energy behavior of the $\pi^-+p\rightarrow\pi^-+n^++n$ cross section. The contribution from single partial waves indicates strong absorption in the $T=\frac{1}{2}$, $P_{\frac{1}{2}}$, $S_{\frac{1}{2}}$, and $D_{\frac{3}{2}}$ states. Ball and Frazer[11] have discussed in more detail the relationship, conjectured in I, between rapidly rising, strong absorption and elastic resonances, which relates these results to the second $\pi-N$ resonance.[12] The experimental situation on inelastic production in single partial waves is quite meager,[13] but there appears to be at least qualitative agreement with these results.

We conclude by remarking that the other results mentioned above[14] all seem to indicate the same qualitative behavior of low-energy $\pi-\pi$ scattering, while the analysis of the τ decay[15] does not seem to be consistent with these results. Since the other analyses of the $T=0$ scattering length indicate $a_0>0$, this should cast some doubt on the validity of the assumptions made in the above analysis of τ decay.

[11] J. S. Ball and W. R. Frazer, Phys. Rev. Letters **7**, 204 (1961).

[12] See the remarks of G. F. Chew, Revs. Modern Phys. **33**, 366 (1961).

[13] W. D. Walker, J. Davis, and W. D. Shephard, Phys. Rev. **118**, 1612 (1961); B. J. Moyer, Revs. Modern Phys. **33**, 367 (1961), particularly Table I.

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Relativistic Lee Model

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A relativistic version of the well-known exactly soluble Lee model of quantum field theory is constructed, which—unlike the original nonrelativistic Lee model—fulfills also the condition of microcausality (field operators commuting or anticommuting for space-like distances). Microcausality is, however, found compatible with solubility only if an indefinite metric is introduced on a much larger scale than was necessary in the nonrelativistic case. Notwithstanding the fact that a formalism of this type would seem to bear still less resemblance to physical reality than does the original Lee model, we arrive at a theory almost identical with the usual version of relativistic charged scalar (or pseudoscalar) meson field theory. From a certain point of view these two formalisms can even be considered identical. In fact, the difference between the realistic theory and the present Lee model hinges merely on whether one uses, for the representation of the Dirac standard ket $|0\rangle$ a physical vacuum or a suitably defined "completely empty space." The form of the presentation follows closely that of the well-known Pauli-Källén description of the nonrelativistic Lee model, up to the point where it is shown that some of the main results can also be obtained with help of a corresponding Chew-Low equation.

I. INTRODUCTION

It goes almost without saying that when exactly soluble models of quantum field theory are constructed, some of the necessary characteristics of a realistic theory must be sacrificed. Such models may nevertheless be useful for the purpose of detailed study of some mathematical properties, equally true in realistic theories, but whose clear meaning may somehow be lost due to the complexity of the problem. Common properties of this type may eventually be of great help in developing new approximation methods to be used even if a solution in a closed form does not exist. Also they may be useful in clarifying some obscure points concerning the already existing calculational techniques, such as for example uniqueness of various types of one-meson approximations, etc. It seems certain however that the physical importance of such generally valid properties is bound to depend strongly on the physical importance of those necessary characteristics of a realistic theory that we have managed to retain in the model. Since we seem to be unable to retain all of them, it must remain a question of "intuition" and also of taste which of them to regard as the most important.

It is the aim of this paper to explore one such possibility, where relativistic covariance and microcausality (in the usual sense of the field operators commuting or anticommuting for space-like distances) are considered the physical characteristics that are most worthwhile retaining, but where we shall be ready to resort to an indefinite metric for the sake of practical solubility.

We shall first of all proceed to show that we are indeed forced to sacrifice the definiteness of the metric of the Hilbert space, but that such a step is then already sufficient for our purpose.

We first recall that the formal reason for the solubility of the original Lee model is that the mathematical structure of its interaction Hamiltonian does not permit the number of particles to increase indefinitely, and thus enables one to have only a limited number of

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amplitudes to deal with within a given "sector." The mathematical condition that must be imposed on the usual type of "triple product" interaction Hamiltonian, to enforce this, is simply that it should not contain, in the same sum, terms of the form $a_i^* a_j a_k$ together with $a_i^* a_j a_k^*$ (where i, j, k stand for any permutation of the three different fields N, V, and Θ.)

