Quasi-Classical Theory of the Spinning Electron*

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We derive a modified Hamilton-Jacobi theory for a classical spinning dipole and show that this classical theory can be put in a form almost identical with the Pauli spin theory. We quantize this theory by requiring that a classical spinor "wave function" be continuous and single-valued, and that it satisfy the usual energy eigenvalue equation. In this manner we deduce the correct energy levels for a hydrogen atom in an external magnetic field. We derive the integral kernel for the Pauli equation from this classical model and discuss the properties of our spinor solutions under canonical transformation. We exhibit the charge-conjugate solution to the classical spin equation.

I. INTRODUCTION

THIS article is the third in a series[1] having a twofold aim. In the first place we wish to show that classical mechanics, when formulated properly, may be used to exhibit most of the qualitative features of nonrelativistic quantum theory; and secondly, that this classical theory is useful in finding eigenvalues and eigenfunctions which approximate, in the sense of the WKB approximation, the eigenvalues and eigenfunctions found in the Schrödinger or Pauli theories.

In the initial article of this series we discussed the early work of Van Vleck who first showed how the wave functions of the WKB approximation to the Schrödinger theory could be determined from the solutions of the classical Hamilton-Jacobi theory.

In the present article we find that the same techniques are applicable to the problem of the spinning electron.

We assume a classical magnetic dipole as our model for the spinning electron, and show that a generalized Hamilton-Jacobi theory of the classical equations of motion of the dipole may be written in a form analogous to the Pauli spin equation.

When we demand that our classical spinor "wave function" satisfy an eigenvalue equation, and that it be continuous and single-valued, we find for the hydrogen atom in a uniform magnetic field that the quasi-classical theory makes the same predictions for the energy and angular momentum eigenvalues as does the usual Pauli theory.

We also formulate a quasi-classical "sum over paths" transformation theory which corresponds to the one that Feynman[2] developed for the Pauli theory. However, unlike Feynman's kernel, our kernel is derived from a specific classical spin theory, in the same way that the kernel is derived for the Schrödinger equation.

We show that our classical spin theory is invariant under canonical transformations and derive the transformation law for the spinor solutions to the classical Pauli equation. Finally we exhibit the charge-conjugate solution to our classical spin equation.

II. CLASSICAL THEORY OF THE SPINNING DIPOLE

As a preliminary to our discussion of the quasi-classical theory of the spinning electron we shall first describe the nonrelativistic equations of motion of a classical spinning electron, a magnetic dipole of negative charge.

The equations of motion of the dipole in the laboratory frame of reference are

$$d\mathbf{s}/dt=-(e/\mu c)\mathbf{s}\times\mathbf{B}, \tag{1}$$

where $\mathbf{s}$ is the spin angular momentum of the dipole, $\mathbf{B}$ the external magnetic field, and μ the mass of the electron.

In this classical description we ignore for the moment the translational motion of the charge. We shall disregard other intrinsic characteristics of the electron such as size and internal forces of cohesion. We assume only that the electron, as a point particle, has associated with it a direction given by the spin vector, $\mathbf{s}$, which can precess in a magnetic field, but whose absolute value remains fixed.

Kramers[3] has shown that the above equations may be placed in Hamiltonian form. Since $|\mathbf{s}|$ is constant there are only two degrees of freedom in the problem. These may be defined as the canonical variables,

$$\xi=s_z,\quad -\omega=-\tan^{-1}(s_x/s_y). \tag{2}$$

The spin coordinate ω is the azimuthal angle of the spin orientation as measured from the y axis in the laboratory frame of reference. The canonically conjugate momentum variable ξ is the component of the spin vector in the z direction. Equations (1) are equivalent to the Hamilton equations

$$d(-\omega)/dt=\partial H_{\mathrm{sp}}/\partial\xi,\quad d\xi/dt=-\partial H_{\mathrm{sp}}/\partial(-\omega), \tag{3}$$

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[1] R. Schiller, preceding papers [Phys. Rev. 125, 1100, 1109 (1962)], hereafter called I and II.

[2] R. P. Feynman, Revs. Modern Phys. 20, 367 (1948), especially pp. 386–387.

[3] H. A. Kramers, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1957), p. 233.

with the appropriate spin Hamiltonian,

$$H_{\rm sp}=\frac{e}{\mu c}[(s^2-\xi^2)^{\frac{1}{2}}(B_x\sin\omega+B_y\cos\omega)+\xi B_z]. \quad (4)$$

The equations of motion may be extended to include the fact that s is a constant of the motion by introducing a cyclic variable conjugate to s, the angle χ.[4] χ is one of the Euler angles used in the description of rigid body motions: It is the azimuthal angle in the frame of reference of the dipole. The equations of motion for these new variables are

$$ds/dt=-\partial H_{\rm sp}/\partial\chi=0,\quad d\chi/dt=\partial H_{\rm sp}/\partial s. \quad (5)$$

The equation for $d\chi/dt$ in (5) provides no additional information; χ has been introduced to complete our equations of motion and to permit us to formulate the theory of the spinning electron so that its Lagrangian vanishes. By definition the spin Lagrangian is

$$L_{\rm sp}=-H_{\rm sp}+\xi\frac{d(-\omega)}{dt}+s\frac{d\chi}{dt}=-H_{\rm sp}+\xi\frac{\partial H_{\rm sp}}{\partial s}+s\frac{\partial H_{\rm sp}}{\partial s}=0. \quad (6)$$

The last result is a consequence of Eqs. (3) and (5) and the fact that $H_{\rm sp}$ is a linear homogeneous function of s and ξ. The appearance of a vanishing Lagrangian in the classical theory in no way restricts the solutions of Eq. (1).

