Theory of Superconductivity

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Recently Eliashberg derived the theory of superconductivity by the Green's function technique. The energy-gap equation was deduced by a self-consistent calculation. The result is similar to the integral equation of Bardeen, Cooper, and Schrieffer except that the effective electron interaction potential is different from the Bardeen-Pines potential as well as the Bogoliubov potential. The purpose of this paper is to discuss these differences. It shows that the "dangerous terms" in the Bogoliubov theory may be calculated in a different manner. The compensation of these dangerous terms leads to the result of Eliashberg. Therefore the self-consistent calculation of Eliashberg is but a different interpretation of the Bogoliubov's idea. However, it is argued that the most reliable way of treating the problem should be the variational method because it gives the lowest ground-state energy. This method gives all the results of Bogoliubov in the lowest order of approximation, but is free from arbitrary interpretations.

INTRODUCTION

An important advance in the theory of superconductivity was made by Bardeen, Cooper, and Schrieffer[1] when they discovered the way of describing the ground state of a superconductor in terms of paired electrons. It was found earlier by Bardeen and Pines[2] that there is an attractive interaction between electrons near the Fermi level as a result of exchange of virtual phonons. The theory of BCS shows that by pairing electron states of opposite momenta and opposite spins, the energy of the electron system can be substantially lowered when the attractive interaction exists. They identified such a state as the superconducting state and proved that most of the properties of a superconductor can be described by a set of wave functions constructed by this principle of pairing.

Bogoliubov[3] proposed a more fundamental way of deriving the theory by using the elegant technique of Bogoliubov transformation to the electron-phonon Hamiltonian of Fröhlich.[4] In this language the BCS ground state is the vacuum state of "quasi-particles" and the BCS excited states are the states containing one, two, etc., quasi-particles.[5] The arbitrary parameters involved in the Bogoliubov transformation are related to the energy gap and are determined by the logic of "compensation of dangerous terms." The condition of compensation is equivalent to the BCS integral equation except that the effective electron interaction potential is different in form.

It is felt that this theory has two weaknesses. In the first place it is easy to show that the dangerous terms are no longer dangerous after energy renormalization. Hence the basic reasoning of the compensation method is rather doubtful. Secondly, it will be shown in the text of this paper that this method is subject to arbitrary interpretations. A different way of evaluating the dangerous terms yields a third form of the effective potential.

The powerful method of Green's functions has also been applied to the theory of superconductivity. Gor'kov[6] applied the method to the Bardeen-Pines Hamiltonian and obtained a theory equivalent to BCS. Eliashberg[7] started from the Fröhlich Hamiltonian and obtained an energy gap equation which is equivalent to the one derived in the present paper by the modified compensation method.

This paper shows how the result of Eliashberg can be derived by the compensation method if the dangerous terms are calculated differently. Thus it provides a simple way of understanding the complicated theory. It also demonstrates that the compensation method is not as reliable as the straightforward variational method because of the arbitrariness involved in its interpretation.

BASIC THEORY

The basic Hamiltonian for the electron-phonon system is (in a system of units with $\hbar=1$)

$$H=\sum_{\mathbf{k},s}\epsilon_k c_{\mathbf{k}s}{}^*c_{\mathbf{k}s}+\sum_{\mathbf{q}}\omega_q b_{\mathbf{q}}{}^*b_{\mathbf{q}}+H_I,$$

where

$$H_I=\sum_{\substack{\mathbf{k},\mathbf{k}'s\\ \mathbf{q}=\mathbf{k}'-\mathbf{k}}}M(\mathbf{k},\mathbf{k}')c_{\mathbf{k}s}{}^*c_{\mathbf{k}'s}b_{\mathbf{q}}{}^*+\text{Herm. conj.}\quad(1)$$

The possible values of $\mathbf{q}$ are restricted by $|\mathbf{q}|\leq q_M$, the maximum wave number of phonons. The electron energy is measured from the Fermi level. The invariant properties of H require

$$M(\mathbf{k},\mathbf{k}')=M(\mathbf{k}',\mathbf{k})^*=M(-\mathbf{k}',-\mathbf{k}).\quad(2)$$

