

Spectral Moments and Continuum Perturbation Theory

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The p th spectral moment is shown to be given exactly by the perturbation expansion of the resolvent operator carried to order p . Any individual term of order n in the perturbation expansion makes an identically zero contribution to the p th moment for n greater than p . The above is true even though perturbation theory is nonconvergent. The result may be useful in cases where perturbation theory converges for high energy but not for low energy. An application to the problem of band structure "tails" in impure crystals is suggested.

I. INTRODUCTION

THE p th moment, M_p , of the eigenvalue spectrum of a bounded matrix H is defined by

$$M_p = \text{Tr} H^p = \sum_j E_j^p,$$

where Tr means trace and E_j are the eigenvalues of $H = H_0 + \lambda H_1$. These moments have the desirable feature of being invariant under unitary transformation so that they may be calculated in any convenient representation. Thus, information about the eigenvalue spectrum may be obtained without the labor of diagonalization. This property has been used by Waller and more recently by Van Vleck¹ in his study of spectral line shapes and also by Montroll² and collaborators in connection with the density of states of lattice vibrations in disordered crystals.

We have discovered a very interesting connection between the moments of a spectrum and the perturbation expansion for the characteristic function (trace of the resolvent operator), an expansion equivalent to the S -matrix expansion. The result, which we have called the Moment-Perturbation Identity is the following: The p th moment may be calculated exactly from the characteristic function expanded as a power series in λ if all terms of order $\leq p$ are retained. Any term of order $> p$ makes an identically zero contribution to the p th moment.

This result is proved in Sec. II. It is also shown in this section that the power series expansion is very "weak" and must be summed to infinite order if perturbed eigenvalues are to be located. The use of Van Hove's³ summation is equivalent to Brillouin-Wigner perturbation theory.⁴ For a discrete spectrum the familiar expansion of the energy as a power series in λ may be obtained.

In Sec. III we review some of Van Hove's results for the properties of the characteristic function in the

limit of a continuous eigenvalue spectrum. For a discrete spectrum, the characteristic function is single-valued with poles on the real axis. For the continuum case, the characteristic function is multivalued with poles off the real axis. A principal sheet of the multivalued function may be defined by making a branch cut on the real axis extending over all eigenvalues. The discrete function goes continuously into the continuum function on the principal sheet as the number of eigenstates tends to infinity. Consequently, the poles of the continuum function off the real axis do not lie on the principal sheet. This fact is also verified explicitly. The poles off the real axis may be interpreted as decaying states.⁵

In Sec. IV Montroll's method² of obtaining the density of states function from the moments is discussed.

In Sec. V the moment-perturbation identity is used to compute partial moments by perturbation theory. It is assumed that perturbation theory is accurate at high energy. The partial moments are computed for the low-energy part of the spectrum. The high-energy cutoff may then become arbitrarily large. An application to the band structure of impure crystals is briefly indicated.

II. PERTURBATION EXPANSIONS AND THE MOMENT-PERTURBATION IDENTITY

We consider the resolvent operator introduced by Van Hove and Hugenholtz⁶

$$R(z) = (H - z)^{-1}, \quad (1)$$

where H is the total Hamiltonian and z is a general complex variable. The resolvent is essentially the Fourier transform of the propagator

$$U \equiv e^{-iHt} = -\frac{1}{2\pi i} \oint e^{-izt} (H - z)^{-1} dz, \quad (2)$$

where the contour encompasses all the eigenvalues of the H matrix,⁷ as shown in Fig. 1(a). The Fourier

¹ I. Waller, *Z. Physik* **79**, 380 (1932); J. H. Van Vleck, *Phys. Rev.* **74**, 1168 (1948).

² A. A. Maradudin, P. Mazur, E. W. Montroll and G. H. Weiss, *Revs. Modern Phys.* **30**, 175 (1958) (see p. 192). E. W. Montroll, *J. Chem. Phys.* **10**, 218 (1942); **11**, 481 (1943).

³ L. Van Hove, *Physica* **21**, 901 (1955).

⁴ P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), p. 1033. These authors call the method the modified iteration-perturbation method.

⁵ L. Van Hove, *Physica* **21**, 517 (1955).