However, since the classical action integral is at the heart of the WKB approximation to the Schrödinger equation, the above calculation shows that in the WKB approximation one must adopt new methods for quantizing the magnetic dipole. In the following paragraphs we shall indicate a possible quantization procedure; in fact one that is really implicit in the usual Pauli theory.

We now indicate how the equations of motion (1) may be rewritten in terms of a two-component spinor. We note that the vector s may be written as

$$\mathbf{s}=\psi^\dagger\boldsymbol{\sigma}\psi, \quad (7)$$

where

$$\psi=s^{\frac{1}{2}}\begin{pmatrix}\cos(\theta'/2)\exp[\frac{1}{2}i(\chi+\omega)]\\ i\sin(\theta'/2)\exp[\frac{1}{2}i(\chi-\omega)]\end{pmatrix}. \quad (8)$$

The components of the vector $\boldsymbol{\sigma}$ are given by the Pauli spin matrices, and the new variable θ' is defined as $s\cos\theta'=\xi$. ψ is a spinor written in invariant form as a function of the Euler angles.[5]

[4] M. Schoenberg of the University of Sao Paulo, Brazil, introduced this coordinate in his competitive thesis, *Principios da Mecanica*, published in 1944 in Sao Paulo.

[5] For a similar representation of a spinor in terms of Euler angles, see H. Goldstein, *Classical Mechanics* (Addison-Wesley Press, Inc., Reading, Massachusetts, 1959), p. 116. Our representation becomes identical with his if his angles ψ and ϕ are replaced by our angles χ and ω, and his body and laboratory frames are interchanged.

The Lagrangian in terms of the spinor is

$$L_{\rm sp}=-\frac{e}{2\mu c}\psi^\dagger\boldsymbol{\sigma}\psi\cdot\mathbf{B}-\frac{1}{2i}\left(\psi^\dagger\frac{d\psi}{dt}-\frac{d\psi^\dagger}{dt}\psi\right). \quad (9)$$

Both $L_{\rm sp}$ and $\delta\int L_{\rm sp}dt$ vanish by virtue of Eqs. (3). The equations of motion may be deduced in characteristic spinor form by varying ψ and $\psi^\dagger$ as independent variables. In terms of the spinor ψ, the equations of motion, (1), are

$$\frac{d\psi}{dt}=-(ie/2\mu c)\mathbf{B}\cdot\boldsymbol{\sigma}\psi. \quad (10)$$

One can easily prove that these equations are identical with Eq. (1). For the proof multiply (10) by $\psi^\dagger\boldsymbol{\sigma}$, take the complex conjugate equation, and add the two equations. The resulting equation is (1).

If we multiply (10) by $i\hbar$, we find that

$$i\hbar d\psi/dt=(e\hbar/2\mu c)\mathbf{B}\cdot\boldsymbol{\sigma}\psi. \quad (11)$$

The constant $\hbar$ is Planck's constant (divided by 2π), but the motion of the dipole has in no way been quantized, for we could have multiplied the spinor by any arbitrary constant. We have introduced $\hbar$ so that these classical equations appear in a form long familiar in quantum mechanics.

However, we shall "quantize" this classical theory if only to indicate the argument which we shall follow when we quantize the quasi-classical Pauli equation.

We require that the spinor ψ satisfy the eigenvalue equation,

$$i\hbar d\psi/dt=E_{\rm sp}\psi, \quad (12)$$

where $E_{\rm sp}$ is the quantized energy of the dipole. For a constant magnetic field along the z axis we find $E_{\rm sp}=\pm e\hbar B/2\mu c$, while from Eq. (4) we have the classical spin energy as $E_{\rm sp}=eBs_z/\mu c$. Thus we see that the requirement (12) leads to the quantization of the spin in the z direction,[6]

$$s_z\equiv\xi=\pm\hbar/2. \quad (13)$$

At first sight, the requirement (12) might be viewed as a novel method of quantization, quite distinct from the usual techniques of quantizing the angular momentum in quantum mechanics. Actually our method of quantization is almost identical with the usual quantization procedure; for instead of focusing attention on the properties of the angular momentum operators, we have made use of the solutions [such as ψ of Eq. (8)] to the angular momentum operator

[6] In our discussion we have shown how to quantize a classical theory of the spinning dipole by introducing a two-component spinor into the theory. It is clear, however, that a higher component spinor could equally well have been introduced in place of the two-component spinor. This would have led to modifications in the form of the classical theory, and would have changed the spin eigenvalues in the quantized theory. We cannot pursue this question any further in the present paper.

equations. This is always possible when one describes the classical spin motion of the dipole; for if the translational motion can be ignored, the spin equations of the quantum theory and the classical theory are identical. Only the requirement (12), or an equivalent eigenvalue equation, distinguishes the classical theory from the quantum theory of the dipole.

III. MODIFIED HAMILTON-JACOBI THEORY

We now consider the full motion of the classical dipole as described by the following equations:

$$\mu\frac{d\mathbf{v}}{dt}=-e\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)-\frac{e}{\mu c}(\nabla\mathbf{B})\cdot\mathbf{s}, \tag{14a}$$

$$\frac{d\mathbf{s}}{dt}=-\frac{e}{\mu c}\mathbf{s}\times\mathbf{B}. \tag{14b}$$

If the applied magnetic field is constant, or purely time-dependent, the translational motion of the electron is independent of its state of rotation. However, we are interested in the general problem when both $\mathbf{B}$ and $\mathbf{E}$ are arbitrary, prescribed fields, functions of the space and time coordinates, and limited only by the requirement that the particle motions they produce should be consistent with the nonrelativistic character of our equations.

The classical Pauli equation is simply the Hamilton-Jacobi theory of the coupled set of equations, (14), in the configuration space of the translational motion.

