Fundamental Constants of the Muon*

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Three precise measurements exist on static properties of the muon. These are the g factor, g-2, and the frequency of the $3D$-$2P$ transition in mesonic phosphorus. They are combined to obtain the best fit to the fundamental constants of the muon.

WITHIN the past year, three measurements of the fundamental properties of the μ meson, all of an accuracy approaching one part in 10^5, have been reported.[1–3] Maximum information can be extracted from the data if they are analyzed as a whole, taking cognizance of the influence which each parameter exerts on the value of all the others. Such a systematic analysis of the fundamental atomic constants has been undertaken by DuMond and Cohen.[4] Although the experimental errors in the muon measurements are larger by an order of magnitude than those in the atomic case, it is perhaps time to begin the application of a similar analysis to determine the fundamental constants of the muon.

It is convenient to choose, as the basic quantities to be determined, the dimensionless ratios $g=g_\mu/g_e$, $m=m_\mu/m_e$, and $e=e_\mu/e_e$. For the ratio of the muon to the electron g factors, there is a theoretical prediction[5]: $g=1.000\,006$. There is at present no theory to predict the mass and charge ratios, although on philosophical grounds the latter might be expected to be unity.

Each of the experimental results on the muon properties can be combined with a separate, independent, and much more accurate measurement of the similar quantity for the electron to yield a certain function of g, m, and e. The quantity g is obtained directly from the results of the CERN group[1] on g_μ, and of the similar experiment of Schupp *et al.*[6] on g_e. In both cases, the fractional difference between spin precession and cyclotron frequencies of the same particle is measured [so-called "g-2"].

$$g=\frac{g_\mu}{g_e}=\frac{1.001145\pm0.000022}{1.0011609\pm0.0000024}=0.999984\pm0.000022. \qquad (1)$$

The measurement of the muon magnetic moment[2] is actually a determination of the ratio of the spin precession frequency of the muon to that of a proton in the same magnetic field. An extremely precise measurement of the analogous ratio for the electron is quoted by DuMond.[7] Correction is made for diamagnetic shielding due to the water molecules of the proton magnetic resonance probe.

$$\frac{ge}{m}=\frac{f_\mu/f_p}{f_e/f_p}=\frac{3.18334\pm0.00005}{658.2107}=0.00483635. \qquad (2)$$

The average of three experiments[3] to measure the wave number, $\tilde{\nu}$, of an x ray emitted by a μ-mesonic atom (in this case the $3D_{\frac{5}{2}}$-$2P_{\frac{3}{2}}$ transition in mesonic phosphorus), together with the Rydberg constant[4] R, gives

$$me^2=\tilde{\nu}/KR=206.76\pm0.02. \qquad (3)$$

Here $K=Z^2\,(n_1^{-2}-n_2^{-2})$ times accurately known correction factors close to unity. Implicitly involved in this ratio is the conversion[4] from Siegbahn units (in terms of which the x-ray wavelengths are calibrated) to centimeters (for the Rydberg).

We now introduce new variables for the purpose of linearizing Eqs. (1)–(3). Let

$$\begin{aligned} X_1&=(g-g_0)/g_0,\\ X_2&=(m-m_0)/m_0,\\ X_3&=(e-e_0)/e_0, \end{aligned} \qquad (4)$$

where g_0, m_0, and e_0 are nominal values of the parameters, chosen to make X_1, X_2, and X_3 so small that their squares may be neglected. Let C_1, C_2, and C_3 be the fractional differences between the right-hand sides of Eqs. (1)–(3), and the same quantities calculated from the nominal values. With three free parameters to fit three experiments, values can always be found to make $C_1=C_2=C_3=0$. If one parameter is suppressed (e.g., e is set equal to unity) then two of the C's can be made

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TABLE I. Solutions for g, m, and e when all are treated as free parameters. Error matrix is expressed in (parts per million)^2.

