Renormalization of Many-Fermion Momentum-Space Distributions in Higher Random Phase Approximations*

N. R. WERTHAMER AND H. SUHL
University of California, San Diego, La Jolla, California
(Received September 5, 1961)

The approach to the many-fermion problem known as the method of higher random phase approximations (RPA) is given a more rigorous formulation. It is shown that the previous heuristic procedures for evaluation of the second RPA are justified, in that expectation values of plane wave operators with respect to the true ground state may validly be replaced by their values in the unperturbed Fermi state. This property of momentum-space occupation renormalization is conjectured to hold also to higher orders of RPA than the second.

I. INTRODUCTION

FOR some years now, the technique for obtaining approximate solutions to the many-body problem, known as the random phase approximation (RPA), has been widely used. Originally developed by Bohm and Pines[1] in connection with the electron gas, it has since been reformulated by others,[2] and applied to a variety of many-body calculations, both normal and superfluid. In every instance, the RPA result was identical to that obtained from a low-order perturbation theory calculation, when augmented by selective resummation of certain infinite classes of diagrams.

In a recent publication,[3] we have generalized the RPA procedure in such a way that it can be extended to higher orders; the usual RPA then becomes the first of an infinite sequence of approximations. It was shown there[3] that the second RPA again could be put in correspondence with a selectively renormalized perturbation treatment taken to an appropriate order.

However, even though it appears plausible that the extended RPA, *in toto*, represents an exact solution to the problem, there is as yet no rigorous proof of this; the RPA procedure exists merely as an *ad hoc* recipe, designed so as to produce a mathematically tractable model. The reproduction of perturbation theory results could not obviously be guaranteed prior to explicit detailed calculation.

In this paper, we attempt to place the extended RPA on a more rigorous foundation, and to justify the heuristic rules previously used for it. We are not able to provide this proof in general, to all orders, but the demonstration is carried through for the second RPA (the order to which correspondence with perturbation theory has previously been exhibited, upon assuming "plausible" rules for interpreting certain averages).

In Sec. II, the extended RPA is reviewed, and the rules previously used for its evaluation highlighted. Section III introduces a more rigorous foundation for the RPA and on this basis Sec. IV contains an explicit validation of the rules for the second RPA.

* Work supported in part by the Air Force Office of Scientific Research.

[1] D. Bohm and D. Pines, Phys. Rev. **92**, 626 (1953).

[2] K. Sawada, Phys. Rev. **106**, 372 (1957); K. Sawada, K. A. Brueckner, N. Fukuda, and R. Brout, *ibid.* **108**, 507 (1957); P. W. Anderson, *ibid.* **112**, 1900 (1958); H. Ehrenreich and M. H. Cohen, *ibid.* **115**, 786 (1959); J. Goldstone and K. Gottfried, Nuovo cimento **13**, 849 (1959).

[3] H. Suhl and N. R. Werthamer, Phys. Rev. **122**, 359 (1961), to be referred to in the following as (I).

II. REVIEW OF EXTENDED RPA

As an example illustrating the use of the extended RPA, we choose the calculation of the energy of a fermion of momentum $\mathbf{k}$ added to an N-fermion system in pair-wise interaction. We note that if $|0\rangle$ is the true ground state of energy E_0, and if there exists an operator $X_{\mathbf{k}}^*$ such that

$$[H,X_{\mathbf{k}}^*]|0\rangle=\omega_{\mathbf{k}}X_{\mathbf{k}}^*|0\rangle, \tag{1}$$

then $X_{\mathbf{k}}^*|0\rangle$ is an eigenstate of the Hamiltonian H with energy $E_0+\omega_{\mathbf{k}}$. When considering the added particle problem, the zeroth approximation to $X_{\mathbf{k}}^*$ is $C_{\mathbf{k}}^*$, a plane-wave creation operator; this choice does not satisfy Eq. (1), however, since with