The most direct method for finding this Hamilton-Jacobi theory is to consider the variables $\mathbf{v}$ and $\mathbf{s}$ as continuous field functions of the space and time coordinates. Thus by $d\alpha/dt$ we mean

$$d\alpha/dt=\partial\alpha/\partial t+(\mathbf{v}\cdot\nabla)\alpha, \tag{15}$$

where α is any function of $\mathbf{q}$ and t.

We now treat Eqs. (14) as field equations, introduce the electromagnetic potentials $\mathbf{A}$ and ϕ in the usual manner, and rewrite the spin equations in their canonical form, Eq. (3).[7] Equations (14) then appear as

$$\frac{\partial\mu\mathbf{v}}{dt}+\nabla(\mu\mathbf{v})\cdot\mathbf{v}=\frac{e}{c}\frac{\partial\mathbf{A}}{\partial t}+e\nabla\phi+\frac{e}{c}[\mathbf{v}\cdot(\nabla\mathbf{A})-(\nabla\mathbf{A})\cdot\mathbf{v}]-\frac{e}{\mu c}(\nabla\mathbf{B})\cdot\mathbf{s}, \tag{16a}$$

$$\partial\xi/\partial t+\mathbf{v}\cdot\nabla\xi=\partial H_{\mathrm{sp}}/\partial\omega, \tag{16b}$$

$$\partial\omega/\partial t+\mathbf{v}\cdot\nabla\omega=-\partial H_{\mathrm{sp}}/\partial\xi, \tag{16c}$$

where $H_{\mathrm{sp}}=(e/\mu c)\mathbf{s}\cdot\mathbf{B}$.

We introduce the Hamiltonian of our system of equations,

$$H=\tfrac{1}{2}\mu v^2-e\phi+H_{\mathrm{sp}}, \tag{17}$$

take the spatial gradient of (17), and find

$$\nabla H=\mu(\nabla\mathbf{v})\cdot\mathbf{v}-e\nabla\phi+\frac{e}{\mu c}(\nabla\mathbf{B})\cdot\mathbf{s}+\frac{\partial H_{\mathrm{sp}}}{\partial\xi}\nabla\xi+\frac{\partial H_{\mathrm{sp}}}{\partial\omega}\nabla\omega. \tag{18}$$

The last two terms arise from the fact that the variables ξ and ω are field functions. We add (18) to (16a) and after some manipulation find that

$$\frac{\partial}{\partial t}\left(\mu\mathbf{v}-\frac{e}{c}\mathbf{A}-\xi\nabla\omega\right)+\nabla\left(\frac{\partial\omega}{\partial t}\xi+H\right)+(\mathbf{v}\cdot\nabla)\left(\mu\mathbf{v}-\frac{e}{c}\mathbf{A}-\xi\nabla\omega\right)-\nabla\left(\mu\mathbf{v}-\frac{e}{c}\mathbf{A}-\xi\nabla\omega\right)\cdot\mathbf{v}=0. \tag{19}$$

These equations may be satisfied as follows:

$$\nabla(\partial S/\partial t+\xi\partial\omega/\partial t+H)=0, \tag{20}$$

and

$$\mu\mathbf{v}-(e/c)\mathbf{A}-\xi\nabla\omega=\nabla S, \tag{21}$$

where S is a scalar function of position that corresponds to Hamilton's characteristic function (the classical action) in an ordinary dynamical theory.

The first equation, (20), is satisfied if we choose S so that

$$\partial S/\partial t+\xi\partial\omega/\partial t+H=0. \tag{22}$$

This choice of S is always possible and the demonstration follows the analogous proof in the usual Hamilton-Jacobi theory.[8]

The second equation, (21), is a generalization of the definition of momentum. We may rewrite this equation in the more familiar form,

$$\mathbf{p}=\mu\mathbf{v}-(e/c)\mathbf{A}=\nabla S+\xi\nabla\omega. \tag{23}$$

In the absence of coupling between translation and rotation we have the usual theorem that the momentum field may always be represented as the gradient of a scalar function (the classical action). When the particle possesses intrinsic angular momentum, we see that, in general, the momentum field can no longer be represented as curl-free as it is in classical dynamics. Indeed the momentum field is rotational, since

$$\nabla\times p=\nabla\xi\times\nabla\omega. \tag{24}$$

The decomposition of the momentum field given in

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[7] In our future discussion we shall disregard the cyclic variable χ as defined by Eq. (5). It was introduced in the classical dipole theory chiefly to indicate that the classical action of this theory vanished. As this variable plays no role in our future discussions we shall suppress it, but it may be readily introduced at any point with very little change in formalism.

[8] See in this connection P. G. Bergmann, *Basic Theories of Physics, Mechanics and Electrodynamics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1949), p. 45.

(23) was first effected by Clebsch[9] in his treatment of vorticity in fluid mechanics. However, unlike the Clebsch decomposition, which describes rotational flow in a real fluid, the presence of "vorticity" in our theory depends on the coupling between spin and translational motion of the electron. Once the motions are decoupled (constant or time-dependent fields), the momentum field becomes irrotational again.