Now, within the framework of a nonrelativistic theory it is possible to describe the "heavy" fermions N and V by field operators $\psi_{N,V}$ ($\psi_{N,V}^*$) containing the annihilation (creation) operators only. The Θ-meson field φ can than be split into φ^+, containing only the creation operators, and φ^-, containing only the annihilation operators, and the above "solubility condition" imposed on the interaction Hamiltonian is fulfilled by discarding φ^+ in the product $\psi_N^* \psi_V \varphi$ and φ^- in its Hermitian conjugate, $\psi_V^* \psi_N \varphi$. What is done with the meson field in the nonrelativistic Lee model is therefore to some extent in line with representing $\psi_{N,V}$ of the "heavy" particles by annihilation operators only, but since at least the Θ meson must be considered a "light" particle, it brings about a nonlocality of the order of m_θ^{-1}, where m_θ is the mass of Θ. Formally this follows from the fact that φ^+ does not commute with φ^- even for space-like distances and the same is true for the interaction densities.

In view of all this, it seems that the only way to make an exactly soluble theory covariant and microcausal is to impose the following conditions on the field operators:

(i) φ^- and $\psi_{N,V}$ should commute or anticommute with φ^+ and $\psi_{N,V}^*$ for (nonzero) space-like distances. (Microcausality.)

(ii) All field operators should contain either only the creation or only the annihilation operators in their free-field expansions (the same in the Schrödinger representation). (Solubility.)

(iii) The above free-field expansions of all the field operators must contain terms corresponding to all possible plane waves, and both positive and negative frequencies.

As far as it is independent of (i), the condition (iii) is a consequence of relativity only. In a truly relativistic theory all masses should be finite and the operators φ^- and $\psi_{N,V}$ should obey—as free fields—the Klein-Gordon and the Dirac equation, respectively. The only possibility of satisfying (i) is then to represent all possible solutions of these equations in the formal expansions of the field operators, i.e., to use both positive and negative frequencies.

It is immediately seen that the conditions (ii) and (iii) are the hardest to reconcile, and that it is only possible to do so as follows:

(a) Fermion Fields $\psi_{N,V}$ ($\psi_{N,V}^*$)

The usual plane-wave expansion of these fields in the interaction representation is

$$\psi_{N,V} = [\Omega]^{-\frac{1}{2}} [m_{N,V}]^{\frac{1}{2}} \sum_{\mathbf{p},i} [\omega_{N,V}(\mathbf{p})]^{-\frac{1}{2}} \times [a_{N,V}(i,+1,\mathbf{p}) u_{N,V}(i,+1,\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + a_{N,V}^*(i,-1,\mathbf{p}) u_{N,V}(i,-1,\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}], \quad (1a)$$

where the ± 1 in the arguments of a's and u's refers to the sign of energy and i is a spin variable. With the help of (1a) the microcausality condition (i) can then be considered a consequence of the following anticommutation relations:

$$\{a_{N,V}(i,\epsilon,\mathbf{k}), a_{N,V}^*(i',\epsilon',\mathbf{k}')\} = \delta_{\mathbf{k}\mathbf{k}'} \delta_{ii'} \delta_{\epsilon\epsilon'}. \quad (2a)$$

Since the anticommutation relations (2a) remain unchanged when the roles played by creation and annihilation operators are inverted, for any number of different sets of indices i, ϵ, $\mathbf{k}$, the condition (ii) is fulfilled without invalidating (i) if we perform the substitution

$$a_{N,V}(i,-1,\mathbf{k}) \rightleftarrows a_{N,V}^*(i,-1,\mathbf{k}),$$
$$a_{N,V}(i,+1,\mathbf{k}) \rightleftarrows a_{N,V}(i,+1,\mathbf{k}),$$

whereby the creation operators for the negative frequencies become annihilation operators and vice versa. Instead of (1a) we get, on the other hand,

$$\psi_{N,V} = [\Omega]^{-\frac{1}{2}} [m_{N,V}]^{\frac{1}{2}} \sum_{\mathbf{p},i} [\omega_{N,V}(\mathbf{p})]^{-\frac{1}{2}} \times u_{N,V}(i,\epsilon,\mathbf{p}) a_{N,V}(i,\epsilon,\mathbf{p}) e^{i\epsilon\mathbf{p}\cdot\mathbf{x}}. \quad (1'a)$$