$$H=\sum_{\mathbf{k}}\epsilon_{\mathbf{k}}C_{\mathbf{k}}^*C_{\mathbf{k}}+\tfrac{1}{2}\sum_{\mathbf{kk'q}}v(\mathbf{q})C_{\mathbf{k'+q}}^*C_{\mathbf{k-q}}^*C_{\mathbf{k'}}C_{\mathbf{k}}, \tag{2}$$

it is found that

$$[H,C_{\mathbf{k}}^*]=\epsilon_{\mathbf{k}}C_{\mathbf{k}}^*+\sum_{\mathbf{k'q}}v(\mathbf{q})C_{\mathbf{k-q}}^*C_{\mathbf{k'+q}}^*C_{\mathbf{k'}}. \tag{3}$$

This is not of the form Eq. (1) due to the presence of the second term, arising from interaction between plane waves.

The RPA proceeds by operating with Eq. (3) on the state $|0\rangle$. The summation of the second term is separated into

$$\sum_{\mathbf{k'}}C_{\mathbf{k-q}}^*C_{\mathbf{k'+q}}^*C_{\mathbf{k'}}=-C_{\mathbf{k}}^*C_{\mathbf{k-q}}^*C_{\mathbf{k-q}}+\sum_{\mathbf{k'}\neq\mathbf{k-q}}C_{\mathbf{k-q}}^*C_{\mathbf{k'+q}}^*C_{\mathbf{k'}},$$

and the operator $C_{\mathbf{k-q}}^*C_{\mathbf{k-q}}$, which is diagonal in a plane-wave occupation representation, is replaced by its expectation value in the true ground state:

$$C_{\mathbf{k-q}}^*C_{\mathbf{k-q}}|0\rangle\rightarrow\langle C_{\mathbf{k-q}}^*C_{\mathbf{k-q}}\rangle_0|0\rangle. \tag{4}$$

It was in fact shown in (I) that this operator has a fluctuation from its c-number mean which is only infinitesimal, ($\lesssim N^{-\frac{1}{2}}$), vanishing in the limit of an infinite system. The first RPA then linearizes Eq. (3)

by discarding the terms $\sum_{k'\neq k-q}$, leaving the identifications with Eq. (1),

$$X_k^{*(1)}\cong C_k^*, \quad \omega_k^{(1)}\cong \epsilon_k - \sum_q v(\mathbf{q})\langle C_{k-q}^* C_{k-q}\rangle_0. \quad (5)$$

The second RPA, however, retains all terms at this stage, and augments Eq. (3) with the equation of motion for the previously discarded trilinear terms; schematically,

$$\begin{aligned}&[H, C_{k-q}^* C_{k'+q}^* C_{k'}]\\ &= (\epsilon_{k-q}+\epsilon_{k'+q}-\epsilon_{k'})C_{k-q}^* C_{k'+q}^* C_{k'}\\ &\quad + \text{linear combination of products } C^*C^*C^*CC. \end{aligned} \quad (6)$$

This is equivalent to choosing as the next approximation for the excitation operator,

$$X_k^{*(2)} = C_k^* + \sum_{k'q} \alpha_{k'q} C_{k-q}^* C_{k'+q}^* C_{k'}, \quad (7)$$

with coefficients α to be determined. In order that the choice Eq. (7) be such as to satisfy Eq. (1), it is necessary to linearize the second term of Eq. (6), much as was done via Eq. (4) for the first RPA; the equations are then closed within the given manifold of states. All bilinear and quadrilinear products of zero total momentum are extracted from the quintuple terms $C^*C^*C^*CC$, and are replaced by their expectation value in the true ground state: in addition to Eq. (4),

$$C_{k-q}^* C_{k'+q}^* C_k C_{k'} \rightarrow \langle C_{k-q}^* C_{k'+q}^* C_k C_{k'}\rangle_0. \quad (8)$$

The remaining quintuple terms in Eq. (6) for which neither of these operations is possible are discarded in the second RPA.