⁶ N. M. Hugenholtz in *The Many-Body Problem*, edited by C. DeWitt (John Wiley & Sons, Inc., New York, 1958), p. 1; also see reference 3.

⁷ For convenience, we will assume in Sec. II that H has a finite number of eigenvalues.

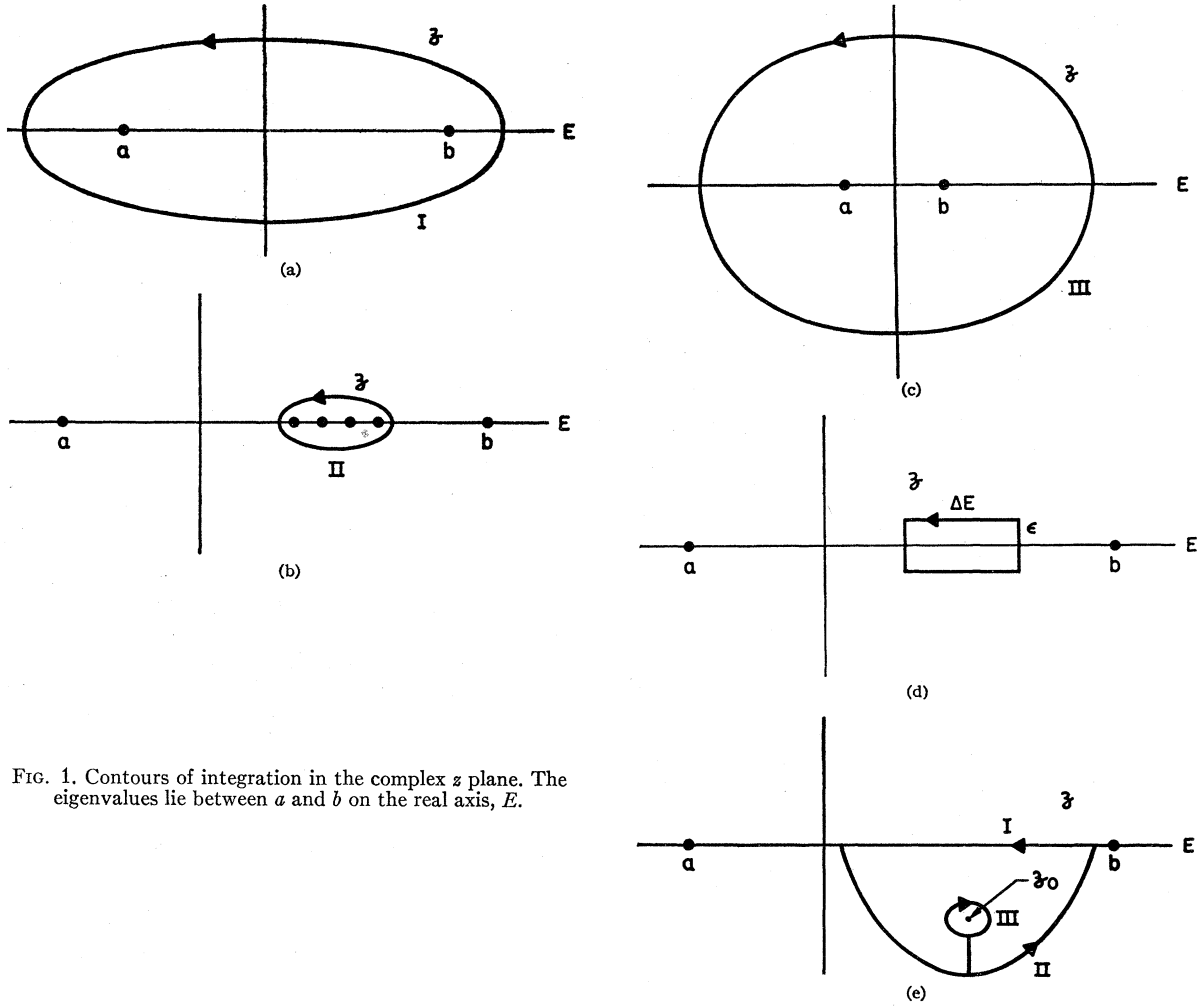


FIG. 1. Contours of integration in the complex z plane. The eigenvalues lie between a and b on the real axis, E .