The above substitution is equivalent, as is well known, to using a form of the theory corresponding to the "prehole" stage of quantum electrodynamics, or to using a "completely empty space" instead of the "Dirac vacuum" as a "standard ket." It means, of course, a serious departure from physical reality, but does not necessitate introducing the indefinite metric as will be the case with the boson field. For reasons similar to those which led to the change of metric in the nonrelativistic Lee model, i.e., for the sake of greater generality (especially in view of the renormalization procedure), we will sometimes introduce a negative metric for the bare V particle. We will therefore use, in the sequel, more general anticommutation relations than (2a), namely:

$$\begin{aligned} \{a_N(i,\epsilon,\mathbf{k}), a_N^*(i',\epsilon',\mathbf{k}')\} &= \delta_{\mathbf{k}\mathbf{k}'} \delta_{ii'} \delta_{\epsilon\epsilon'}, \\ \{a_V(i,\epsilon,\mathbf{k}), a_V^*(i',\epsilon',\mathbf{k}')\} &= \epsilon_1 \delta_{\mathbf{k}\mathbf{k}'} \delta_{ii'} \delta_{\epsilon\epsilon'}. \end{aligned} \quad (2'a)$$

(b) Boson Field φ

In the case of the field φ^- describing the Θ mesons, we have first to fulfill (iii) by supplying the necessary negative frequencies in φ^- and only then try to fulfill (ii) in some possibly analogous way to what was done with the fermion fields. We can fulfill (iii) either by restoring the original (Majorana type) φ of the neutral theory or by introducing antiparticles Θ^- different from the original Θ's, thus arriving at a charged theory. In

case of the neutral theory φ becomes, however, identical with its Hermitian conjugate and it is then impossible to satisfy (ii). We are therefore left with the charged theory and we have

$$\varphi=[2\Omega]^{-\frac{1}{2}}\sum_{\mathbf{k}}[\omega_\theta(\mathbf{k})]^{-\frac{1}{2}} \times\{a_\theta(+1,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+a_\theta^*(-1,\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\}. \quad (1\text{b})$$

(φ^- and φ^+ will henceforth be called φ and φ^*, respectively.) The microcausality condition (i) is fulfilled if

$$[a_\theta(\epsilon,\mathbf{k}),a_\theta^*(\epsilon',\mathbf{k}')]=\delta_{\mathbf{k}\mathbf{k}'}\delta_{\epsilon\epsilon'}. \quad (2\text{b})$$

We are now in the position to try to satisfy (ii). To have in (1b) the annihilation operators only we must, in analogy to what we have already done with the fermion fields, perform the substitutions

$$a_\theta(+1,\mathbf{k})\rightarrow a_\theta(+1,\mathbf{k}),\quad a_\theta^*(+1,\mathbf{k})\rightarrow a_\theta^*(+1,\mathbf{k}),$$
$$a_\theta(-1,\mathbf{k})\rightarrow a_\theta^*(-1,\mathbf{k}),\quad a_\theta^*(-1,\mathbf{k})\rightarrow a_\theta(-1,\mathbf{k}),$$

Because, however [contrary to (2a)], the commutators change sign if the order of terms is inverted, the above substitutions lead to

$$\varphi=[2\Omega]^{-\frac{1}{2}}\sum_{\mathbf{k},\epsilon}[\omega_\theta(\mathbf{k})]^{-\frac{1}{2}}a_\theta(\epsilon,\mathbf{k})e^{i\epsilon\mathbf{k}\cdot\mathbf{x}}, \quad (1'\text{b})$$

and to

$$[a_\theta(\epsilon,\mathbf{k}),a_\theta^*(\epsilon',\mathbf{k}')]=\epsilon\delta_{\mathbf{k}\mathbf{k}'}\delta_{\epsilon\epsilon'}. \quad (2'\text{b})$$

For the negative frequencies this leads to the well-known Gupta[1] representation involving an indefinite metric, and is precisely the result which we mentioned at the beginning.

We shall presently see that working in the basis of eigenstates belonging to the negative eigenvalues of $a_\theta^*(-1,\mathbf{k})a_\theta(-1,\mathbf{k})$ of reference 1, is equivalent to introducing negative-energy states for the Θ mesons, which will be the closest possible analogs of the "negative-energy fermions" of the "pre-hole theory." They will indeed share most of the characteristic properties of these "negative-energy fermions" except that they will also be "ghosts." The standard ket $|0\rangle$ of Gupta bears a similar relation to the standard ket of the charged meson theory as does our completely empty space to the Dirac physical vacuum. From this formal point of view the results of (a) and (b) are very much in line with each other and can be expressed by saying that the relativistic charged theory is soluble if "all vacuum effects" (fermion as well as meson) are neglected.