In a similar manner, the nth RPA trial form would be

$$\begin{aligned}X^{*(n)} &= C^* + \sum \alpha_3 C^*C^*C\\ &\quad + \sum \alpha_5 C^*C^*C^*CC + \cdots + \sum \alpha_{2n-1} C^* \cdots C. \end{aligned} \quad (9)$$

The commutator $[H, X^{*(n)}]$ is computed, all operator products in the commutator with finite expectation value in the true ground state are replaced by that value, and the uncontractable products of $2n+1$ plane-wave operators are discarded. The requirement that $X^{*(n)}$ satisfy Eq. (1) now leads to coupled eigenvalue equations for the α_i and $\omega^{(n)}$.

So far, however, there is one important step remaining before ω is in principle completely determined: this is to outline a method for calculating the as yet unknown true ground state expectations. In (I) it was conjectured that these quantities would differ only slightly from their values taken with respect to the unperturbed Fermi state, namely

$$\langle C_k^* C_k\rangle_0 \cong \Theta(k_F - k), \text{ all others } \cong 0.$$

Here $\Theta(x)$ is the unit step function,

$$\Theta(x) = 1 \quad \text{for} \quad x>0, \quad =0 \quad \text{for} \quad x<0.$$

When these approximations were made, it was found that all calculated results precisely matched those obtained previously from selectively resummed perturbation theory. Thus, although the formulation of the RPA summarized above indicates that all true ground state expectation values are to be retained, and somehow calculated self-consistently, nevertheless, the question remains as to whether correct results are indeed obtained simply by using Fermi state values. The next sections demonstrate, at least to the lowest few orders, that this conjecture is in fact correct.

III. REFORMULATION OF THE EXTENDED RPA

We proceed by discussing the excitation operators X with somewhat greater care, particularly with regard to particle number.[4] If single added fermion excitations are being considered, then the operators X^* must be restricted to those raising particle number by one; if $|0;N\rangle$ is the true N-particle ground state, then $X^*|0;N\rangle$ is an excited state of the $(N+1)$-particle system.

Furthermore, if any sort of perturbation description for these excited states is to be valid, they must be obtainable as an adiabatic transform of a corresponding plane-wave state; the complete set of all exact-excitation quantum numbers must contain (at least) the set of unperturbed quantum numbers. Thus, not only are there excitation operators X_k which for vanishing interaction go over into C_k but also operators $X_{kk'k''}$ which go over into $C_k^* C_{k'} C_{k''}$, etc. The X's may be labelled by the set of quantum numbers $[\mathbf{k}] = \mathbf{k}_1 \cdots \mathbf{k}_{2\nu+1}$ such that

$$X_{[k]} \rightarrow C_{k_1}^* \cdots C_{k_\nu}^* C_{k_{\nu+1}} \cdots C_{k_{2\nu+1}}.$$

Next, since all $(N+1)$-particle excited states have energies $E_{[k]}(N+1)$ greater than the ground state energy,

$$E_{[k]}(N+1) > E_0(N+1) \equiv E_0(N) + \mu,$$

it follows that

$$X_{[k]}^*|0;N\rangle = 0 \quad \text{for} \quad \omega_{[k]} < \mu, \quad (10)$$

where

$$\mu = dE_0(N)/dN \quad (11)$$

is the chemical potential, or separation energy. A similar result holds for the $(N-1)$-particle system: since

$$\begin{aligned}&E_{[k]}(N-1) > E_0(N-1) = E_0(N) - \mu,\\ &X_{[k]}|0;N\rangle = 0 \quad \text{for} \quad \omega_{[k]} > \mu. \end{aligned} \quad (12)$$

In addition, since the excited eigenstates are orthonormal, therefore

$$\begin{aligned}\langle 0;N|X_{[k]}^* X_{[k']}|0;N\rangle &= \delta_{[k],[k']}\Theta(\mu - \omega_{[k]}),\\ \langle 0;N|X_{[k']} X_{[k]}^*|0;N\rangle &= \delta_{[k],[k']}\Theta(\omega_{[k]} - \mu), \end{aligned} \quad (13)$$

and thus the excitation operators obey an anticommutation relation,[5]

$$\langle 0;N|[X_{[k]}^*, X_{[k']}]_+|0;N\rangle = \delta_{[k],[k']}. \quad (14)$$

---

[4] This discussion closely follows that given by K. Sawada, Phys. Rev. **119**, 2090 (1960).

