\cos\Theta\cos\Phi-i\sin\Phi), && S_{13}=\sin\Theta\cos\Phi,\\
&S_{21}+iS_{22}=e^{i\Psi}(-\cos\Theta\sin\Phi+i\cos\Phi), && S_{23}=\sin\Theta\sin\Phi,\\
&S_{31}+iS_{32}=e^{i\Psi}\sin\Theta, && S_{33}=\cos\Theta,
\end{aligned} \tag{25}$$

$$\textstyle\sum_\rho S_{\rho\sigma}S_{\rho\tau}=\delta_{\sigma\tau}. \tag{26}$$

Then, (23) with (20) can be satisfied by setting

$$X_\rho{}^{\mathrm{II}}=-2^{-\frac{1}{2}}(S_{\rho1}+iS_{\rho2})\quad(\rho\neq0). \tag{27}$$

In (19), we adopt

$$\mathbf{L}^2=-\left\{\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\sin\Theta\frac{\partial}{\partial\Theta}+\frac{1}{\sin^2\Theta}\frac{\partial^2}{\partial\Phi^2}\right\}. \tag{28}$$

A straightforward calculation gives

$$\textstyle\sum_\rho X_\rho{}^{\mathrm{II}*}\mathbf{L}^2X_\rho{}^{\mathrm{II}}=\mathbf{L}^2+\dfrac{1}{\sin^2\Theta}\left(2\cos\Theta\dfrac{1}{i}\dfrac{\partial}{\partial\Phi}+1\right), \tag{29}$$

$$\textstyle\sum_\rho X_\rho{}^{\mathrm{I}*}\mathbf{L}^2X_\rho{}^{\mathrm{II}}=e^{i\Psi}\left\{\dfrac{\partial}{\partial\Theta}+\dfrac{1}{\sin\Theta}\left(\dfrac{1}{i}\dfrac{\partial}{\partial\Phi}+\cos\Theta\right)\right\}. \tag{30}$$

For the sake of brevity, we omit the corresponding formulas for the $\mathbf{L}$ components and the bare particle isospin which together define the total isospin.

For the evaluation of (29) and (30) it turns out to be very helpful to consider the spherical top whose coordinates are Θ,Φ,Ψ (note that Ψ is now a variable), and whose angular momentum vector (in the "body-fixed frame") is given by

$$P_3=\frac{1}{i}\frac{\partial}{\partial\Psi},$$

$$P_1\pm iP_2=e^{\pm i\Psi}\left\{\pm\frac{\partial}{\partial\Theta}+\frac{1}{\sin\Theta}\frac{1}{i}\left(\frac{\partial}{\partial\Phi}-\cos\Theta\frac{\partial}{\partial\Psi}\right)\right\}, \tag{31}$$

$$\mathbf{P}^2=-\left\{\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\sin\Theta\frac{\partial}{\partial\Theta}+\frac{1}{\sin^2\Theta}\left(\frac{\partial^2}{\partial\Phi^2}-2\cos\Theta\frac{\partial^2}{\partial\Phi\partial\Psi}+\frac{\partial^2}{\partial\Psi^2}\right)\right\}. \tag{32}$$

The eigenfunctions of $\mathbf{P}^2$ and P_3 have the form[3]

$$e^{in\Psi}e^{im\Phi}F_{nml}(\Theta). \tag{33}$$

[3] The detailed structure of the functions $F(\theta)$ will not be needed. See, e.g., A. Sommerfeld, *Atombau und Spektrallinien* (Friedrich Vieweg und Sohn, Braunschweig, 1939), Vol. II, p. 161ff.

Comparing (32) with (29) [and (28)], one sees that the eigenfunctions (33) with $n=-1$ are also eigenfunctions of the operator (29). The eigenvalues of (29) are therefore $l(l+1)$ with $l=1, 2, \cdots$ [$l\neq 0$, because $l\geqslant |n|$ in (33)]:

$$(\mathrm{II}|U^\dagger \mathbf{L}^2 U|\mathrm{II})=l(l+1), \quad l\geqslant 1. \tag{34}$$

Note, also, that the functions (33) with $n=0$ are identical with the eigenfunctions of $\mathbf{L}^2$ (28), that is to say, with the eigenfunctions of type I, belonging to the eigenvalues (22). We shall immediately make use of this fact.