transform may be inverted

$$(H-z)^{-1} = -i \lim_{t \rightarrow \infty} \int_0^t e^{-i(H-z)t'} dt'. \quad (3)$$

Equation (3) only defines the function for those z for which

$$e^{zt} \rightarrow 0; \quad t \rightarrow \infty. \quad (4)$$

The general function may be defined by analytic continuation. The trace of the resolvent operator may be written

$$\Omega(z) \equiv \text{Tr}(H-z)^{-1} = \sum_i \frac{1}{E_j - z}, \quad (5)$$

since the trace is invariant under unitary transformation. In this form we see that the characteristic function $\Omega(z)$ is an analytic function with simple poles on the real axis at the eigenvalues of the matrix H . The number of states, n , in any domain of the real axis may

be found by a contour integral which encloses that domain. See Fig. 1(b).

$$n = -\frac{1}{2\pi i} \oint_{\text{II}} \Omega(z) dz. \quad (6)$$

The moments of the eigenvalue spectrum are defined by

$$M_p \equiv \text{Tr} H^p = \sum_j E_j^p. \quad (7)$$

Alternatively, the moments may be determined from the characteristic function $\Omega(z)$ using Eq. (5).

$$M_n = -\frac{1}{2\pi i} \oint_I z^n \Omega(z) dz, \quad (8)$$

where the contour encircles all the eigenvalues of H .

Expanding the sum in Eq. (5) in powers of $1/z$, we find

$$\Omega(z) = -\frac{1}{z} \sum_{p=0}^{\infty} \frac{M_p}{z^p}. \quad (9)$$

Hence the analytic function Ω is uniquely defined (using analytic continuation) in terms of its moments. In view of Eq. (5) the radius of convergence of the expansion is determined by

$$z > |E|_{\max}. \quad (10)$$

Hence Eq. (9) cannot be used in Eq. (6) to find the number or density of states.

We now consider the perturbation problem, writing

$$H = H_0 + \lambda H_1, \quad (11)$$

where both H_0 and H_1 are assumed bounded. The resolvent operator may be expanded in powers of λ , following Van Hove,³ as

$$(H - z)^{-1} = (H_0 - z)^{-1} \sum_{n=0}^{\infty} \{-\lambda H_1 (H_0 - z)^{-1}\}^n. \quad (12)$$

We calculate the trace of Eq. (12) using the unperturbed basis functions diagonalizing H_0 . Let ϵ_j be the eigenvalues of H_0 and h_{ij} , the matrix elements of H_1 in the unperturbed basis. Then we obtain

$$\Omega(z) = \sum_{j_1} \frac{1}{\epsilon_1 - z} + \sum_{n \geq 1; j_1, j_2, \dots, j_n} (-\lambda)^n \times \frac{h_{j_1, j_2} h_{j_2, j_3} \dots h_{j_n, j_1}}{(\epsilon_{j_1} - z)^2 (\epsilon_{j_2} - z) \dots (\epsilon_{j_n} - z)}. \quad (13)$$

Subsequently, j_1 will be written simply 1. The ambiguity between n and j_n should not cause confusion. It should be observed that if the number of eigenstates is finite as we are assuming and if the h_{ij} are bounded, the expansion in Eq. (13) obviously converges for sufficiently large z .

Now substituting Eq. (11) in Eq. (7) and expanding, we obtain

$$M_p = \sum_{\substack{n; 1, 2, \dots, n \\ r_1, r_2, \dots, r_{n+1}}} \lambda^n h_{12} h_{23} \dots h_{n1} \epsilon_1^{r_1 + r_{n+1}} \epsilon_2^{r_2} \dots \epsilon_n^{r_n}, \quad (14)$$

where $r_1 + r_2 + \dots + r_{n+1} + n = p$. We have used the invariance of the trace to calculate the moments with the unperturbed basis. We may also calculate the p th moment using the expansion of Eq. (13) in Eq. (8). The result is

$$M_p = -\frac{1}{2\pi i} \sum_{n \geq 1; 1, 2, \dots, n} (-\lambda)^n h_{12} \dots h_{n1} \oint_{\text{III}} z^p \times \frac{1}{(\epsilon_1 - z)^2 (\epsilon_2 - z) \dots (\epsilon_n - z)} dz - \frac{1}{2\pi i} \oint_{\text{III}} \frac{z^p}{\epsilon_1 - z} dz. \quad (15)$$

The contour in Eq. (15) may be taken for very large z [see Fig. 1(c)] where the expansion of Eq. (13) certainly converges. Thus, the expansions of Eqs. (14) and (15) must obviously be identically equal.