II. HAMILTONIAN OF THE PROBLEM AND THE EQUATIONS OF MOTION

It is hardly necessary to write down the complete Hamiltonian and commutation rules of our formalism, which is simply that of the relativistic charged theory with the roles of all the negative-energy creation and annihilation operators exchanged. To make the disucssion general enough, however, and anticipating—as in the case of a nonrelativistic Lee model—a need of sometimes introducing indefinite metric for bare V particles, it is sufficient to say that the corresponding part of the zero Hamiltonian should be multiplied by ϵ_1,[2] and that the commutation relation for ψ_V for space-like separations should in general be

$$\{\psi_V(\mathbf{x},0),\psi_V^*(\mathbf{x}',0)\}=\epsilon_1\delta(\mathbf{x}-\mathbf{x}'). \quad (2.1)$$

The interaction Hamiltonian is given by

$$H_I=g_0\int d^3x\{\bar{\psi}_VQ_0\psi_N\varphi+\bar{\psi}_NQ_0\psi_V\varphi^*\}$$
$$=\sum_{\mathbf{k},\epsilon}\{J(\epsilon,\epsilon\mathbf{k})a_\theta(\epsilon,\epsilon\mathbf{k})+J^*(\epsilon,\epsilon\mathbf{k})a_\theta^*(\epsilon,\epsilon\mathbf{k})\}, \quad (2.2)$$

where

$$Q_0=\tfrac{1}{2}(\epsilon_0+1)-\tfrac{1}{2}(\epsilon_0-1)\gamma_5 \quad \text{with} \quad \gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3, \quad (23)$$

which also defines the important operator $J(\epsilon,\mathbf{k})$. Going over to the Heisenberg representation and performing exactly the same formal manipulations as in the case of ordinary charged meson theory (as far as formal manipulations with commutators are concerned, they are completely independent of whether some operators are renamed or not), we obtain the following "local" equations of motion:

$$(\partial+m_V)\psi_V=-g_0\epsilon_1Q_0\psi_N\varphi, \quad (2.4)$$

$$(\partial+m_N)\psi_N=-g_0Q_0\psi_V\varphi^*, \quad (2.5)$$

$$(\Box-m_\theta^2)\varphi=g_0\bar{\psi}_NQ_0\psi_V, \quad (2.6)$$

where

$$\partial=\gamma_0\partial/\partial x_0+\boldsymbol{\gamma}\cdot\nabla. \quad (2.7)$$

To obtain solutions for the first two sectors of our model we will follow closely the methods applied by Pauli and Källén[3] to the nonrelativistic Lee model. Apart from the slight complication caused by the spin of N and V this is rather a straightforward matter so that we shall skip many unnecessary repetitions and concentrate our attention mainly on the covariant aspects of the results.

[1] See, e.g., J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), pp. 103–110.

[2] In addition to those ϵ's (with primed superscripts) introduced to distinguish between positive and negative energies four additional special ϵ's will be used as explained by the following:

$\epsilon_0\begin{cases}=+1 \text{ in case of original scalar coupling,}\\=-1 \text{ in case of original pseudoscalar coupling;}\end{cases}$

$\epsilon_1\begin{cases}=+1 \text{ for positive definite metric of "bare" } V,\\=-1 \text{ for indefinite metric of "bare" } V;\end{cases}$

$\epsilon_2\begin{cases}=+1 \text{ if the "relative" parity of } N \text{ to } V \text{ is not changed in the process of "dressing" of the latter,}\\=-1 \text{ in case the above "relative" parity does change;}\end{cases}$

$\epsilon(\mathbf{V})\begin{cases}=+1 \text{ in case the corresponding } |\mathbf{V}\rangle \text{ has positive norm,}\\=-1 \text{ if } |\mathbf{V}\rangle \text{ is a "ghost" state.}\end{cases}$

[3] W. Pauli and G. Källén, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 12, (1953).

III. THE ($V\rightleftarrows N+\Theta$) SECTOR

We shall start with the mass eigenvalue problem of the dressed **V** particle and then consider the N-Θ scattering.

(a) Bound-State Problems of the V Particle

The properly normalized [i.e., with the norm equated to $\epsilon(\mathbf{V})$] exact solution of the four-momentum eigenvalue problem is given by the formulas:

$$\langle V(\mathbf{p})|\mathbf{V}(\mathbf{p})\rangle = -Z^{-\frac{1}{2}}[m_V \mathbf{m}_V]^{\frac{1}{2}} \times[\omega_V(\mathbf{p})\omega_V(\mathbf{p})]^{-\frac{1}{2}}u_V(\mathbf{p})\beta\mathbf{u}_V(\mathbf{p}), \quad (3.1)$$