$$g_\mu/g_e=0.999984\pm0.000022,$$
$$m_\mu/m_e=206.7630\pm0.0078,$$
$$e_\mu/e_e=0.999993\pm0.000035,$$
$$V_{ij}=\begin{pmatrix} 490 & 327 & -163 \\ 327 & 1439 & 947 \\ -163 & 947 & 1193 \end{pmatrix}.$$

to vanish but the third, in general, will not. Equations (1)–(3) then become

$$X_1 \cong C_1, \tag{5}$$

$$X_1 - X_2 + X_3 \cong C_2, \tag{6}$$

$$X_2 + 2X_3 \cong C_3. \tag{7}$$

The $\cong$ sign indicates that in an overdetermined situation, equality cannot in general be fulfilled, and the solution we seek is the best fit to all the data. The best fit is determined by the criterion that

$$M = W_1(C_1-X_1)^2 + W_2(C_2-X_1+X_2-X_3)^2 + W_3(C_3-X_2-2X_3)^2 \tag{8}$$

be a minimum. Here W_1, W_2, and W_3 are the relative weights assigned to the three input data, Eqs. (1)–(3). Differentiating M in turn with respect to each of the X's and setting equal to zero, leads to the normal equations:

$$(W_1+W_2)X_1 - W_2X_2 + W_2X_3 = W_1C_1 + W_2C_2, \tag{9}$$

$$-W_2X_1 + (W_2+W_3)X_2 + (-W_2+2W_3)X_3 = -W_2C_2 + W_3C_3, \tag{10}$$

$$W_2X_1 + (-W_2+2W_3)X_2 + (W_2+4W_3)X_3 = W_2C_2 + 2W_3C_3. \tag{11}$$

Solution of these equations yields the most probable values of X_1, X_2, and X_3, and hence of g, e, and m. The inverse of the matrix of the left-hand side of Eqs. (9)–(11) is the error matrix V_{ij}. The diagonal elements of the error matrix are equal to the square of the standard deviation of the corresponding X_i. The off-diagonal terms have the following interpretation: The standard deviation, Δf, in any function of (g,m,e) is given by

$$(\Delta f)^2 = \sum_{ij} V_{ij} \frac{\partial f}{\partial X_i}\frac{\partial f}{\partial X_j}. \tag{12}$$

TABLE II. Best fit for g and m when e_μ/e_e is taken as unity. Error matrix is expressed in (parts per million)^2.

$$g_\mu/g_e=0.999983\pm0.000022,$$
$$m_\mu/m_e=206.7638\pm0.0054,$$
$$V_{ij}=\begin{pmatrix} 468 & 457 \\ 457 & 687 \end{pmatrix}.$$

The weights of the input data are taken to be reciprocal of the square of the assigned experimental error, expressed as a fraction.

$$W_1=2.04\times10^9,\quad W_2=4.05\times10^9,\quad W_3=0.10\times10^9. \tag{13}$$

With three free parameters, an exact solution for g, m, and e can be obtained by algebraic manipulation of Eqs. (1)–(3). Table I lists these values together with the error matrix.

If e is set equal to unity, the normal equations are obtained by deleting the terms in X_3 of Eqs. (9) and (10), and omitting Eq. (11) altogether. Table II gives the best-fit values of g and m in this case, together with the error matrix.

The importance of the values of these constants is far-ranging. The value of g can be compared directly with theoretical prediction, as indicated above. The apparent equality of muon and electron charge can be used (assuming conservation of charge in muon decay) to set an upper limit to the electric charge of the neutrino that is coupled with muons (if that neutrino is distinct from the beta-decay neutrino). The mass of the muon is at present the cornerstone for the mass values of all the unstable particles. The most accurate value for the pion mass is obtained[8] from the muon mass and the "Q value" of the decay $\pi\rightarrow\mu+\nu$.[9] The present knowledge of the π-μ mass difference ($\Delta m=33.93\pm0.05$ Mev) limits the accuracy of the pion mass, which becomes, using Table II:

$$m_\pi=139.58\pm0.05 \text{ Mev}.$$

The masses of all the strange particles include the mass of one or more pions in their total.

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