The motion of a spinning dipole is completely determined when solutions are known of the following three equations:

$$\frac{\partial S}{\partial t}+\xi\frac{\partial\omega}{\partial t}+\frac{1}{2\mu}\left(\nabla S+\xi\nabla\omega+\frac{e}{c}\mathbf{A}\right)^2-e\phi+H_{\mathrm{sp}}=0, \tag{25a}$$

$$\frac{\partial\xi}{\partial t}+\frac{1}{\mu}\nabla\xi\cdot\left(\nabla S+\xi\nabla\omega+\frac{e}{c}\mathbf{A}\right)=\frac{\partial H_{\mathrm{sp}}}{\partial\omega}, \tag{25b}$$

$$\frac{\partial\omega}{\partial t}+\frac{1}{\mu}\nabla\omega\cdot\left(\nabla S+\xi\nabla\omega+\frac{e}{c}\mathbf{A}\right)=-\frac{\partial H_{\mathrm{sp}}}{\partial\xi}. \tag{25c}$$

Solutions of Eqs. (25) are equivalent in the Hamilton-Jacobi sense to the ordinary differential equations, (14). These three equations replace the single Hamilton-Jacobi equation of classical mechanics. The two additional equations arise from the internal degrees of freedom and the fact that we have constructed an ensemble in the configuration space of the translational motion.

IV. EQUATION OF CONTINUITY FOR PARTICLES

Since these equations, (25), describe an ensemble in which the number of particles remains the same, our Hamilton-Jacobi equations must define a law of continuity. We shall now derive this law.

It is not difficult to show that, for the motion of a single particle, the three functions S, ω, and ξ appearing in the partial differential equations, (25), admit solutions which depend on five constants of the motion. A simple analysis (carried out in Sec. VIII) based on the Hamilton-Jacobi theory in the configuration space of the coordinates $\mathbf{q}$ and ω, proves that this is the general situation. We shall single out three of these five constants of the motion and ignore the other two in the calculations to follow. For arbitrary external fields it is difficult to say which of the three constants might be preferred, but for the case we treat, we find that the only sensible constants are those associated with the translational degrees of freedom. A necessary requirement in the choice of these constants is that the determinant

$$\left\|\frac{\partial}{\partial q^i}\left(\frac{\partial S}{\partial\alpha_k}+\xi\frac{\partial\omega}{\partial\alpha_k}\right)\right\|$$

not vanish.

We differentiate Eq. (25a) with respect to these three constants α_i, and after some manipulation find that

$$\frac{\partial}{\partial t}\left(\frac{\partial S}{\partial\alpha_i}+\xi\frac{\partial\omega}{\partial\alpha_i}\right)+\frac{1}{\mu}\frac{\partial}{\partial q^k}\left(\frac{\partial S}{\partial\alpha_i}+\xi\frac{\partial\omega}{\partial\alpha_i}\right)\times\left(\frac{\partial S}{\partial q_k}+\xi\frac{\partial\omega}{\partial q_k}+\frac{e}{c}A^k\right)+\frac{\partial\xi}{\partial\alpha_i}\left(\frac{d\omega}{dt}+\frac{\partial H_{\mathrm{sp}}}{\partial\xi}\right)+\frac{\partial\omega}{\partial\alpha_i}\left(\frac{\partial H_{\mathrm{sp}}}{\partial\omega}-\frac{d\xi}{dt}\right)=0. \tag{26}$$

The last two terms vanish when we make use of the last two equations of (25). We now differentiate Eq. (26) with respect to the spatial coordinates q^i and find

$$\frac{\partial}{\partial t}\frac{\partial}{\partial q^i}\left(\frac{\partial S}{\partial\alpha_i}+\xi\frac{\partial\omega}{\partial\alpha_i}\right)+\frac{1}{\mu}\frac{\partial}{\partial q^i}\frac{\partial}{\partial q^k}\left(\frac{\partial S}{\partial\alpha_i}+\xi\frac{\partial\omega}{\partial\alpha_i}\right)\times\left(\frac{\partial S}{\partial q_k}+\xi\frac{\partial\omega}{\partial q_k}+\frac{c}{c}A^k\right)+\frac{1}{\mu}\frac{\partial}{\partial q^k}\left(\frac{\partial S}{\partial\alpha_i}+\xi\frac{\partial\omega}{\partial\alpha_i}\right)\times\frac{\partial}{\partial q^i}\left(\frac{\partial S}{\partial q_k}+\xi\frac{\partial\omega}{\partial q_k}+\frac{e}{c}A^k\right)=0. \tag{27}$$

If we define the matrix ϕ_{ik} as

$$\phi_{ik}=\frac{\partial}{\partial q^k}\left(\frac{\partial S}{\partial\alpha_i}+\xi\frac{\partial\omega}{\partial\alpha_i}\right), \tag{28}$$

then there always exists an inverse to this matrix for a well-defined mechanical problem and a propitious choice of the constants α_i. Call this inverse ϕ^{ik}. It satisfies the equation

$$\phi_{ij}\phi^{ki}=\delta_j{}^k=\phi_{ji}\phi^{ik}. \tag{29}$$

If we now multiply Eq. (27) by the inverse ϕ^{ji}, and sum over i and j, we find that

$$\phi^{ji}\frac{\partial\phi_{ij}}{\partial t}+\phi^{ji}\frac{\partial\phi_{ij}}{\partial q_k}v_k+\frac{\partial v_k}{\partial q_k}=0, \tag{30}$$

where

$$v_k=\frac{1}{\mu}\left(\frac{\partial S}{\partial q^k}+\xi\frac{\partial\omega}{\partial q^k}+\frac{e}{c}A_k\right).$$

If the determinant $\|\phi_{ij}\|=D$ is introduced, we observe that Eq. (30) becomes

$$\frac{\partial D}{\partial t}+\frac{\partial}{\partial q^k}(Dv^k)=0. \tag{31}$$

Equation (31) is a law of continuity in the configuration space of the translational motion. This law is not independent of Eqs. (25): Its appearance reflects the fact

[9] Clebsch's theorem is discussed in H. Lamb, *Hydrodynamics* (Cambridge University Press, New York, 1953), p. 248.

that the Hamilton-Jacobi theory is an ensemble theory which is structured so that the ensemble population remains fixed.