Following Bogoliubov one makes the transformation

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957). Hereafter referred to as BCS.
[2] J. Bardeen and D. Pines, Phys. Rev. **99**, 1140 (1955).
[3] N. N. Bogoliubov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 58 (1958), Soviet Phys.—JETP **7**, 41 (1958); see also G. Rickayzen, Phys. Rev. **111**, 817 (1958).
[4] H. Fröhlich, Phys. Rev. **79**, 845 (1950), Proc. Roy. Soc. (London) **A215**, 291 (1952).
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to quasi-particle operators defined by

$$\alpha_{\mathbf{k}}=\cos\theta_k\, c_{\mathbf{k}\uparrow}-\sin\theta_k\, c_{-\mathbf{k}\downarrow}^*,$$
$$\beta_{-\mathbf{k}}=\cos\theta_k\, c_{-\mathbf{k}\downarrow}+\sin\theta_k\, c_{\mathbf{k}\uparrow}^*, \tag{3}$$

where the θ_k's are real. It is easy to verify that α and β are fermion operators. The resulting Hamiltonian can be shown to be

$$H=U+H_0+H_1+H_2, \tag{4}$$

where

$$U=2\sum_{\mathbf{k}}\epsilon_k\sin^2\theta_k,$$

$$H_0=\sum_{\mathbf{k}}E_k(\alpha_{\mathbf{k}}^*\alpha_{\mathbf{k}}+\beta_{\mathbf{k}}^*\beta_{\mathbf{k}})+\sum_{\mathbf{q}}\Omega_q b_{\mathbf{q}}^*b_{\mathbf{q}},$$

$$H_1=\sum_{\mathbf{k},\mathbf{k}'}M(\mathbf{k},\mathbf{k}')\sin(\theta_k+\theta_{k'})(\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*+\beta_{-\mathbf{k}}\alpha_{\mathbf{k}'})\times(b_{\mathbf{q}}^*+b_{-\mathbf{q}})+\sum_{\mathbf{k},\mathbf{k}'}M(\mathbf{k},\mathbf{k}')\cos(\theta_k+\theta_{k'})\times(\alpha_{\mathbf{k}}^*\alpha_{\mathbf{k}'}+\beta_{\mathbf{k}}^*\beta_{\mathbf{k}'})(b_{\mathbf{q}}^*+b_{-\mathbf{q}}),$$

$$H_2=\sum_{\mathbf{k}}\epsilon_k\sin2\theta_k(\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}}^*+\beta_{-\mathbf{k}}\alpha_{\mathbf{k}})+\sum_{\mathbf{k}}[\epsilon_k\cos2\theta_k-E_k](\alpha_{\mathbf{k}}^*\alpha_{\mathbf{k}}+\beta_{\mathbf{k}}^*\beta_{\mathbf{k}})+\sum_{\mathbf{q}}[\omega_q-\Omega_q]b_{\mathbf{q}}^*b_{\mathbf{q}}.$$

The term H_0 is added in the Hamiltonian and then subtracted out in H_2. The purpose of doing this is to perform the energy renormalization in a self-consistent manner. The renormalized particle and phonon energies are E_k and Ω_q, respectively.

The effect of electron-phonon interaction is investigated by a canonical transformation method similar to that of Bardeen and Pines.[2] One looks for a transformation $\exp(-iS)$ such that, to the lowest order, it eliminates from the new Hamiltonlan,

$$H'=\exp(-iS)H\exp(iS), \tag{5}$$

the terms linear in $b_{\mathbf{q}}^*$ and $b_{\mathbf{q}}$. Expanding the exponentials, one finds that

$$H'=U+H_0+H_1+H_2+i[H_0,S]+\{i[H_1,S]-\tfrac{1}{2}[[H_0,S],S]\}+\cdots.$$

The appropriate generating function is found from

$$H_1+i[H_0,S]=0, \tag{6}$$

because this eliminates the term H_1 which is linear in phonon operators. And the new Hamiltonian is therefore

$$H'=U+H_0+H_2+\tfrac{1}{2}i[H_1,S]+\cdots.$$

From Eq. (6) it is rather straightforward to find

$$S=\sum_{\mathbf{k},\mathbf{k}'}[A(\mathbf{k},\mathbf{k}')\alpha_{\mathbf{k}}^*\alpha_{\mathbf{k}'}+B(\mathbf{k},\mathbf{k}')\beta_{\mathbf{k}}^*\beta_{\mathbf{k}'}+C(\mathbf{k},\mathbf{k}')\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*+D(\mathbf{k},\mathbf{k}')\beta_{-\mathbf{k}}\alpha_{\mathbf{k}'}]b_{\mathbf{q}}^*+\text{Herm. conj.}, \tag{8}$$