$$\langle N(\mathbf{p}-\mathbf{k}),\theta(\mathbf{k})|\mathbf{V}(\mathbf{p})\rangle = g_0\epsilon Z^{-\frac{1}{2}}[\mathbf{m}_V m_N]^{\frac{1}{2}}[2\Omega]^{-\frac{1}{2}}[\omega_N(\mathbf{p}-\mathbf{k})\omega_\theta(\mathbf{k})\omega_V(\mathbf{p})]^{-\frac{1}{2}} \times\{i(p-k)+m_N\}^{-1}u_N(\mathbf{p}-\mathbf{k})\beta Q_0\mathbf{u}_V(\mathbf{p}), \quad (3.2)$$

where ϵ is $+1$ or -1 according to whether Θ has positive or negative energy, u's are spinors normalized according to $\bar{u}u=1$ and

$$p=\mp\omega_V(\mathbf{p})\gamma_0+\mathbf{p}\cdot\gamma;\quad k=\mp\omega_\theta(\mathbf{k})\gamma_0+\mathbf{k}\cdot\gamma,$$

all other amplitudes being identically zero as they should in a Lee model.

The renormalization factor Z is

$$Z=|(d/d\mathbf{m}_V)[\mathbf{S}_V(i\epsilon_2\mathbf{m}_V)]^{-1}|. \quad (3.3)$$

The mass eigenvalues of possible dressed **V**'s (there are in general several such eigenvalues possible, starting with a single m_V) are obtained as real positive solutions of the algebraic equation

$$[\mathbf{S}_V(i\epsilon_2\mathbf{m}_V)]^{-1}=0, \quad (3.4)$$

where $\mathbf{S}_V(p)$ denotes the exact physical propagator of **V**. If ϵ_2 were not already included in (3.4) we would have to deal with negative mass eigenvalues for $\epsilon_2=-1$ but then the well-known transformation

$$u_m\rightarrow\gamma_5 u_{-m}, \quad (3.5)$$

would restore the positive sign of a mass in an appropriate equation for u_m. As (3.5) changes the relative parity of **V** as compared to V, the appearance of such negative eigenvalues would mean that the change of parity is enforced by the process of "dressing." The "physical" propagator $\mathbf{S}_V(p)$ can be given here in a closed form[4]:

$$[\mathbf{S}_V(p)]^{-1}=m-m_V+(g_0/4\pi)^2\epsilon_1 \times Q_0 m(m+m_N+m_\theta)(m+m_N-m_\theta)m^{-3}Q_0 \times\{k_{\max}'-(\pi/4m)[-\Delta(m)]^{\frac{1}{2}}\}, \quad (3.6)$$

where the "mass operator"

$$m=-ip \quad (3.7)$$

is introduced, rather than p, because its eigenvalues are $\epsilon_2\mathbf{m}_V$, so that the explicit form of the algebraic equation (3.4) may be read off directly from (3.6). $k_{\max}'$ is a covariant cutoff parameter (maximal momentum of either of the virtual particles forming a "bubble" in the rest frame of **V**). m is given by

$$m=[(m)^2]^{\frac{1}{2}}=[-p^2]^{\frac{1}{2}}, \quad (3.8)$$

and has to be replaced by $\mathbf{m}_V$ in the eigenvalue equation. Finally

$$\Delta(m)=(m+m_N+m_\theta)(m+m_N-m_\theta) \times(m-m_N+m_\theta)(m-m_N-m_\theta), \quad (3.9)$$

and causes the appearance of four square-root type branch points $-m_N-m_\theta$, $-m_N+m_\theta$, m_N-m_θ, m_N+m_θ in $\mathbf{S}_V(im)$ considered as an analytic function of m and consequently, as we shall presently see, also in the N-Θ scattering amplitude.[5]

(b) Renormalization

No infinite mass-renormalization is necessary in our model.[6] It follows easily from the fact that any stable (real) eigenvalues $\mathbf{m}_V$ must fall outside scattering branch cuts, and therefore must lie within the regions $(-m_N-m_\theta, -m_N+m_\theta)$ and $(m_N-m_\theta, m_N+m_\theta)$. The real reason that such eigenvalues do exist, even for $k_{\max}'\rightarrow\infty$ is, however, a nontrivial consequence of Θ ghosts.

Infinite coupling constant renormalization is, however, necessary. As usual, we define the renormalized coupling constant by

$$g_0\rightarrow g=[Z(\mathbf{m}_V)]^{-\frac{1}{2}}g_0. \quad (3.10)$$

As there is in general more than one stable **V** isobar possible, we have a free choice which one to use in the argument of Z. The most convenient choice, which we shall follow here, is to use as the "standard" **V** one which is a "normal" particle and not a "ghost."