Namely, turning now to the matrix elements (30) linking states I and II, we observe that the operator P_1+iP_2 (31), when applied to a function (33) with $n=-1$, reduces precisely to the operator (30). On account of the factor $e^{i\Psi}$ in P_1+iP_2, this operator indeed links the states $n=-1$ with those $n=0$, or type I, whereas it is diagonal in m and l. The general formula for angular momenta gives immediately

$$(n=0, m, l|P_1+iP_2|n=-1, m, l)=[l(l+1)]^{\frac{1}{2}}, \quad (l\geqslant 1),$$

so this is the value of the matrix element (30) (independent of the charge m), except for the phase factor $e^{i\Psi}$ which can now be set $=1$.

Collecting the results, we have for each $l\geqslant 1$ and $|m|\leqslant l$ the 2×2 matrix

$$U^\dagger \mathbf{L}^2 U=\left\|\begin{matrix} l(l+1)+1 & [l(l+1)]^{\frac{1}{2}} \\ [l(l+1)]^{\frac{1}{2}} & l(l+1)\end{matrix}\right\| \tag{35}$$

[first rows and columns refer to type I, see Eq. (22)], whereas for $l=0$ there is only one uncoupled solution of type I: $U^\dagger \mathbf{L}^2 U=1$.

4. SCALAR THEORY. DISCUSSION

We go back to the (transformed) "kinetic energy" (17) to which the "potential energy" (16) has to be added, namely, $V_-^{\mathrm{I}}(r)$ or $V_-^{\mathrm{II}}(r)$ for the state vectors of type I, II, respectively.

In the case of strict doublet symmetry, $g=g'$, $\gamma=\gamma'$, the two potentials coincide exactly, and also the corresponding factors of $\mathbf{L}^2$ in (17) [see (18)] are the same. Except for this factor (which we call B) and an additive constant $(-A)$, the energy levels are given immediately by the eigenvalues of the matrix (35):

$$\begin{aligned} E_{l,\pm}&=-A+B[l(l+1)+\tfrac{1}{2}\pm(l+\tfrac{1}{2})] \\ &=-A+B(l+\tfrac{1}{2}\pm\tfrac{1}{2})^2 \quad (g=g'). \end{aligned} \tag{36}$$

For $l=0$, only the upper sign applies Every energy level occurs twice, with two neighboring l values: $E_{l,+}=E_{l+1,-}$. This result agrees, of course, with that of the analysis in terms of Y, Z doublets, mentioned earlier.

If g'/g departs from unity, the above considerations make it clear that the energies will now be given as the eigenvalues of a matrix which has the following structure:

$$U^\dagger H U = \left\|\begin{matrix} -A+B[l(l+1)+1] & C[l(l+1)]^{\frac{1}{2}} \\ C[l(l+1)]^{\frac{1}{2}} & -A'+B'l(l+1)\end{matrix}\right\|. \tag{37}$$

A, A' are essentially the two valley depths ($\propto g^2, g'^2$), and B, B' are the factors (18) ($\propto g^{-2}, g'^{-2}$). C is of the same order as B, B', or smaller owing to an incomplete overlap of the oscillatory wave functions {these are proportional to $\exp[-\frac{1}{2}\mu(r-r_K)^2]$, where r_K, $K=\mathrm{I}, \mathrm{II}$, are the respective valley radii}.

The crucial point is that the coupling through the off-diagonal elements in (37) becomes ineffective as soon as

$$|A-A'|\gg C. \tag{38}$$

We then arrive at the "normal cases," as we called them in Sec. 2, the lowest levels being

$$\text{for } g>g': \quad E_{l,-}{}^{\mathrm{I}}=-A+B[l(l+1)+1], \tag{39}$$

$$\text{for } g<g': \quad E_{l,-}{}^{\mathrm{II}}=-A'+B'l(l+1), \quad (l\neq 0). \tag{40}$$

Varying the parameter g'/g through 1, how quickly do the spectra (39), (36), (40) change into each other? We give the answer for the "high cutoff" version of the theory in which the same matrix (37) appears but with different values for the constants. The orders of magnitude are then

$$A/g^2=A'/g'^2\sim 1/a,$$
$$Bg^2=B'g'^2\sim\mu,$$

where $\mu=$meson mass, $a=$source radius (cutoff momentum $1/a\gg\mu$). Condition (38) is satisfied if

$$|(g-g')/g|\gg a\mu/g^4. \tag{41}$$

Since the strong-coupling approximation requires $g\gg 1$ if $a\mu\ll 1$, the right-hand side of (41) is very small indeed, and goes to zero, even for finite g, if one lets the source radius tend to zero (this leaves B finite in the static scalar theory). The resonance which characterizes the case of doublet symmetry becomes, in this limit, infinitely sharp.

As an illustration, we have in Fig. 1 schematically plotted the $l=0, 1, 2$ energy levels $[+\frac{1}{2}(A+A')]$ versus g'/g (near $g'/g=1$). The whole transition takes place in a narrow interval of order $a\mu/g^4$.