where

$$A(\mathbf{k},\mathbf{k}')=\frac{iM(\mathbf{k},\mathbf{k}')\cos(\theta_k+\theta_{k'})}{E_k-E_{k'}+\Omega_q}=B(\mathbf{k},\mathbf{k}'),$$

$$C(\mathbf{k},\mathbf{k}')=\frac{iM(\mathbf{k},\mathbf{k}')\sin(\theta_k+\theta_{k'})}{E_k+E_{k'}+\Omega_q}, \tag{9}$$

$$D(\mathbf{k},\mathbf{k}')=\frac{iM(\mathbf{k},\mathbf{k}')\sin(\theta_k+\theta_{k'})}{-E_k-E_{k'}+\Omega_q}.$$

The commutator $[H_1,S]$ consists of products of four fermion operators and two boson operators. It may be written as a sum of commutators, of which one term is

$$[\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}}^*b_{\mathbf{q}}^*,\beta_{-\mathbf{p}}\alpha_{\mathbf{p}'}b_{\mathbf{q}'}]=\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*b_{\mathbf{q}}^*\beta_{-\mathbf{p}}\alpha_{\mathbf{p}'}b_{\mathbf{q}'}-\beta_{-\mathbf{p}}\alpha_{\mathbf{p}'}b_{\mathbf{q}'}\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*b_{\mathbf{q}}^*. \tag{10}$$

In the second product one applies the commutation rules of boson and fermion operators to rearrange the operators in the normal order, i.e., all creation operators stand to the left of all annihilation operators. The reason for doing this will become clear later. The result of this manipulation is

$$\beta_{-\mathbf{p}}\alpha_{\mathbf{p}'}b_{\mathbf{q}'}\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*b_{\mathbf{q}}^*=\delta_{\mathbf{k}\mathbf{p}'}\delta_{\mathbf{k}'\mathbf{p}}-\delta_{\mathbf{k}\mathbf{p}'}\delta_{\mathbf{k}'\mathbf{p}}(\alpha_{\mathbf{k}}^*\alpha_{\mathbf{k}}+\beta_{-\mathbf{k}'}^*\beta_{-\mathbf{k}'}-b_{\mathbf{q}}^*b_{\mathbf{q}})+\delta_{\mathbf{q}\mathbf{q}'}\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*\beta_{-\mathbf{p}}\alpha_{\mathbf{p}'}-\delta_{\mathbf{k}'\mathbf{p}}\alpha_{\mathbf{k}}^*\alpha_{\mathbf{p}'}b_{\mathbf{q}}^*b_{\mathbf{q}'}-\delta_{\mathbf{k}\mathbf{p}'}\beta_{-\mathbf{k}'}^*\beta_{-\mathbf{p}}b_{\mathbf{q}}^*b_{\mathbf{q}'}+\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*\beta_{-\mathbf{p}}\alpha_{\mathbf{p}'}b_{\mathbf{q}}^*b_{\mathbf{q}'}. \tag{11}$$

When expanding the commutator the last term on the right-hand side of Eq. (11) is cancelled out. The remaining terms are discussed separately. The first term contains no operators and is the result of contracting operators in pairs. In analogy to quantum electrodynamics a contracted pair corresponds to an internal line of a Feynman diagram, i.e., virtual creation and annihilation of a particle. Therefore this term corresponds to a vacuum fluctuation process. Similarly the next three terms are the self-energy terms of quasi-particles and phonons. The last group of three terms represent collisions between particles and between particles and phonons. The nature of the vertices of the diagrams are given by the members of the commutator under consideration. For this commutator the vertices denote separately the excitation out of vacuum and the annihilation of two particles and a phonon. Therefore the various terms in the normal order expansion are denoted by the diagrams in Fig. 1. The purpose of rearranging the operators in normal order is to separate out the contribution of each diagram. Similarly the commutator below has the normal order expansion:

$$[\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*b_{\mathbf{q}'}^*,\beta_{-\mathbf{p}'}^*\beta_{-\mathbf{p}}b_{\mathbf{q}}]=-\delta_{\mathbf{k}\mathbf{p}'}\delta_{\mathbf{k}'\mathbf{p}}\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}}^*-\delta_{\mathbf{q}\mathbf{q}'}\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*\beta_{-\mathbf{p}'}^*\beta_{-\mathbf{p}}-\delta_{\mathbf{p}\mathbf{k}'}\alpha_{\mathbf{k}}^*\beta_{-\mathbf{p}'}^*b_{\mathbf{q}}^*b_{\mathbf{q}'}, \tag{12}$$

and the Feynman diagrams in Fig. 2. The diagram Fig. 2(a) is one of the dangerous diagrams of Bogoliubov. In this manner all the commutators involved in $[H_1,S]$

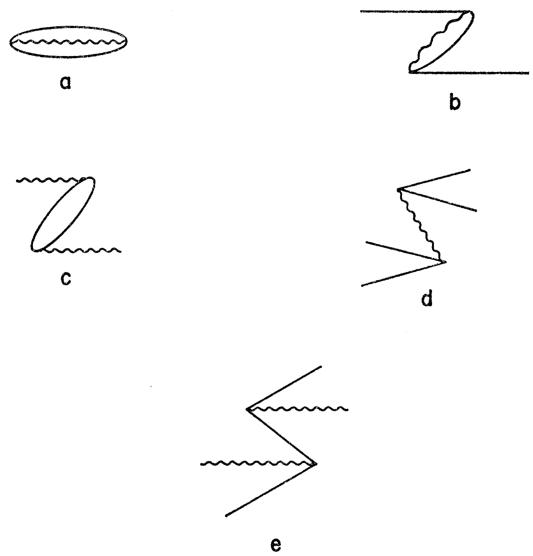

FIG. 1. The Feynman diagrams of the normal order expansion of the commutator in Eq. (10).

can be expanded and identified with a Feynman diagram. In general there are many terms in the expansion of $[H_1,S]$ corresponding to the same diagram.

In Bogoliubov theory special attention is given to the pair creation terms of Fig. 2(a). These terms come from expanding the commutators

$$[\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*b_{\mathbf{q}}^*,\beta_{-\mathbf{p}'}^*\beta_{-\mathbf{p}}b_{\mathbf{q}'}],$$

and

$$[\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}'}^*b_{\mathbf{q}}^*,\alpha_{\mathbf{p}}^*\alpha_{\mathbf{p}'}b_{\mathbf{q}'}].$$

In each case one may take the first member of the commutator from H_1 and the second member from S or vice versa. The first choice gives an energy denominator that appears in the coefficients $A(\mathbf{k},\mathbf{k}')$ and $B(\mathbf{k},\mathbf{k}')$. The second choice gives an energy denominator that appears in $C(\mathbf{k},\mathbf{k}')$. Collecting these results one finds the coefficient of $\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}}^*$ in the expansion of $\frac{1}{2}i[H_1,S]$ to be

$$\begin{aligned}\tfrac{1}{2}i\sum_{\mathbf{k}'}\{&-M(\mathbf{k},\mathbf{k}')\sin(\theta_k+\theta_{k'})B(-\mathbf{k}',-\mathbf{k})^*\\&-M(\mathbf{k}',\mathbf{k})\sin(\theta_k+\theta_{k'})A(\mathbf{k}',\mathbf{k})^*\\&+M(\mathbf{k},\mathbf{k}')\cos(\theta_k+\theta_{k'})C(\mathbf{k}',\mathbf{k})\\&+M(-\mathbf{k},-\mathbf{k}')\cos(\theta_k+\theta_{k'})C(\mathbf{k},\mathbf{k}')\}\\=-\tfrac{1}{2}\sum_{\mathbf{k}'}\bigg\{&\frac{|M(\mathbf{k},\mathbf{k}')|^2\sin2(\theta_k+\theta_{k'})}{E_{k'}-E_k+\Omega_q}\\&+\frac{|M(\mathbf{k},\mathbf{k}')|^2\sin2(\theta_k+\theta_{k'})}{E_{k'}+E_k+\Omega_q}\bigg\}.\end{aligned}\tag{13}$$

The pair-annihilation term $\beta_{-\mathbf{k}}\alpha_{\mathbf{k}}$ has the same coefficient.