According to (3.4) we then have

$$g^{-2}-\epsilon_1 g_0^{-2}=(\partial/\partial\mathbf{m}_V)B(\mathbf{m}_V,k_{\max}'), \quad (3.11)$$

where

$$B(\mathbf{m}_V,k_{\max}') =\epsilon_0(16\pi^2)^{-1}(\mathbf{m}_V)^{-2}(\epsilon_0\epsilon_2\mathbf{m}_V+m_N+m_\theta) \times(\epsilon_0\epsilon_2\mathbf{m}_V+m_N-m_\theta) \times\{k_{\max}'-(\pi/4\mathbf{m}_V)[-\Delta(\mathbf{m}_V)]^{\frac{1}{2}}\}. \quad (3.12)$$

[4] It actually follows from $[\mathbf{S}_V(p)]^{-1}=[S_V(p)]^{-1}+\Sigma^*(p)$, with $\Sigma^*(p)=(2\pi)^{-4}i\epsilon_1 g_0^2 Q_0\int\Delta_\theta^{\mathrm{Ret}}(k)S_N^{\mathrm{Ret}}(p-k)d^4k\, Q_0$, as the only proper self-energy graph possible in our model. "Causal" functions replace "retarded" in a corresponding expression in a realistic theory.

[5] It is perhaps worth mentioning at this point that the "bubble" graph based on "causal" rather than "retarded" functions in the formula for Σ^* of reference 4 would lead to logarithmic branch points at $-m_N-m_\theta$ and m_N+m_θ, but without a branch cut extending from $-m_N+m_\theta$ to m_N-m_θ (in the physical sheet). This is obviously connected with the fact that the "realistic" theory would not allow for scattering between particles of positive and (nonexistent) negative energies. The closest analog of this cut in the realistic theory is furnished by the cut which appears in "crossed" scattering amplitudes, in the usage of Mandelstam relations.

[6] Relativistic models of this type have recently been studied, by means of dispersion techniques, by F. Zachariasen, Phys. Rev. **121**, 1851 (1961).

(3.11) has exactly the same form as the corresponding formula of Pauli and Källén and only the possibility of changing the metric of "bare" V and having $\epsilon_1=-1$ enables g_0 to remain real for finite g and for $k_{\max}'\to\infty$.

We can now solve (3.11) with respect to g_0 and, consequently, express all previous equations in terms of the renormalized coupling constant g. If this is done in (3.4), we obtain the following formula for the finite "mass correction" of the "standard" $\mathbf{V}$:

$$\begin{aligned}\delta m_V &= m_V-\epsilon_2\mathbf{m}_V\\ &= g^2B(\mathbf{m}_V,k_{\max}')\\ &\qquad\times[g^2(\partial/\partial\mathbf{m}_V)B(\mathbf{m}_V,k_{\max}')-1]^{-1}. \quad (3.13)\end{aligned}$$

(c) N-Θ Scattering and Chew-Low Approach

N-Θ scattering states are solutions of the four-momentum eigenvalue problem belonging to the continuous spectrum of m lying on the branch cuts of the analytic function $\mathbf{S}_V$, extending from $-\infty$ to $-m_N-m_\theta$, from $-m_N+m_\theta$ to m_N-m_θ, and from m_N+m_θ to $+\infty$, along the real axis and also from $-i|\mathbf{p}|$ to $+i|\mathbf{p}|$ along the imaginary axis.[7] The appropriate normalization for these solutions is a δ function in the three-momenta of the incoming (outgoing) particles and they are furthermore labelled "out" ("in") to denote that no incoming (outgoing) scattered waves are present. Solutions so specified can be explicitly given by formulas similar to (3.1–2). With their help we have

$$\begin{aligned}&\langle(N(\mathbf{p}-\mathbf{k}'),\Theta(\mathbf{k}'))\text{ in}|J(\mathbf{k})|N(\mathbf{p}-\mathbf{k})\rangle\\ &\quad= g_0^2m_N\epsilon(2\Omega)^{-1}[\omega_N(\mathbf{p}-\mathbf{k})\omega_\theta(\mathbf{k})\omega_N(\mathbf{p}-\mathbf{k}')\omega_\theta(\mathbf{k}')]^{-\frac{1}{2}}\\ &\qquad\times\bar{u}_N(\mathbf{p}-\mathbf{k}')Q_0\mathbf{S}_V(p_{\text{out}})Q_0u_N(\mathbf{p}-\mathbf{k}), \quad (3.14)\end{aligned}$$

where

$$p_{\text{out}}\equiv(\epsilon'\omega_\theta(\mathbf{k}')+\epsilon\omega_N(\mathbf{p}-\mathbf{k}')+i\eta,\ \mathbf{p}), \quad (3.15)$$