V. CLASSICAL PAULI THEORY

The four equations in (25) and (31) contain the four variables S, ξ, ω, and D. We shall now show that these four equations may be written in a form similar to the Pauli equation.

We introduce the two-component complex quantity (a spinor with respect to the spin variables),

$$\psi = Re^{iS/\hbar}\begin{pmatrix} \cos(\theta'/2)\exp(i\omega/2) \\ i\sin(\theta'/2)\exp(-i\omega/2) \end{pmatrix}, \qquad (32)$$

where $R=D^{\frac{1}{2}}$ and $\cos\theta'=\xi/(\hbar/2)$ (we assume that the z component of the spin angular momentum has a maximum value $\hbar/2$). S and ω are defined through Eqs. (25).

Our task is to find that equation satisfied by the spinor which is equivalent to the four equations, (25) and (31). We may accomplish this most simply by deriving the equation for ψ from a variational principle, and then showing that the variation of the four variables S, ξ, ω, and D in the same action integral leads to our four equations.

As Lagrangian density for our system of equations we choose

$$\mathfrak{L} = \frac{i\hbar}{2}\left(\psi^\dagger\frac{\partial\psi}{\partial t} - \frac{\partial\psi^\dagger}{\partial t}\psi\right) - \frac{1}{2\mu}\left(-i\hbar\nabla\psi^\dagger + \frac{e}{c}\mathbf{A}\psi^\dagger\right) \cdot\left(i\hbar\nabla\psi + \frac{e}{c}\mathbf{A}\psi\right) + e\phi\psi^\dagger\psi - \frac{e\hbar}{2\mu c}\psi^\dagger\boldsymbol{\sigma}\cdot\mathbf{B}\psi + \frac{\hbar^2}{8\mu}\nabla(\psi^\dagger\boldsymbol{\sigma}\psi) : \nabla(\psi^\dagger\boldsymbol{\sigma}\psi)\frac{1}{\psi^\dagger\psi}. \qquad (33)$$

The variation of the independent variable $\psi^\dagger$ results in an equation similar in form to the Pauli equation,

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2\mu}\left(-i\hbar\nabla + \frac{e}{c}\mathbf{A}\right)^2\psi - e\phi\psi + \frac{e\hbar}{2\mu c}\boldsymbol{\sigma}\cdot\mathbf{B}\psi - \frac{\hbar^2}{8\mu}\frac{\delta}{\delta\psi^\dagger}\int\frac{\nabla(\psi\boldsymbol{\sigma}\psi) : \nabla(\psi\boldsymbol{\sigma}\psi)}{\psi^\dagger\psi}d\mathbf{q}. \qquad (34)$$

Our classical equation, (34), differs from the Pauli equation only because of the presence of the final term.

It is of interest that the Pauli spin energy operator, $(e\hbar/2\mu c)\boldsymbol{\sigma}\cdot\mathbf{B}$, in Eq. (34), has the correct value for the magnetic moment of the electron. This value appears because of the representation of the spinor in Eq. (32), and the definition of the z component of the spin angular momentum as $\xi=\frac{1}{2}\hbar\cos\theta'$.

In order to show that Eq. (34) is equivalent to the four equations, (25) and (31), we rewrite the Lagrangian, (27), in terms of the variables S, ω, ξ, and D. The Lagrangian density becomes

$$\mathfrak{L} = -D\left[\frac{\partial S}{\partial t} + \xi\frac{\partial\omega}{\partial t} + \frac{1}{2\mu}\left(\nabla S + \xi\nabla\omega - \frac{e}{c}\mathbf{A}\right)^2 - e\phi + \frac{e}{\mu c}\mathbf{s}\cdot\mathbf{B}\right]. \qquad (35)$$

Variation of $\int\mathfrak{L}dtdq$ with respect to the four independent variables leads to the correct four equations, (25) and (32).

Before proceeding to the quantization of this theory we should like to briefly comment on the fact that we have introduced Planck's constant $\hbar$ into the classical theory by giving the z component of the spin angular momentum the maximum value $\hbar/2$. It is a curiosity of this classical theory that the coupling between the spin and translational motion permits a "classical" measurement of $\hbar$.

One might for example determine $\hbar$ by means of a Stern-Gerlach experiment. The lateral spread of a beam of silver atoms in an inhomogeneous magnetic field may be used to measure the magnetic moment of the electron, $es_z/\mu c$, and this measurement would establish the value of s_z. Of course, on the basis of classical theory, it would remain an unexplained mystery why only two lines appeared on the photographic plate and not a continuous smear of atoms. However great this mystery, the experiment would still determine $\hbar$, if the other constants e, μ, and c were known.

VI. QUANTIZATION OF THE CLASSICAL PAULI EQUATION

We may quantize the above presented classical theory by requiring that the classical wave function ψ given by Eq. (32), be a continuous and single-valued function of position. For stationary states we also demand that ψ be an eigenfunction of the energy operator, i.e., that ψ satisfy an equation of the form,

$$i\hbar\partial\psi/\partial t = H_{\mathrm{op}}\psi = E\psi. \qquad (36)$$

To illustrate the method of quantization we treat the case of the hydrogen atom in a uniform magnetic field B oriented along the z axis. The equations of motion for the dipole, (3), may be integrated immediately to yield

$$\xi = \text{const}, \quad \omega = -(e/\mu c)Bt + \omega_0. \qquad (37)$$

We shall omit the phase angle ω_0 as it plays no role in our considerations.

Since the spin and translational motion are completely independent of each other for a constant magnetic field, the function S satisfies the usual classical Hamilton-Jacobi equation so that there exist solutions

of the form

$$S=-E_0t+\int \mathbf{p}\cdot d\mathbf{q}. \tag{38}$$

E_0 is the energy of a spinless charge in a magnetic field.