According to Bogoliubov the pair-creation terms give vanishing energy denominators in higher order perturbation calculations and should therefore be compensated out. The compensation is carried out up to the second order by requiring the quantity in Eq. (13) to cancel the coefficient of the corresponding term in H_2. This gives the following condition for every $\mathbf{k}$:

$$\epsilon_k\sin2\theta_k-\tfrac{1}{2}\sum_{\mathbf{k}'}\left[\frac{|M(\mathbf{k},\mathbf{k}')|^2}{E_{k'}-E_k+\Omega_q}+\frac{|M(\mathbf{k},\mathbf{k}')|^2}{E_{k'}+E_k+\Omega_q}\right]\sin2(\theta_k+\theta_{k'})=0.\tag{14}$$

The thus far undetermined parameters θ_k are determined by this set of equations. It is easy to reduce Eq. (14) to the integral equation of BCS for the energy gap function Δ_k, namely

$$\Delta_{\mathbf{k}}=\tfrac{1}{2}\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\frac{\Delta_{\mathbf{k}'}}{(\Delta_{\mathbf{k}'}^2+\xi_{\mathbf{k}'}^2)^{\frac{1}{2}}},\tag{15}$$

where

$$V_{\mathbf{k}\mathbf{k}'}=\frac{|M(\mathbf{k},\mathbf{k}')|^2}{E_{k'}-E_k+\Omega_q}+\frac{|M(\mathbf{k},\mathbf{k}')|^2}{E_{k'}+E_k+\Omega_q},\tag{16}$$

$$\xi_{\mathbf{k}}=\epsilon_k-\tfrac{1}{2}\sum_{\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}\cos2\theta_k,\tag{17}$$

$$\Delta_{\mathbf{k}}=\xi_{\mathbf{k}}\tan2\theta_k.\tag{18}$$

The form of the potential $V_{\mathbf{k}\mathbf{k}'}$ is equivalent to the one found by Eliashberg.[7] Therefore the result of Eliashberg can be derived by the compensation method if the pair creation terms are calculated differently.

The analogy between the present calculation and the self-consistent calculation of Eliashberg can be illustrated as follows. In the formalism of Eliashberg the unperturbed Hamiltonian is taken as non-diagonal in the representation of free electrons. Certain quantities are added to the electron Hamiltonian and then sub-

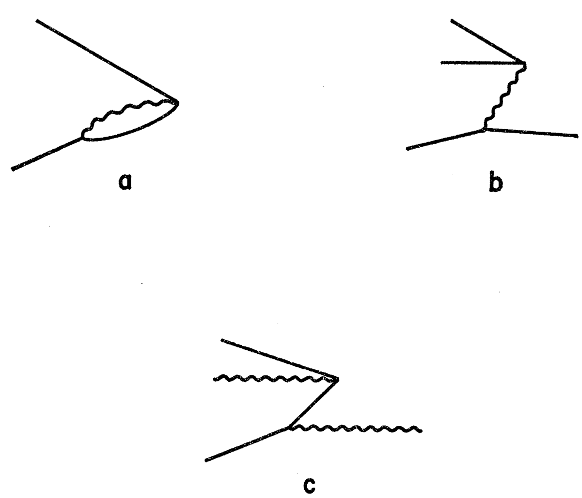

FIG. 2. The Feynman diagrams of the normal order expansion of the commutator in Eq. (12).

tracted out, i.e., in the notation of reference 7,

$$H_0=\tilde{H}_{\mathrm{el}}+H_{\mathrm{ph}},$$

where

$$\tilde{H}_{\mathrm{el}}=H_{\mathrm{el}}+(\tilde{H}_{\mathrm{el}}-H_{\mathrm{el}}).$$

It is required that $\tilde{H}_{\mathrm{el}}$ should be diagonalizable by a transformation to quasi-particle representation, and that the residual term $\tilde{H}_{\mathrm{el}}-H_{\mathrm{el}}$ should cancel the self-energy part due to electron-phonon interaction. Therefore H_0 is entirely equivalent to the unperturbed Hamiltonian in the present calculation and $\tilde{H}_{\mathrm{el}}-H_{\mathrm{el}}$ the renormalization Hamiltonian H_2. The cancellation of the diagonal part of $\tilde{H}_{\mathrm{el}}-H_{\mathrm{el}}$ is just the energy renormalization process for bare electrons (not quasi-particles). The cancellation of the nondiagonal part is in principle equivalent to the compensation of pair creation terms since they all come from the assumption of pair correlation. Therefore the two methods are basically the same but expressed in different notations.