5. PSEUDOSCALAR THEORY

If ψ stands for a pseudoscalar field, and $g'\neq 0$, parity conservation demands (in the static approximation) the $\sigma\cdot\nabla$ coupling:

$$H_1=g(\Lambda^\dagger\sigma\mathbf{\Sigma}+\mathbf{\Sigma}^\dagger\sigma\Lambda)\cdot\nabla\psi_{\mathrm{av}}+g'i(\mathbf{\Sigma}^\dagger\times\sigma\mathbf{\Sigma})\cdot\nabla\psi_{\mathrm{av}}. \tag{42}$$

Instead of the 4×4 matrix (10) we now have an 8×8 matrix which may still be formally written like (10) except that, in every element of (10), q_ρ must be re-

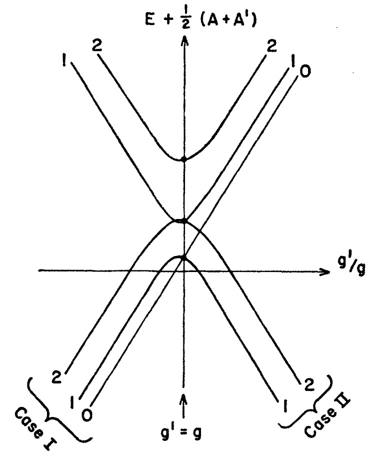

FIG. 1. Energy levels $E_l+\frac{1}{2}(A+A')$ for $l=0$, 1, 2, plotted vs g'/g. The numerals refer to l values (isotopic spin).

placed with a 2×2 spin matrix:

$$q_\rho \to \sum_i \sigma_i q_{i\rho} \equiv Q_\rho. \tag{43}$$

Instead of three, there are now nine variables $q_{i\rho}$ describing the bound mesons (these can be bound in three p states), and their contribution to H_0 is, instead of (7):

$$H_0=\tfrac{1}{2}\sum_{i\rho}(p_{i\rho}^2+\mu^2 q_{i\rho}^2), \tag{44}$$

corresponding to a nine-dimensional harmonic oscillator.

We have not succeeded in determining the rotational spectra for arbitrary values of g'/g. For $g'=0$, the solution is comparatively simple; this is reported in the Appendix. Starting from this case, one can extend the theory to small values of g'/g by perturbation theory, but this does not allow an extrapolation up to $g'/g\approx 1$. On the other hand, the case $g'=g$ (doublet symmetry) can be handled by the transformation to Y, Z states, because then H_1 splits into two 4×4 matrices, and the problem reduces to the well-known nucleon-pion strong-coupling problem.[2] The lowest energy levels obey Eq. (36), where again l= isospin, but now the bound mesons contribute also to the angular momentum (in x space); actually the spin j of the (coinciding) levels $E_{l,+}$ and $E_{l+1,-}$ is $l+\frac{1}{2}$. Both spin and isospin are invariants under adiabatic changes of g'/g; this enables us to connect the lowest levels at $g'=0$ and $g'=g$.

As g'/g approaches unity, a resonance situation like in the scalar theory must be expected. The reason is that, whereas for $g'=0$ there is *one* lowest valley (doubled by spin degeneracy), there are *two* coincident ones for $g'=g$, as a consequence of Y-Z degeneracy. This suggests that the two valley depths are essentially proportional to g^2 and g'^2, respectively, and "meet" in the singular case of doublet symmetry. The resonance width should again be proportional to g^{-4} (in the high-cutoff version: relative width $\sim(a/g)^4$, which is $\ll 1$ if the strong-coupling criterion is satisfied, viz., $g\gg a$ in the case of $\sigma\cdot\nabla$ coupling).

If we should try to interpret our present empirical knowledge about particles and resonant states of strangeness -1 in terms of this theory, assuming at least approximate global symmetry, we would have to identify [see Eq. (36)] the states[4] $0+$, $1-$, and $1+$, respectively, with the particles Λ, Σ, and the $Y_1{}^*$ resonance. The Σ-Λ mass difference must then be attributed to a deviation from global symmetry which could mean $g'/g\neq 1$, or alternatively, the interaction with the nucleon-Kaon states may well break the symmetry. (Note that the eigenfunctions $0+$ and $1-$ involve different mixtures of bare Λ and Σ components.) An encouraging feature is the well known fact that the $Y_1{}^*-\Lambda$ mass difference matches the nucleon-pion $(p,\frac{3}{2},\frac{3}{2})$ resonance energy rather closely.

