excited level at 140 kev. Their first excited level in Sn^{117} is also a little low in energy compared with the experimental value. The 7/2+ and 5/2+ observed states in Sn^{115} are quite high in energy in comparison with their lowest predicted 7/2+, and 5/2+ levels and may probably not be pure quasi-particle states. The second 7/2+ excited level due to Kisslinger and Sorensen has a large phonon admixture and lies at 1.5 Mev above the 1/2+ ground state. Our second excited level with a spin of 7/2+ could be the second 7/2+ excited level of their level structure of Sn^{115}.

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Use of Normal Coordinates in Nuclear Physics*

F. J. SMITH† AND J. G. BRENNAN
Catholic University of America, Washington, D. C.
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It is shown that normal coordinates are not difficult to use in nuclear theory if it is assumed that the masses of the nucleons are equal. In the shell model with harmonic oscillator functions they have the advantage that they exclude the spurious states due to the motion of the center of mass.

THE use of normal coordinates to describe a system of particles is usually restricted by cumbersome algebra. However, this algebra is much simplified in nuclear physics where the masses of the nucleons may often be taken as equal.

Consider a system of particles with equal masses moving in the field of a potential V. If $\mathbf{r}_i$ is the position vector of the ith particle relative to some fixed origin, the Schrödinger equation of the system is

$$[-\tfrac{1}{2}\textstyle\sum_i \nabla_i^2+V(\mathbf{r}_1,\mathbf{r}_2,\cdots)]\psi(\mathbf{r}_i)=E\psi(\mathbf{r}_i). \quad (1)$$

To separate the motion of the center of mass, a linear transformation is made to a set of normal coordinates, one of which is the position vector of the center of mass, $\mathbf{R}$. The matrix equation of the transformation can be written

$$\boldsymbol{\varrho}=B\mathbf{r}', \quad (2)$$

where $\boldsymbol{\varrho}$ represents the set of normal coordinates $\boldsymbol{\varrho}_j$, and $\mathbf{r}'$ the set of coordinates $\mathbf{r}_i$. The matrix, B, is nonsingular. If $\partial/\partial\boldsymbol{\varrho}$ and $\partial/\partial\mathbf{r}'$ represent the sets of gradients $\partial/\partial\boldsymbol{\varrho}_j$ and $\partial/\partial\mathbf{r}_i$, then

$$\frac{\partial}{\partial\boldsymbol{\varrho}_j}=\sum_i\left(\frac{\partial\mathbf{r}_i}{\partial\boldsymbol{\varrho}_j}\right)\frac{\partial}{\partial\mathbf{r}_i}.$$

So

$$\frac{\partial}{\partial\boldsymbol{\varrho}}=(B^{-1})^\dagger\frac{\partial}{\partial\mathbf{r}'},$$

and

$$\sum_i \nabla_i^2=\left(\frac{\partial}{\partial\mathbf{r}'}\right)^\dagger\left(\frac{\partial}{\partial\mathbf{r}'}\right)=\left(\frac{\partial}{\partial\boldsymbol{\varrho}}\right)^\dagger BB^\dagger\left(\frac{\partial}{\partial\boldsymbol{\varrho}}\right). \quad (3)$$

If there are to be no cross derivatives, $BB^\dagger$ must be diagonal. If $\boldsymbol{\varrho}_n$ is chosen as $\boldsymbol{\varrho}_n=n\mathbf{R}$, where n is the total number of particles in the system, then $B_{ni}=1$ for all i. It follows that

$$\textstyle\sum_i B_{ji}=0 \quad \text{if} \quad j\neq n,$$

and that

$$\textstyle\sum_i B_{ji}B_{ki}=0 \quad \text{if} \quad k\neq j.$$

These conditions may be satisfied by numerically simple matrices: for a single system of four particles the matrices take the form

$$B=\begin{bmatrix}1&-1&0&0\\1&1&-2&0\\1&1&1&-3\\1&1&1&1\end{bmatrix},$$

$$B^{-1}=\begin{bmatrix}1/2&1/2\times3&1/3\times4&1/4\\-1/2&1/2\times3&1/3\times4&1/4\\0&-1/3&1/3\times4&1/4\\0&0&-1/4&1/4\end{bmatrix}, \quad (4)$$

$$BB^\dagger=\begin{bmatrix}1\times2&0&0&0\\0&2\times3&0&0\\0&0&3\times4&0\\0&0&0&4\end{bmatrix}.$$

If the system consists of two (or more) groups, as in a collision problem, it is useful also to separate out the motion of one group relative to the other. This requires

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† Now at the Queen's University of Belfast, North Ireland.

the use of the position vector, $\mathbf{r}$, joining the centers of mass of the groups, as a normal coordinate. This is only possible, in general, if the masses of the particles are equal; the two row vectors of B corresponding to $\mathbf{R}$ and $\mathbf{r}$ would not be orthogonal.