VARIATIONAL METHOD

The BCS wave function is a trial wave function containing a set of parameters. In the present calculation these parameters are contained in the Bogoliubov transformation as θ_k's. The compensation method gives one way of determining the parameters, while the variational method is another way. However the variational method guarantees the smallest energy attainable. Therefore, in principle, it should be the better method. Such a calculation is the subject of this section.

The correction to the ground state energy due to electron-phonon interaction comes entirely from the vacuum-fluctuation terms discussed in the last section. To the second order the ground-state energy is

$$\begin{aligned}E_0&=U+\tfrac{1}{2}i\langle 0|[H_1,S]|0\rangle\\&=2\sum_{\mathbf{k}}\epsilon_k\sin^2\theta_k-\sum_{\mathbf{k},\mathbf{k}'}\frac{|M(\mathbf{k},\mathbf{k}')|^2\sin^2(\theta_k+\theta_{k'})}{E_{k'}+E_k+\Omega_q}.\end{aligned}\tag{19}$$

One should choose θ_k such that the energy is a minimum. The condition of minimization is easily found to be

$$\epsilon_k\sin2\theta_k-\sum_{\mathbf{k}'}\frac{|M(\mathbf{k},\mathbf{k}')|^2\sin2(\theta_k+\theta_{k'})}{E_{k'}+E_k+\Omega_q}=0.\tag{20}$$

This is equivalent to the compensation condition in reference 3. The connection between the Bogoliubov method and the variational method was first pointed out by Yosida.[8]

The energy of an elementary excitation is determined as follows: One starts from a wave vector of a state containing one quasi-particle and evaluates the energy of the state by a perturbation calculation. Then one minimizes this energy with respect to the θ_k's and calculates the minimized energy. The energy of an elementary excitation is the difference between the above calculated energy and the ground state energy. If one does this calculation one will find that the energy gap equation is not effectively changed. This is to be expected because the energy gap is a collective phenomenon of all the electrons near the Fermi level. The effect of one single excited particle should therefore be entirely negligible.

If one ignores the change in energy gap due to a single excited particle, one finds that the energy of the particle can be determined self-consistently by requiring the renormalization terms in H_2 to cancel the self-energy terms in

$$\tfrac{1}{2}i[H_1,S].$$

This gives

$$E_k=\epsilon_k\cos2\theta_k-\sum_{\mathbf{k}'}\left[\frac{|M(\mathbf{k},\mathbf{k}')|^2\cos^2(\theta_k+\theta_{k'})}{E_{k'}-E_k+\Omega_q}-\frac{|M(\mathbf{k},\mathbf{k}')|^2\sin^2(\theta_k+\theta_{k'})}{E_{k'}+E_k+\Omega_q}\right],\tag{21}$$

where the last term is found by collecting the coefficients of the appropriate terms in the normal order expansion of the commutator

$$\tfrac{1}{2}i[H_1,S].$$

For particles near the Fermi level, Bogoliubov showed that

$$E_k\cong(\Delta_{\mathbf{k}}^2+\xi_{\mathbf{k}}^2)^{\frac{1}{2}},\tag{22}$$

where $\Delta_{\mathbf{k}}$ and $\xi_{\mathbf{k}}$ are given in Bogoliubov's paper. Therefore the energy denominator for the pair creation terms $\alpha_{\mathbf{k}}^*\beta_{-\mathbf{k}}^*$ is $2E_k$, a quantity that never vanishes as long as $\Delta_{\mathbf{k}}$ is not zero. This shows that the "dangerous terms" become harmless after energy renormalization.