APPENDIX

Here, we study the interaction (42) with

$$g'=0.$$

For the diagonalization of H_1 we have to solve the linear equations

$$\sum_\rho Q_\rho X_\rho=\lambda X_0,$$
$$Q_\rho X_0=\lambda X_\rho \quad (\rho\neq 0),$$

where Q_ρ stands for the spin matrices defined in (43) $[Q_0\equiv 0]$, and the X_ρ $(\rho=0\cdots 3)$ are two-component spinors. One concludes immediately

$$(\lambda^2-r^2)X_0=0,$$

where

$$r^2=\sum_\rho Q_\rho{}^2=\sum_{i\rho} q_{i\rho}{}^2. \tag{45}$$

The solutions are, as in Sec. 2, of two types:

$$\text{I: } X_0\neq 0,\quad \lambda=\pm r,\quad X_\rho=\lambda^{-1}Q_\rho X_0;\quad (\rho\neq 0) \tag{46}$$

$$\text{II: } X_0=0,\quad \lambda=0,\quad \sum_\rho Q_\rho X_\rho=0. \tag{47}$$

The lowest potential valley results from $\lambda=-r$, $V(r)=\frac{1}{2}\mu^2r^2-\gamma r$, and the corresponding eigenvectors (two of them, for spin degeneracy) may be taken as

$$X_0{}^s=-2^{-\frac{1}{2}}u_s,\quad X_\rho{}^s=2^{-\frac{1}{2}}(Q_\rho/r)u_s,\quad (\rho\neq 0), \tag{48}$$

where u_s stands for either one of two orthogonal unit spinors. Then, the "kinetic energy" will have to be transformed as follows:

$$U^\dagger TU\equiv\sum_\rho X_\rho{}^{s\dagger}TX_\rho{}^{s'}=\frac{1}{2}\left(T+\sum_\rho \frac{Q_\rho}{r}T\frac{Q_\rho}{r}\right), \tag{49}$$
$$T=\tfrac{1}{2}\sum_{i\rho} p_{i\rho}{}^2.$$

In the last expression (49) the spinor factors have been dropped, to the effect that $u_s{}^\dagger\sigma_i u_{s'}$ is re-interpreted as σ_i.

Since we are now dealing with a spherical potential in 9 dimensions, it will be helpful to recall some formulas valid in N-dimensional spaces, with Cartesian

[4] The $+$, $-$ signs do *not* refer to parity! Doublet symmetry implies equal parity of all states in the series, and would be ruled out if experimental evidence to the contrary were secured.

coordinates q_n ($n=1\cdots N$) and $r^2=\sum_n q_n^2$. Define

$$L_{mn}=\frac{1}{i}\left(q_m\frac{\partial}{\partial q_n}-q_n\frac{\partial}{\partial q_m}\right),\quad \mathbf{L}^2=\sum_{m<n}L_{mn}^2.$$

One proves easily

$$\mathbf{L}^2=r^2\left(-\Delta+\frac{\partial^2}{\partial r^2}\right)+(N-1)r\frac{\partial}{\partial r},$$

where $\Delta=\sum_n \partial^2/\partial q_n^2=-2T$; hence

$$T=-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{N-1}{r}\frac{\partial}{\partial r}\right)+\frac{1}{2r^2}\mathbf{L}^2. \tag{50}$$

As simplest eigenfunctions of $\mathbf{L}^2$ one finds

$$\begin{aligned} &F=f(r), && \mathbf{L}^2F=0, && (L=0)\\ &F_n=q_nf(r), && \mathbf{L}^2F_n=(N-1)F_n, && (L=1)\\ &F_{mn}=(q_mq_n-N^{-1}\delta_{mn}r^2)f(r), && \mathbf{L}^2F_{mn}=2NF_{mn}, && (L=2) \end{aligned} \tag{51}$$

and so on. The general eigenvalue formula is

$$\mathbf{L}^2=L(L+N-2),\quad L=0,1,2,\cdots. \tag{52}$$

Another formula will be needed:

$$[\mathbf{L}^2,q_n]=-2i\sum_m q_mL_{mn}+(N-1)q_n, \tag{53}$$

and finally, following from (53):

$$\sum_n q_n[\mathbf{L}^2,q_n]=(N-1)r^2. \tag{54}$$

Returning to (49), the expression (50) (with $N=9$) may now be used for T. Since $[\partial/\partial r,Q_\rho/r]=0$, we obtain

$$\begin{aligned} U^\dagger TU&=-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{8}{r}\frac{\partial}{\partial r}\right)+\frac{1}{2r^2}U^\dagger\mathbf{L}^2U\\ U^\dagger\mathbf{L}^2U&=\tfrac{1}{2}(\mathbf{L}^2+r^{-2}\textstyle\sum_\rho Q_\rho\mathbf{L}^2Q_\rho)\\ &=\mathbf{L}^2+\tfrac{1}{2}r^{-2}\textstyle\sum_{ij\rho}\sigma_i\sigma_jq_{i\rho}[\mathbf{L}^2,q_{j\rho}]. \end{aligned} \tag{55}$$