ϵ' and ϵ denoting the sign of energy of Θ and N, respectively, and where $J(\mathbf{k})$ are defined by (3.2). As is well known,[8] the knowledge of matrix elements (3.14) is sufficient to calculate the S matrix, which then is given by

$$\begin{aligned}&\langle N(\mathbf{p}'-\mathbf{k}'),\Theta(\mathbf{k}')|S|N(\mathbf{p}-\mathbf{k}),\Theta(\mathbf{k})\rangle\\ &\quad=\epsilon'\delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{k}-\mathbf{k}')+i\delta^4(p-p')\\ &\qquad\times[\omega_N(\mathbf{p}-\mathbf{k})\omega_\theta(\mathbf{k})\omega_N(\mathbf{p}'-\mathbf{k}')\omega_\theta(\mathbf{k}')]^{-\frac{1}{2}}\\ &\qquad\times\bar{u}_N(\mathbf{p}'-\mathbf{k}')Tu_N(\mathbf{p}-\mathbf{k}), \quad (3.16)\end{aligned}$$

where passage has already been made to continuous $\mathbf{k}$ and where the invariant scattering amplitude is given by

$$T=-(8\pi)^{-1}\epsilon'\epsilon g_0^2m_NQ_0\mathbf{S}_V(p_{\text{out}})Q_0. \quad (3.17)$$

(3.17) shows once more how basic is the role played by the "physical" propagator $\mathbf{S}_V$ in our model. This equation may bear some resemblance to a realistic case as it is at least a completely covariant formula describing scattering of relativistic particles. Useful as (3.17) may be, as a hint concerning the form of T in a "realistic" theory, it fails to reproduce the dependence on the other variable a scattering amplitude should involve, namely the momentum transfer, the formal reason for this being the already mentioned lack of "crossed reactions."

One can easily prove by an explicit calculation that the S matrix given by (3.16–17) is pseudo-unitary, as it should be, because our Hamiltonian is a self-adjoined operator. Whenever, on grounds of energy conservation, a transition to a ghost state is impossible, the S matrix within the corresponding energy interval is simply unitary. It follows therefore that it is actually unitary along the right-hand cut, extending from m_N+m_θ to $+\infty$, because a collision of two particles of positive energy can never lead to a state of two particles of the same masses with the sign of energy of even one of the particles reversed.

The appearance of $\mathbf{S}_V$ in all the most important formulas here can now be used as a means for investigating the uniqueness of results obtained by applying some recently developed general methods which avoid the use of the explicit form of the interaction, such as the "Chew-Low method."

For purpose of applying the latter we now propose to consider (3.14), not as a result of an explicit calculation based on our model, but as an "ansatz" to determine $\mathbf{S}_V$, together with a similarly constructed "ansatz" for the matrix elements of J involving $\mathbf{V}$ states:

$$\begin{aligned}&\langle\mathbf{V}(\mathbf{p})|J(\mathbf{k})|N(\mathbf{p}-\mathbf{k})\rangle\\ &\quad=g_0(2\Omega)^{-\frac{1}{2}}[Z(\mathbf{m}_V)]^{-\frac{1}{2}}[m_N\mathbf{m}_V]^{\frac{1}{2}}\epsilon_1\\ &\qquad\times[\omega_\theta(\mathbf{k})\omega_N(\mathbf{p}-\mathbf{k})\boldsymbol{\omega}_V(\mathbf{p})]^{-\frac{1}{2}}\\ &\qquad\times\mathbf{u}_V(\mathbf{p})Q_0u_N(\mathbf{p}-\mathbf{k}). \quad (3.18)\end{aligned}$$

If a Chew-Low equation[9] is now set up for our model (which is here of course identical with its one-meson approximation) and if (3.14) and (3.18) are used, lengthy straightforward calculation leads to the following equation for $\mathbf{S}_V$:

$$\begin{aligned}\mathbf{S}_V(im)=&-\sum_{\mathbf{V}}\epsilon(\mathbf{V})[Z(\mathbf{m}_V)]^{-1}[\epsilon_2\mathbf{m}_V-m]^{-1}\\ &-g_0^2(4\pi)^{-4}\int_L dm'(m-m')^{-1}[4(m')^2]^{-1}\\ &\times[\Delta(m')]^{\frac{1}{2}}\{[m^2+\mathbf{p}^2]^{\frac{1}{2}}[(m')^2+\mathbf{p}^2]^{-\frac{1}{2}}C[(m')^2]\\ &+[(m')^2+\mathbf{p}^2]^{\frac{1}{2}}D[(m')^2]\}, \quad (3.19)\end{aligned}$$

where the complex integration path L is given by

[7] Actually if $\mathbf{S}_V(im)$ and not simply $S_V(im)$ is considered as an analytic function of m, the latter two branch points also appear on account of the square-root type dependence of $\boldsymbol{\omega}_m(\mathbf{p})$ on m. The corresponding imaginary branch cut is of course a peculiarity of an unrealistic scattering of two particles with opposite signs of energy and corresponds to the case when $k^2>[\omega_N(\mathbf{p}-\mathbf{k})-\omega_\theta(\mathbf{k})]^2$, i.e., when the rest mass of the system comprising these particles is imaginary.