We now require that the wave function ψ satisfy the eigenvalue equation,

$$i\hbar\partial\psi/\partial t=E\psi=H_{\text{op}}\psi, \tag{39}$$

where the total energy E is the eigenvalue,

$$E=E_0+(\hbar e/2\mu c)B\sigma_z. \tag{40}$$

E_0 is the energy of translation and σ_z is the Pauli spin matrix,

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

H_{op} is defined by the right hand side of Eq. (34).

There are two possible solutions to (40),

$$\begin{aligned}E_+&=E_0+(e\hbar/2\mu c)B,\\ E_-&=E_0-(e\hbar/2\mu c)B.\end{aligned} \tag{41}$$

If we compare the two possible values of the quantized spin energy with the classical spin energy, $E_{\text{sp}}=(e\xi/\mu c)B$, we find that the requirement (39) has led to the quantization of the z component of the spin angular momentum, $s_z\equiv\xi=\pm\hbar/2$.

The quantized energy E_0 of the translational motion is found by requiring that the wave function ψ be continuous and single-valued. Since the spin and center-of-mass motion are decoupled in the example considered, the results of paper I may be applied. It is easy to show by the methods of the WKB approximation that the energy of translation E_0, for moderate magnetic fields, has the usual quantum theory values,

$$E_0=-(\mu e^4/2\hbar^2n^2)+(me\hbar/2\mu c)B.$$

The total energy is then

$$E_\pm=-\frac{\mu e^4}{2\hbar^2n^2}+\frac{e\hbar}{2\mu c}B(m\pm1). \tag{42}$$

n is the total quantum number and m the magnetic quantum number.

The two basic solutions from which the WKB solutions to this particular problem are constructed are:

$$\psi_+=\begin{pmatrix}R\exp(iS/\hbar+i\omega/2)\\ 0\end{pmatrix},\quad \psi_-=\begin{pmatrix}0\\ R\exp(iS/\hbar-i\omega/2)\end{pmatrix}. \tag{43}$$

VII. CLASSICAL KERNELS

In the "sum over paths" formulation of nonrelativistic quantum mechanics one makes use of the fact that solutions to classical problems generate solutions of the Schrödinger equation.[10] We should now like to show that the same principle holds in the Pauli spin theory.

We seek the kernel $\mathbf{K}$ which transforms a Pauli state function from time t_0 to time t,

$$\psi(\mathbf{q},t)=\int \mathbf{K}(\mathbf{q},t;\mathbf{q}_0,t_0)\psi(\mathbf{q}_0,t_0)d\mathbf{q}_0. \tag{44}$$

The kernel $\mathbf{K}$ must be a 2 by 2 matrix since ψ is a spinor. When $t=t_0$, $\mathbf{K}$ should reduce to the delta function, $\mathbf{I}\cdot\delta(q-q_0)$.

We first introduce another spinor $\bar\psi$,

$$\bar\psi=Re^{iS/\hbar}\begin{pmatrix}i\sin(\theta'/2)\exp(i\omega/2)\\ \cos(\theta'/2)\exp(-i\omega/2)\end{pmatrix}, \tag{45}$$

which is the charge conjugate spinor with respect to the spin variables θ' and ω. This spinor describes a dipole whose spin orientation is opposite to the spin orientation given by ψ, i.e., $\bar\psi^\dagger\boldsymbol{\sigma}\bar\psi=-\psi^\dagger\boldsymbol{\sigma}\psi$.

We now assume that the spin magnitude is small, so that, in a first approximation to the true motion, we may neglect the final term in the force law, (14a). Under these circumstances both ψ and $\bar\psi$ satisfy the same equation, (34). Since the translational motion is decoupled from the spin motion, S represents the classical action of a particle without spin and satisfies the usual Hamilton-Jacobi equation,

$$\partial S/\partial t+(1/2\mu)(\nabla S-(e/c)\mathbf{A})^2-e\phi=0.$$

To construct the kernel $\mathbf{K}$ we seek solutions of this equation of the form $S=S(q^i,t;q_0^i,t_0)$, where q_0^i gives the position of the particle at the time t_0. The probability amplitude R equals the square root of the Van Vleck determinant,

$$R=\left\|\frac{\partial^2S}{\partial q^i\partial q_0^k}\right\|^{\frac12}.$$

The spinor parts of the wave functions ψ and $\bar\psi$ are φ and $\bar\varphi$,

$$\varphi=\begin{pmatrix}\cos(\theta/2)\exp(i\omega/2)\\ i\sin(\theta'/2)\exp(-i\omega/2)\end{pmatrix},\quad \bar\varphi=\begin{pmatrix}i\sin(\theta'/2)\exp(i\omega/2)\\ \cos(\theta'/2)\exp(-i\omega/2)\end{pmatrix}, \tag{46}$$

These spinors satisfy the same equation

$$\begin{aligned}d\varphi/dt&=-(ie/2\mu c)\mathbf{B}\cdot\boldsymbol{\sigma}\varphi,\\ d\bar\varphi/dt&=-(ie/2\mu c)\mathbf{B}\cdot\boldsymbol{\sigma}\bar\varphi.\end{aligned} \tag{47}$$

From the work of Pauli[11] we know that the proba-

[10] See I, Sec. III B.

[11] W. Pauli, *Feldquantisierung* (Akad. Buchgenossenshaft, Zurich, 1957), 2nd ed., p. 139.

bility amplitude, R, is a function of the time alone, when we consider infinitesimal displacements of the particle.