The terms $i=j$ in the sum reduce to the expression (54) $[n=i,\rho]$ and hence give only an additive constant 4. A more important contribution comes from the terms $i\neq j$ [e.g., $\sigma_1\sigma_2=-\sigma_2\sigma_1=i\sigma_3$] which may be evaluated by means of (53). The result can be written as follows:

$$U^\dagger\mathbf{L}^2U=\mathbf{L}^2+\sum_i\sigma_il_i+4, \tag{56}$$

where l_i is the "orbital" angular momentum (in the 3-dimensional x space) defined as

$$l_1=\frac{1}{i}\sum_\rho\left(q_{2\rho}\frac{\partial}{\partial q_{3\rho}}-q_{3\rho}\frac{\partial}{\partial q_{2\rho}}\right),\ \cdots,\ \cdots,$$

$$[l_1,l_2]=il_3,\ \cdots,\ \cdots.$$

TABLE I. Lowest levels in case $g'=0$.

L	l, j, i	$U^\dagger\mathbf{L}^2U-4$	Symbol
0	$s, \frac{1}{2}, 0$	0	Λ
1	$p, \frac{1}{2}, 1$	6	Σ
	$p, \frac{3}{2}, 1$	9	Y_1^*
2	$d, \frac{3}{2}, 2$	15	Y_2^*
	$d, \frac{3}{2}, 0$	15	Y_0^*
	$p, \frac{1}{2}, 1$	16	
	$s, \frac{1}{2}, 2$	18	
	$p, \frac{3}{2}, 1$	19	
	$d, \frac{5}{2}, 2$	20	
	$d, \frac{5}{2}, 0$	20	

For the eigenvalues of (56), we use (52) and introduce, as usual, the "total" angular momentum (in 3-space) $j_i=l_i+\frac{1}{2}\sigma_i$, with the result

$$U^\dagger\mathbf{L}^2U=L(L+7)+j(j+1)-l(l+1)-\tfrac{3}{4}+4. \tag{57}$$

Note that (contrary to Secs. 3, 4 where l denoted the isospin) l here has the customary meaning of orbital angular momentum, whereas the isospin i enters in (57) only via the number L which may be called an "angular momentum in 9-space."

To find out what j, l, and i values are compatible with a given L, one has to examine how the corresponding eigenfunctions (51) transform under rotations in ordinary space and in charge space. For instance, $q_n=q_{i\rho}$ is a vector in both spaces, so the 9 states $L=1$ necessarily belong to $l=1$, $i=1$. The same is true for the 9 functions $(q_{i\rho}q_{j\sigma}-q_{j\rho}q_{i\sigma})[i\neq j,\ \rho\neq\sigma]$ which belong to $L=2$, whereas $(\sum_\rho q_{i\rho}q_{j\rho}-\frac{1}{3}\delta_{ij}r^2)$ has $l=2$, $i=0$, $L=2$; and so on. In Table I we list all states $L=0$, 1, 2, with their respective l, j, and i quantum numbers (instead of $l=0,1,2$, we write the symbols s, p, d, as usual) with their "rotational" excitation energies (in units $\frac{1}{2}r^{-2}$), viz., $U^\dagger\mathbf{L}^2U-4$.

In the last column of the table we have inserted a tentative identification symbol which should be understood as follows. Imagine that we "switch on" the g' part of the interaction (42) which then would modify the level spacings. We know (see Sec. 5) that, as $g'\to g$, Λ and Σ merge into one energy level, and so also do Y_1^* and Y_2^*. As for Y_0^*, no low-lying level with zero isospin ($\neq\Lambda$) exists for $g'=g$; presumably this means that the Y_0^* level is shifted far upward as one comes into the resonance region $g'/g\approx1$. Optimistically, one might hope to find a g'/g value which would give qualitatively the observed level spacings. This might even include the state Y_0^* if the experimental indications for the existence of such a resonance are confirmed.

Further exploration of the cases $g'/g\neq0$, 1 is obviously desirable. Even for $g'/g=\alpha\ll1$, and to first order in α, the potential valley in 9-space loses its spherical symmetry (the determinant $|q_{i\rho}|$ appears, besides r, in the potential). This may indicate the nature of the mathematical problem.