If the particles have equal masses and if the system consists of two groups, one with n_1 particles and the other with n_2 particles, then $\mathbf{r}$ takes the form

$$\mathbf{r}=\frac{1}{n_1}\sum_{i=1}^{i=n_1}\mathbf{r}_i+\frac{1}{n_2}\sum_{i=n_1+1}^{i=n_1+n_2}\mathbf{r}_i. \tag{5}$$

This imposes new restrictions on the matrix B which can be easily satisfied. For example, if $n_1=2$ and $n_2=3$, the matrix may take the form

$$B=\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ -3 & -3 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$BB^{\dagger}=\begin{bmatrix} 1\times 2 & 0 & 0 & 0 & 0 \\ 0 & 1\times 2 & 0 & 0 & 0 \\ 0 & 0 & 2\times 3 & 0 & 0 \\ 0 & 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}. \tag{6}$$

Using matrices such as those in Eqs. (4) and (6), transformations can be easily made from one system of coordinates to the other for simple and multiple systems of coordinates.

The advantage of using such normal coordinates has been demonstrated in resonating group structure by Butcher in his formulation of alpha-alpha scattering.[1] However, normal coordinates have apparently not been used in shell-model theory although they are not difficult to work with; yet they separate out the motion of the center of mass and exclude the spurious states found by Elliott and Skyrme.[2] This is demonstrated below where some of the results of Elliott and Skyrme are reproduced.

If it is assumed that the system of nucleons are moving independently of one another in the field of a simple harmonic oscillator potential, the Hamiltonian may be written[2]

$$\begin{aligned} H&=-\tfrac{1}{2}\textstyle\sum_i \nabla_i^2+\tfrac{1}{2}\sum_i(\mathbf{r}_i-\mathbf{R})^2 \\ &=-\tfrac{1}{2}\textstyle\sum_i \nabla_i^2+\tfrac{1}{2}\sum_i \mathbf{r}_i^2-\tfrac{1}{2}nR^2, \end{aligned} \tag{7}$$

where the units are appropriately chosen. If one changes to normal coordinates using a matrix B similar to that in Eq. (4), then since $(BB^{\dagger})_{ik}=\delta_{ik}k(k+1)$ for $k\neq n$, where δ_{ik} is the Kronecker delta function,

$$\sum_i r_i^2=\varrho^{\dagger}(B^{-1})^{\dagger}B^{-1}\varrho=\sum_{k=1}^{k=n-1}\rho_k^2(k+k^2)^{-1}+nR^2,$$

and

$$\sum_i \nabla_i^2=\left(\frac{\partial}{\partial\varrho}\right)^{\dagger}BB^{\dagger}\left(\frac{\partial}{\partial\varrho}\right)=\sum_{k=1}^{k=n-1}\nabla_{\rho_k}^2(k+k^2)+n\nabla_R^2.$$

So

$$\begin{aligned} H=&-\tfrac{1}{2}n\nabla_R^2 \\ &+\{-\tfrac{1}{2}\sum_{k=1}^{k=n-1}(k+k^2)\nabla_{\rho_k}^2+\tfrac{1}{2}\sum_{k=1}^{k=n-1}\rho_k^2/(k^2+k)\}. \end{aligned}$$

The Schrödinger equation is now easily separated and solved. The eigenfunctions for the internal states of the nucleus are given by

$$\begin{aligned} \psi=&\exp\{-\tfrac{1}{2}\sum_k \rho_k^2/(k^2+k)\} \\ &\times\prod_{k=1}^{k=n-1}[H_{m_1^k}(\xi_k)H_{m_2^k}(\eta_k)H_{m_3^k}(\zeta_k)], \end{aligned}$$

where

$$\varrho_k=(k^2+k)^{\frac{1}{2}}\{\mathbf{i}\xi_k+\mathbf{j}\eta_k+\mathbf{k}\zeta_k\},$$

and $H_m(x)$ is a Hermite polynomial of order m. The energy values are

$$E=N+(n-1)\tfrac{3}{2},$$

where

$$N=\sum_{k=1}^{k=n-1}(m_1^k+m_2^k+m_3^k),$$

the degree of the polynomial of the wave function. These wave functions are linear combinations of degenerate shell-model wave functions[3] corresponding to the shell-model energy level, $N+n(3/2)$. These results are in agreement with those of Elliott and Skyrme.[2]

The normal coordinate energy level has a degeneracy $(N+3n-4)!/N!(3n-4)!$, while the corresponding degeneracy in the shell model is higher, $(N+3n-1)!/N!(3n-1)!$ The extra number corresponds to those linear combinations of shell-model wave functions which are spurious. All of these are automatically excluded if normal coordinates are used.

[1] A. C. Butcher and J. M. McNamee, Proc. Phys. Soc. (London) **74**, 529 (1959).

[2] J. P. Elliott and T. H. R. Skyrme, Proc. Roy. Soc. (London) **A232**, 561 (1955).

[3] This can be proved using the generating functions for Hermite polynomials; see I. Bloch and Yu-Chang Hsieh, Phys. Rev. **96**, 382 (1954).