If we examine the solutions ψ and $\bar{\psi}$ for infinitesimal time intervals when the translational and spin motion are decoupled, we find that both solutions satisfy the Pauli equation, a linear differential equation. The following matrices are also solutions of the Pauli equation:

$$\mathsf{K}_1=\begin{pmatrix}\psi_1 & 0\\ \psi_2 & 0\end{pmatrix},\quad \mathsf{K}_2=\begin{pmatrix}0 & \bar{\psi}_1\\ 0 & \bar{\psi}_2\end{pmatrix}, \tag{48}$$

with

$$\psi\equiv\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.$$

The sum $\mathsf{K}=\mathsf{K}_1+\mathsf{K}_2=\mathbf{\Phi}R\exp(iS/\hbar)$ also satisfies the Pauli equation. The spin matrix Φ,

$$\mathbf{\Phi}=\begin{pmatrix}\cos(\theta'/2)\exp(i\omega/2) & i\sin(\theta'/2)\exp(i\omega/2)\\ i\sin(\theta'/2)\exp(-i\omega/2) & \cos(\theta'/2)\exp(-i\omega/2)\end{pmatrix}, \tag{49}$$

obeys the ordinary differential equation

$$d\mathbf{\Phi}/dt=-(ie/2\mu c)\mathbf{B}\cdot\boldsymbol{\sigma}\mathbf{\Phi}. \tag{50}$$

For short time intervals we find that the solutions for S, R, and $\mathbf{\Phi}$ are as follows:

$$S=\frac{\mu}{2}\frac{(\mathbf{q}-\mathbf{q}_0)^2}{(t-t_0)}-V(\mathbf{q})(t-t_0)-\frac{e}{2c}(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{A}(\mathbf{q}_0)-\frac{e}{2c}(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{A}(\mathbf{q}), \tag{51}$$

$$R=(t-t_0)/\mu^{-\frac{3}{2}},$$

$$\mathbf{\Phi}=\mathbf{\Phi}_0(1-ie/2\mu c)\mathbf{B}\cdot\boldsymbol{\sigma}(t-t_0),$$

where $\mathbf{\Phi}_0$ is a constant matrix.

We now construct the kernel $\mathsf{K}=\mathbf{\Phi}R\exp(iS/\hbar)$ from the solutions in Eq. (51). The proof that this is the sought-for integral kernel of the Pauli equation has been given by Feynman,[12] who first discovered it by an *ad hoc* procedure.

VIII. CLASSICAL TRANSFORMATION THEORY

In Secs. III and IV we showed how the coupled set of equations of motion, (14), could be rewritten as a set of modified Hamilton-Jacobi equations. In Sec. V we proved that these modified equations could be replaced by a single partial differential equation for a spinor. However, our entire analysis has been restricted by a special choice of coordinate system. We now wish to show that the theory is invariant under canonical transformations.

[12] See reference 2.

We are interested in the transformation law for the spinor wave function ψ. It proves somewhat awkward to find this transformation law from the quasi-classical theory presented in Secs. III–V. In those sections we made use of the reduced configuration space of the variables q^i, while for our present purposes it proves more convenient to introduce the expanded configuration space of the coordinate variables q^i and ω.

In terms of these variables the canonical theory takes its usual form. Hamilton's equations are

$$d\mathbf{q}/dt=\partial\mathcal{H}/\partial\mathbf{p},\quad d\mathbf{p}/dt=-\partial\mathcal{H}/\partial\mathbf{q},$$
$$-d\omega/dt=\partial\mathcal{H}/\partial\xi,\quad d\xi/dt=\partial\mathcal{H}/\partial\omega, \tag{52}$$

with

$$\mathcal{H}=\frac{1}{2\mu}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2-e\phi+\frac{e}{\mu c}\times[(s^2-\xi^2)^{\frac{1}{2}}(B_x\cos\omega+B_y\sin\omega)+\xi B_z]. \tag{53}$$

Calculation shows that Hamilton's equations, (52), are identical with the equations of motion, (14). The corresponding Hamilton-Jacobi equation is

$$\partial\mathcal{S}/\partial t+\mathcal{H}=0, \tag{54}$$

where $\mathbf{p}=\partial\mathcal{S}/\partial\mathbf{q}$ and $\xi=-\partial\mathcal{S}/\partial\omega$.

A complete integral of this equation is of the form,

$$\mathcal{S}=\mathcal{S}(\mathbf{q},\omega,t,\alpha_\mu), \tag{55}$$

where the α_μ are four constants of integration. Four other constants may be found by differentiating $\mathcal{S}$ with respect to the α_μ,

$$\beta_\mu=\partial\mathcal{S}/\partial\alpha_\mu. \tag{56}$$

Choose one of these equations for the four constants, say $\beta_4=\partial\mathcal{S}/\partial\alpha_4$, and solve it for the variable ω. ω then becomes a function of q^i, α_μ, β_4, and t. The spin coordinate is now a function of the space-time variables and five constants of the motion. From the equation $\xi=-\partial\mathcal{S}/\partial\omega$, ξ may also be found as a function of these same variables.