The renormalized phonon energy is found in a similar manner. The result is

$$\Omega_q=\omega_q-\sum_{\substack{\mathbf{k},\mathbf{k}'\\ \mathbf{k}=\mathbf{k}'+\mathbf{q}}}\left\{\frac{|M(\mathbf{k},\mathbf{k}')|^2}{E_{k'}+E_k-\Omega_q}+\frac{|M(\mathbf{k},\mathbf{k}')|^2}{E_{k'}+E_k+\Omega_q}\right\}\sin^2(\theta_k+\theta_{k'}).\tag{23}$$

All the results in this section agree with Bogoliubov.

DISCUSSION

It is interesting to compare the BCS theory with the Bogoliubov or Eliashberg theories. In BCS theory one first treats the electron-phonon interaction to obtain a pair-interaction Hamiltonian, then makes a transformation to quasi-particles and shows that this leads to a state which can be identified as the superconducting state. In the other two theories one takes into account the quasi-particle properties in the very beginning and considers the quasi-particle-phonon interaction. The difference between the Bardeen-Pines potential and the Bogoliubov potential comes entirely from this difference

[8] K. Yosida, Phys. Rev. **111**, 1255 (1958).

in procedure. In the crudest approximation all these theories give the same answer. However, it is felt that the theory of Bogoliubov is more fundamental since it is derived from the variational point of view.

Note added in proof. In the present discussion the electron-phonon interaction is assumed to be so weak that a second-order perturbation calculation is adequate. When the interaction is strong, the method of Eliashberg allows, in principle, a summation to all orders of perturbation. On the other hand there exists no extension of Bogoliubov method for the strong coupling case.

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Specific Heat of Ferrites at Liquid Helium Temperatures*†

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The specific heat of lithium, cobalt, magnesium, and nickel ferrite has been measured from 1.8°K to 5°K using a calorimetric technique similar to that used by Clement. A "heat switch" was used to cool the samples rather than a helium exchange gas, thereby avoiding helium desorption effects. In all cases the specific heat C could be described by the relation $C=\alpha_M T^{\frac{3}{2}}+\beta T^3$, where T is the temperature and α_M and β are constants. The Debye temperatures were computed from β and were in good agreement with values obtained at liquid nitrogen temperatures. The $T^{\frac{3}{2}}$ temperature dependence verifies the spin-wave theory for these compounds although the values of α_M were consistently larger than anticipated from observed values of the Curie temperatures. Possible explanations for this are discussed. An extra term proportional to T^{-2} was observed for cobalt ferrite and this was identified as the nuclear contribution to the specific heat. From the magnitude of this term the magnetic field at the nucleus of the cobalt ion was evaluated and found to be approximately 410 koe.

I. INTRODUCTION

SPIN-WAVE theory[1,2] predicts that for a ferromagnetic or a ferrimagnetic material there should be a contribution to the specific heat which is proportional to the temperature to the three-halves power. Measurements, by Kouvel,[3] of the temperature dependence of the specific heat of the ferrite Fe_3O_4 indicated the presence of this term. This work constituted one of the first direct verifications of the spin-wave theory. It was, however, pointed by Kouvel that the exchange parameters obtained by low-temperature specific heat measurements did not agree with high-temperature measurements of the same parameters.

The temperature dependence of the specific heat has been measured for a wide class of ferrites, among which are nickel, magnesium, lithium, and cobalt ferrites, in order (a) to test the $T^{\frac{3}{2}}$ law over a wide range of ferrites and (b) to evaluate exchange parameters from low-temperature data.

In all the samples measured, the $T^{\frac{3}{2}}$ dependence of the spin-wave specific heat was observed. However, its magnitude was, as with the results of Kouvel, larger than anticipated from high-temperature determinations of the exchange integral.

II. THEORY

We now evaluate the specific heat in a ferrite in the absence of a dc applied magnetic field. We will consider the effects of exchange interactions and the interactions between the spins and the dipole field.

Kittel and Herring[4] have shown that the dispersion relation for spin waves under these assumptions and in a medium of high symmetry can be written

$$\omega^2(K)=(\eta K^2)(\eta K^2+4\pi M_s\gamma\sin^2\theta_K), \qquad (1)$$

where $\eta=(A/M_s)(ge/mc)$, A is the Landau-Lifshitz[5] exchange constant, M_s is the saturation magnetization, $\gamma=e/mc$, g is the spectroscopic splitting factor, and θ_K is the angle between the direction of propagation of a spin wave and the easy direction of magnetization.

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