It is easily shown that, in fact, the identity between Eqs. (14) and (15) holds for every individual term in the sum. This is true by virtue of the identity

$$\frac{(-1)^{n+1}}{2\pi i} \oint z^p \frac{1}{(\epsilon_1 - z)^2 (\epsilon_2 - z) \dots (\epsilon_n - z)} dz = \sum_{r_1, r_2, \dots, r_{n+1}} \epsilon_1^{r_1 + r_{n+1}} \epsilon_2^{r_2} \dots \epsilon_n^{r_n}, \quad (16)$$

where $r_1 + r_2 + \dots + r_n + r_{n+1} = p - n$. This identity is easily proved by expanding the denominator in the integral in powers of $1/z$ and letting $z \rightarrow \infty$.

We may state the above result in a less general but useful form:

Theorem. The perturbation expansion of the characteristic function [Eq. (13)] carried to order p gives the moments of the spectrum correct to order p . Any partial summation of higher order terms does not spoil the exact result. The last statement follows from the fact that any term of order $n > p$ makes a zero contribution to the p th moment.

It is very fortunate that partial summations do not spoil the moment-perturbation identity because the expansion of Eq. (13) is a very poor representation of the characteristic function. To demonstrate this we use the cyclic symmetry of the numerator in Eq. (13) to write (13) in the form

$$\Omega(z) = \sum_1 \frac{1}{\epsilon_1 - z} + \frac{d}{dz} \sum_{n; 1, 2, \dots, n; n \geq 1} \frac{(-\lambda)^n}{n} \times \frac{h_{12} h_{23} \dots h_{n1}}{(\epsilon_1 - z) (\epsilon_2 - z) \dots (\epsilon_n - z)}. \quad (17)$$

We have averaged over the cyclic permutations of indices 1 through n . If we now attempt to represent Ω by a sum of terms to any finite order n and substitute the result in Eq. (6) we find we get the *unperturbed* density of states. The second term on the right of Eq. (17) gives an identically zero contribution. We may interpret this result as follows: The close connection between the resolvent operator and the U matrix is shown by Eq. (2). A power series expansion of the U matrix to order n corresponds to allowing a maximum of n scattering events to take place. But a finite number of scattering events in infinite time will lead to zero energy shift, hence to the unperturbed density of states. Clearly we must sum an infinite number of terms in Eq. (13). This has been accomplished by Van Hove³

who wrote Eq. (13) in the form⁸

$$\Omega(z) = \sum_1 1 / (\epsilon_1 - z) \left\{ 1 - \sum'_{n; 2, 3, \dots, n; n \geq 1} (-\lambda)^n \frac{h_{12} h_{23} \cdots h_{n1}}{(\epsilon_1 - z)(\epsilon_2 - z) \cdots (\epsilon_n - z)} \right\}. \quad (18)$$

The prime on the sum indicates that no intermediate state is to be identical with the initial state 1. The infinite number of such occurrences in Eq. (13) are just exactly accounted for in the series expansion of the denominator in Eq. (18). The roots of the denominator are the poles of the characteristic function which now occur at shifted positions. In fact, the roots of the denominator correspond to the elementary perturbation expansion of the energy as a power series in λ provided we solve for the root by successive approximations starting with $z = \epsilon_1$ as zeroth approximation. If we do not expand by successive approximations, we have the Brillouin-Wigner form of perturbation theory.⁴

Actually the denominator in Eq. (18) has in general a very large number of roots. To see this, let the eigenstates of the unperturbed problem, φ_i , be expanded in terms of the perturbed eigenstates, ψ_j

$$\varphi_i = \sum_j s_{ji} \psi_j. \quad (19)$$

Calculating the trace in Eq. (5) with the basis φ_i yields

$$\Omega(z) = \sum_i \left(\sum_j \frac{|s_{ij}|^2}{E_j - z} \right). \quad (20)$$

The term in parentheses corresponds to the denominator in Eq. (18). Hence, in general, every perturbed eigenvalue occurs as a root in the denominator.