We now assume that $\mathcal{S}$ depends implicitly on these same space-time variables through ω, so that

$$d\mathcal{S}/dt=\partial\mathcal{S}/\partial t+(\partial\mathcal{S}/\partial\omega)(\partial\omega/\partial t)=\partial\mathcal{S}/\partial t-\xi\partial\omega/\partial t,$$

or

$$\partial\mathcal{S}/\partial t=d\mathcal{S}/dt+\xi\partial\omega/\partial t. \tag{57}$$

Similarly, differentiation with respect to the spatial variables q^i yields

$$\partial\mathcal{S}/\partial q^i=d\mathcal{S}/dq^i+\xi\partial\omega/\partial q^i. \tag{58}$$

We now insert ω as a function of its variables in $\mathcal{S}$, so that $\mathcal{S}$ becomes an explicit function of the space-time variables,

$$\mathcal{S}(\mathbf{q},t,\omega(q,t,\alpha_\mu,\beta_4),\alpha_\mu)=S(\mathbf{q},t,\alpha_\mu,\beta_4). \tag{59}$$

After we have made this substitution we find

$$d\mathcal{S}/dt = \partial S/\partial t = \partial\mathcal{S}/\partial t - \xi\partial\omega/\partial t, \tag{60}$$

and

$$d\mathcal{S}/d\mathbf{q} = \partial S/\partial\mathbf{q} = \partial\mathcal{S}/\partial\mathbf{q} - \xi\partial\omega/\partial\mathbf{q}. \tag{61}$$

In terms of the quantities S, ξ, and ω, which are functions of the space-time variables, we find that the Hamilton-Jacobi equation, (54), becomes

$$\frac{\partial S}{\partial t} + \xi\frac{\partial\omega}{\partial t} + \frac{1}{2\mu}\left(\nabla S + \xi\nabla\omega + \frac{e}{c}\mathbf{A}\right)^2 - e\phi + H_{\rm sp} = 0. \tag{62}$$

Equation (62) is the same relation (14a) we found by another method in Sec. III. The Eqs. (14b) and (14c) also follow from the corresponding two relations in Hamilton's equations, (52).

Thus, the modified Hamilton-Jacobi theory is equivalent to the usual Hamiltonian theory in an expanded configuration space. The transformation laws of the modified theory are easily found once we know the laws of transformation of the Hamiltonian theory of Eqs. (52)–(54). Under canonical transformation, the Hamilton-Jacobi function $\mathcal{S}$ is transformed into $\mathcal{S}'$ by means of the generating function $\mathfrak{F}$,

$$\mathcal{S}(\mathbf{q},\omega,t,\alpha_\mu) = \mathcal{S}'(\mathbf{Q},\Omega,t,\alpha_\mu) + \mathfrak{F}(q,Q,\omega,\Omega,t). \tag{63}$$

The transformation equations are

$$\begin{aligned} &\mathbf{p} = \partial\mathcal{S}/\partial\mathbf{q} = \partial\mathfrak{F}/\partial\mathbf{q}, && \mathbf{P} = \partial\mathcal{S}'/\partial\mathbf{Q} = -\partial\mathfrak{F}/\partial\mathbf{Q},\\ &\xi = -\partial\mathcal{S}/\partial\omega = -\partial\mathfrak{F}/\partial\omega, && \Xi = -\partial\mathcal{S}'/\partial\Omega = \partial\mathfrak{F}/\partial\Omega,\\ &\mathcal{H} = \mathcal{H}' + \partial\mathfrak{F}/\partial t, \end{aligned} \tag{64}$$

where the transformed variables Q, Ω, P, Ξ correspond to the original variables q, ω, p, ξ.

The transformed spinor ψ' may be found by introducing the new constant $\beta_4' = \partial\mathcal{S}'/\partial\alpha_4$, and then finding Ω, Ξ, and S' as functions of the space-time variables. The density $D' = R'^2$,

$$D' = \left|\left|\frac{\partial}{\partial Q^i}\left(\frac{\partial S}{\partial\alpha_k} + \Xi\frac{\partial\Omega}{\partial\alpha_k}\right)\right|\right|, \tag{65}$$

satisfies a continuity equation in the reduced configuration space of the variables Q^i,

$$\frac{\partial D'}{\partial t} + \frac{\partial}{\partial Q^k}\left(D'\frac{\partial H'}{\partial P_k}\right) = 0, \tag{66}$$

with $P_i = \partial S'/\partial Q^i + \Xi\partial\Omega/\partial Q^i$.

The transformed spinor ψ' is

$$\psi' = R' e^{iS'/\hbar}\begin{pmatrix} \cos(\Theta/2)\exp(i\Omega/2) \\ i\sin(\Theta/2)\exp(-i\Omega/2) \end{pmatrix}, \tag{67}$$

with $(\hbar/2)\cos\Theta = \Xi$.

The transformed Hamiltonian operator corresponding to the Hamiltonian operator of Eq. (34) may be found by requiring that the equations resulting from this operator acting on ψ' lead to the four transformed equations for R', S', Ω, and Ξ.

IX. CHARGE CONJUGATE SOLUTIONS

The spinor ψ,

$$\psi = R e^{iS/\hbar}\begin{pmatrix} \cos(\theta'/2)\exp(i\omega/2) \\ i\sin(\theta'/2)\exp(-i\omega/2) \end{pmatrix}, \tag{32}$$

satisfies the quasi-classical equation, (34). The classical function $\psi_{\rm c.c.}$,

$$\psi_{\rm c.c.} = R e^{-iS/\hbar}\begin{pmatrix} i\sin(\theta'/2)\exp(i\omega/2) \\ \cos(\theta'/2)\exp(-i\omega/2) \end{pmatrix}, \tag{68}$$

satisfies the same equation if e is replaced by $-e$. $\psi_{\rm c.c.}$ is the charge-conjugate solution. The solution $\psi_{\rm c.c.}$ is not to be confused with the solution $\bar{\psi}$ of Eq. (45). The latter is the charge-conjugate solution for the spin variables alone.

X. CONCLUSION

In this paper we have shown how one might construct a classical Hamilton-Jacobi theory of a spinning dipole so that this theory is the WKB approximation to the Pauli equation. It thus appears that for many problems of interest in nonrelativistic quantum theory, purely classical methods are adequate to obtain the asymptotic solutions of the Schrödinger or Pauli wave equations. In a future paper we shall deal with the Dirac equation by similar methods.