[8] Compare, e.g., G. C. Wick, Revs. Modern Phys. 27, 339 (1955).

[9] What is actually meant is an equation of the type given, e.g., in J. Hamilton, *The Theory of Elementary Particles* (Clarendon Press, Oxford, 1959), Eq. (103), p. 331.

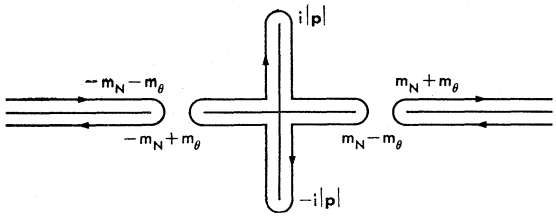

FIG. 1. The integration path L in the complex m' plane of Eq. (3.19).

Fig. 1, whereas $C(m^2)$ and $D(m^2)$ are defined from the identity

$$C(m^2)+[m^2+\mathbf{p}^2]^{\frac{1}{2}}D(m^2) \equiv \mathbf{S}_V(im)Q_0m(m+m_N+m_\theta) \times(m+m_N-m_\theta)Q_0[\mathbf{S}_V(im)]_-, \quad (3.20)$$

where $[S_V(im)]_-$ is the value of $\mathbf{S}_V(im)$ at the opposite side of the cut. $[\Delta(m)]^{\frac{1}{2}}$ and $[m^2+\mathbf{p}^2]^{\frac{1}{2}}$ are both assumed positive above the right-hand cut. In a way almost identical with that used by Chew and Low to obtain the solution for partial wave amplitudes[10] it now follows from (3.19) that the discontinuity of $[\mathbf{S}_V(im)]^{-1}$ across the cut is actually that given by (3.6), whereas the positions and residues of its poles are given by $\mathbf{m}_V$ and $Z(\mathbf{m}_V)$, respectively, thus proving (3.3). It must however be remembered that the latter are arbitrary parameters in (3.19) and, although the solution for $\mathbf{S}_V$ is uniquely determined by these parameters, it might differ from (3.8). This sort of ambiguity is essentially the same as that pointed out for the first time by Castillejo, Dalitz and Dyson[11] for the static Chew-Low model.

IV. SECOND SECTOR

So far the author has not succeeded in extending the results concerning the second sector of the present model much beyond those already discussed by Pauli and Källén in the case of the original nonrelativistic Lee model, although explicit solutions for the latter have recently been given by Amado.[12] As in the nonrelativistic case it is possible to express explicitly the $N+2\Theta$ amplitudes by the $V+\Theta$ amplitudes and therefore to write down a closed-form exact equation for the latter, which is essentially the Bethe-Salpeter equation based on the ladder diagram given by Fig. 2. (This diagram is the only possible for a Lee model.) Starting with this equation, it is furthermore possible to demonstrate that the scattering matrix R obeys the equation

$$R+R^*=RR^*. \quad (4.1)$$

As in the nonrelativistic case, it then follows that there is no energy range where the corresponding S matrix may be purely unitary, because, contrary to the result for the first sector expressed by (3.16–17), even if two "realistic" particles $\mathbf{V}$ and Θ collide there is always a possibility of creating an N and two Θ's, one of which might have negative energy and hence provide a transition to a ghost state.

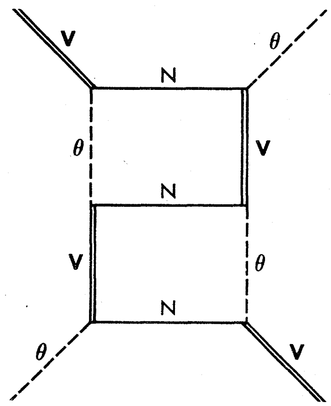

FIG. 2. Typical ladder diagram corresponding to the second sector.

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[10] G. F. Chew and F. Low, Phys. Rev. **101**, 1570 (1956).
[11] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).
[12] R. Amado, Phys. Rev. **122**, 696 (1961).