III. THE CONTINUUM LIMIT

In continuum problems we do not desire the detailed information about eigenvalues afforded by Eq. (6). We assume the roots are very numerous (continuous in the limit) and that we are interested in the density of roots but not their precise location. We pick a contour for Eq. (6) as shown in Fig. 1(d).

Assuming a distribution of simple poles, we find that the end contours in Fig. 1(d) contribute $\epsilon \ln \epsilon$, which is negligible. [This result is *not* true for Ω in the form of Eq. (13).] The density of states, $\rho(E)$, may then be written

$$\rho(E) = (1/2\pi i) \{ \Omega(E + i\epsilon) - \Omega(E - i\epsilon) \}. \quad (21)$$

For a dense but finite number of roots, Eq. (21) only makes sense if ϵ is large compared to the separation between adjacent roots. Only then will Ω be approximately constant on the contour of Fig. 1(d).

⁸ Van Hove has shown that in continuum problems the special importance of the initial state results from the existence of "diagonal singularities." These singularities also call for special consideration of iterated "excited" states. The iterated excited states may be summed over in a manner very analogous to the summation in Eq. (18). For the results we refer to Van Hove's paper.³

In going to the continuum limit Eq. (5) is replaced by

$$\Omega(z) = \int_a^b \frac{\rho(E)}{E - z} dE. \quad (22)$$

The passage to the limit completely alters the analytic character of the function $\Omega(z)$. If $\rho(E)$ has no singularities on the real axis between a and b , it is easy to see that a and b are branch points for the multivalued (logarithmic type) function $\Omega(z)$.⁹ If we make a cut along the real axis from a to b , we define the "principal value" of the function $\Omega(z)$ which is what we obtain by passing to the limit in Eq. (5). The multivalued character of Ω cannot be obtained from Eq. (5) but rather is inferred by analytic continuation. In particular, the function can have no pole off the real axis on the principal sheet since there is no such pole in Eq. (5). The continuous multivalued function $\Omega(z)$ does in general have poles off the real axis as we shall discuss shortly. It is important to clarify this point since the moments have been determined using a large contour which we wish to shrink down to surround the branch cut so that we can use Eq. (21). If there were a pole off the real axis this "shrinkage" could not be performed.

Returning to Eq. (18), we consider only second order terms (taking $h_{ii} = 0$)

$$\Omega(z) = \sum_1 1 / [(\epsilon_1 - z) - \lambda^2 \sum_2' (h_{12} h_{21}) / (\epsilon_2 - z)]. \quad (23)$$

It is easily demonstrated geometrically that the denominator in Eq. (23) has exactly n roots, where n is the number of eigenstates of the Hamiltonian, as we should expect from Eq. (20).

In the continuum limit, Eq. (23) may be written

$$\Omega(z) = \int \frac{\rho_0(\epsilon_1, \alpha_1) d\epsilon_1 d\alpha_1}{(\epsilon_1 - z) - G(\epsilon_1, z)}, \quad (24)$$

$$G(\epsilon_1, z) = K(\epsilon_1, z) + iJ(\epsilon_1, z), \quad (24')$$

$$K(\epsilon_1, E) = \lambda^2 \int_P \frac{h_{12} h_{21} \rho_0(\epsilon_2, \alpha_2) d\epsilon_2 d\alpha_2}{\epsilon_2 - E}, \quad (25)$$

$$J(\epsilon_1, E \pm i\gamma) = \pm \pi \lambda^2 \int h_{12} h_{21} \delta(\epsilon_2 - E) \rho_0(\epsilon_2, \alpha_2) d\epsilon_2 d\alpha_2; \quad (26)$$

γ is an infinitesimal positive number, $\rho_0(\epsilon_1)$ is the unperturbed density of states, α refers to the variables independent of the energy, \int_P means principal value. K and J are real functions. For z near the real axis, K is clearly an energy shift term arising from virtual transitions while J arises from energy conserving transitions

⁹ To see this, expand $\rho(E)$ in a power series and integrate.

as shown by the δ function in Eq. (26). This term is often described as a "broadening."

If J were zero in Eq. (24), the use of the relationship

$$\lim_{\gamma \rightarrow 0} \frac{1}{x - (\epsilon + i\gamma)} - \frac{1}{x - (\epsilon - i\gamma)} = 2\pi i \delta(x - \epsilon) \quad (27)$$

is suggested by Eq. (21). Making a change of variables

$$y = \epsilon_1 - K(\epsilon_1, E), \quad (27')$$

we obtain

$$\frac{dn}{dE} = \frac{dn}{d\epsilon_1} \frac{d\epsilon_1}{dy} \delta(E - y). \quad (28)$$

The energy levels of the unperturbed spectrum have each been shifted by the amount $-K(\epsilon_1, E)$. No such interpretation can be given to the J term. It represents the fact that we cannot follow in detail the correspondence between the unperturbed and the perturbed energy levels. If we attempt to do so by ordinary perturbation theory we obtain divergent results.

When J is nonzero but small, it appears that quite generally the denominator in Eq. (23) will have a root off the real axis. By our earlier discussion this root cannot be on the principal sheet,¹⁰ and obviously cannot be interpreted as an eigenvalue of the perturbed Hamiltonian. However, if we return to the propagator of Eq. (2) we find that we can interpret the root in an approximate fashion as corresponding to a decaying state. This interpretation has previously been given by Van Hove.⁵

Returning to Eq. (2) and calculating the diagonal component of U with respect to the unperturbed basis, using the results just derived, we find

$$\langle 1 | e^{-iHt} | 1 \rangle = -\frac{1}{2\pi i} \oint_{\text{I}} \frac{e^{-izt} dz}{(\epsilon_1 - z) - G(\epsilon_1, z)}. \quad (29)$$

The contour surrounds the branch cut. The integral in this form may in principle be evaluated explicitly with the use of Eqs. (24'), (25), and (26). However, there appears to be no precise way of using the pole or poles off the axis to evaluate the integral. If the pole is near the real axis, then this region makes a very large contribution to the integral (in the limit a δ -function contribution, as we have already seen). In this case we approximate the integral by

$$\langle 1 | e^{-iHt} | 1 \rangle \simeq +\frac{1}{2\pi i} \int_{-\infty}^{\infty} dE \times e^{-iEt} \left\{ \frac{1}{(\epsilon_1 - E) - G(\epsilon_1, E + i\gamma)} - \frac{1}{(\epsilon_1 - E) - G(\epsilon_1, E - i\gamma)} \right\}. \quad (30)$$

¹⁰ The integral in Eq. (26) is positive definite. Hence the root on the sheet continued from the upper half plane lies below the real axis and vice versa, as we should expect.

The integrand is an analytic function which may be evaluated using the contour in Fig. 1(e). The circle must be chosen so that e^{-iEt} is small on contour II. This fact dictates the choice made which has the result that the contribution to the integral comes from the pole of the first term in the integrand of Eq. (30). The result is then

$$\langle 1 | e^{-iHt} | 1 \rangle \simeq \exp[-i(\epsilon_1 - K(\epsilon_1, z_0))t - J(\epsilon_1, z_0)t]; \quad (31)$$

z_0 is the location of the "pole off axis." From this result, K appears very clearly as an energy shift while J causes the state to decay which leads to broadening. The interpretation cannot be made precise since contour II cannot be extended indefinitely on account of the singularities at the ends of the branch cut. It is, of course, impossible to get rid of these singularities because of the necessarily multivalued character of $\Omega(z)$.

Equation (31) provides an interpretation for the continuum model which is *qualitatively* different from the discrete model. The unperturbed state $|1\rangle$ propagates with the perturbed energy; this is essentially the adiabatic theorem. In spite of such qualitative differences arising from the existence of a second sheet, the important quantitative result embodied in Eq. (21) should be the same for both models with the understanding that ϵ be small compared to the scale of variation of $\rho(E)$, but large compared to the separation of the poles in the discrete case.

IV. MOMENT INVERSION BY MONTROLL'S METHOD

We now briefly consider the question of how one obtains the density-of-states function if one has calculated the moments. The problem is easily solved, following Montroll,² if one can assume a power series (or Legendre polynomial) expansion for the density-of-states function. We write

$$\rho(E) = \sum_n \alpha_n P_n(E), \quad (33)$$

where $P_n(E)$ are the Legendre polynomials. The bounded eigenvalue domain is assumed to be -1 to 1 for simplicity, though any other bounded domain could easily be used with the appropriate scale factor and shift of origin. Using the orthogonality of the P 's, the α_n may be written

$$\alpha_n = (n + \frac{1}{2}) \int_{-1}^1 P_n(E) \rho(E) dE. \quad (34)$$

With the use of Eq. (22) and Cauchy's theorem, Eq. (34) may be written

$$\alpha_n = \frac{(n + \frac{1}{2})}{2\pi i} \oint P_n(z) \Omega(z) dz, \quad (35)$$

where the contour surrounds the interval -1 to 1 . Substituting in (33), we find

$$\rho(E) = \sum_n \frac{(n + \frac{1}{2})}{2\pi i} \oint P_n(z) P_n(E) \Omega(z) dz. \quad (36)$$

The contour integral terms are easily found if the moments are known.

It seems likely that in many cases of interest the expansion of Eq. (33) is not justifiable or will be slowly converging. In such cases, the method of moment inversion may not be useful by itself but may be of value as a supplement to other methods.

V. PARTIAL MOMENTS OF INFINITE SPECTRA

We would like to use the moment method in cases where the spectrum has an infinite or at least very large upper bound. In this case, we are led to construct partial moments of the spectrum. To illustrate the use of partial moments, consider the problem of a free electron in interaction with a medium of randomly located impurities. The eigenvalue spectrum may, at least approximately, be considered to have a lower bound, but the upper bound should be infinite or at least very large in order to characterize the problem. The customary use of perturbation theory to find the density of states in this problem is most accurate at high energy where the electron-impurity interaction is small compared to the electron's kinetic energy. In a problem of this type, it appears that the moment-perturbation identity may have a useful application. We can construct the "partial moments," $M_p(\eta)$ of the low-energy part of the spectrum with the use of perturbation theory:

$$M_p(\eta) \simeq \int_{E_{\min}}^{\eta} \rho_p(E) E^p dE. \quad (37)$$

The subscript p on the density of states denotes that it is to be calculated from Eq. (21) using the perturbation expansion of Eq. (13), where all terms of order $n \leq p$ are included. Any partial summation over higher order terms is allowable. The accuracy with which the partial moments are given by Eq. (37) is determined by the accuracy of $\rho_p(E)$ in the energy range $\eta \leq E \leq E_{\max}$, since by virtue of the moment-perturbation identity, the total moment as given by perturbation theory is exact. Having obtained partial moments, we are now able to let E_{\max} become arbitrarily large without difficulty.

The error in M_p , δM_p , may be expected to be of the order

$$\delta M_p \simeq \eta^p \{ \delta \rho(E) \} \Delta E, \quad (38)$$

where $\delta \rho(E)$ is the error in the density of states due to truncating the perturbation expansion and ΔE is an "interaction range" representing the energy range of the unperturbed states which contribute significantly to the perturbed eigenstate at η , the cutoff energy.

Note added in proof.—The density of states function for the impurity problem at high energies can be approximately described by the series

$$\rho(E) = \rho_0(E) \left[1 + \sum_{n=1}^{\infty} \alpha_n E^{-n} \right],$$

where the m th order of perturbation theory contributes to α_n for $n \geq m$. Clearly, this result is consistent with the moment theorem. The exact function $\rho(E)$ appears to contain logarithmic terms or perhaps even more complicated functions but we believe that the above simple form correctly describes the convergence of the series. This problem will be discussed in more detail in a forthcoming publication.

The partial moments might conceivably be used in connection with a "moment inversion" scheme such as that given by Eq. (36). Because of the close relation between moments and perturbation theory it seems doubtful that such a method would be applicable when perturbation theory is not. The partial moments may be more useful if a reasonable form with adjustable parameters for the density-of-states function can be guessed. In this way it is similar to the commonly used variational method. Unfortunately, we do not know of any criterion whereby the relative accuracy of different trial functions may be judged.

We are on firmer ground if we use the "partial moments" to interpolate rather than extrapolate. In the impurity problem, the states deep in the tail may be calculated from the point of view of isolated clusters of impurities. If this approximation proves feasible, the "partial moments" can be used to interpolate between the low-energy and high-energy approximations to the density of states.

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