[5] In this sense, the X^*, X might be interpreted as quasi-particle creation and destruction operators, but with a serious restriction:

Equations (13) provide the basis for investigating the renormalization of the momentum space distribution function. Suppose it can be shown that for every diagonal product $C_\mathbf{k}^* C_\mathbf{k}$ of plane-wave operators which occurs in an RPA equation of motion, there arises from higher order equations linear combinations of other products with finite expectation, C^*C^*CC, $C^*C^*C^*CCC$, etc., such that the sum of all such contributions just gives $X_\mathbf{k}^* X_\mathbf{k}$. The fluctuation of this operator from its mean value is negligible for large systems; hence, Eqs. (13) imply that $X_\mathbf{k}^* X_\mathbf{k}$ operating on the true ground state is just the unit step function $\Theta(\mu-\omega_\mathbf{k})$, suggestive of an occupation of all eigenstates up to a new "Fermi level." But this result is precisely that obtained from the "plausible" procedure adopted in previous RPA work and outlined above: that of setting all plane-wave operator products with finite expectation in the true ground state equal to their values with respect to the *unperturbed* Fermi state. The only difference between the two procedures in evaluating actual formulas might arise if K, defined by $\omega_K=\mu$, were not equal to the unperturbed Fermi momentum k_F; since, however, the shape of the container, the boundary conditions on the wave functions, and hence the wave vectors $\mathbf{k}$ associated with excitations in the box, are not affected by turning on interparticle interactions, and since the number of particles certainly remains fixed, the Fermi *momentum* cannot change in the process either.[6]

IV. VERIFICATION OF RENORMALIZATION IN SECOND RPA

The hypothesis of the previous paragraph, regarding the renormalization of the momentum space distribution function, has not so far been proved to all higher RPA orders. It can be demonstrated, however, in the second RPA. The initial step is to exhibit the expressions obtained in (I) for the second RPA to the operator $X_\mathbf{k}$ and its associated excitation energy $\omega_\mathbf{k}$:

$$X_\mathbf{k}^{(2)}=C_\mathbf{k}-\sum_{\mathbf{k}'\mathbf{q}} \frac{v(\mathbf{q})}{1-v(\mathbf{q})\chi(\mathbf{k},\mathbf{q})} \frac{C(\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}')}{\epsilon'(\mathbf{k}'+\mathbf{q},\mathbf{k};\mathbf{k}+\mathbf{q},\mathbf{k}')}, \tag{15}$$

$$\omega_\mathbf{k}^{(2)}=-\epsilon_\mathbf{k}+\sum\nolimits_\mathbf{q} v(\mathbf{q}) n_{\mathbf{k}+\mathbf{q}}/[1-v(\mathbf{q})\chi(\mathbf{k},\mathbf{q})]. \tag{16}$$

The notation summarized from (I) is

$$\begin{aligned} n_\mathbf{k} &\equiv \Theta(k_F-k),\\ C(\mathbf{p};\mathbf{q},\mathbf{r}) &\equiv C_\mathbf{p}^* C_\mathbf{q} C_\mathbf{r},\\ \epsilon'(\mathbf{p},\mathbf{q};\mathbf{r},\mathbf{s}) &\equiv \epsilon_\mathbf{p}'+\epsilon_\mathbf{q}'-\epsilon_\mathbf{r}'-\epsilon_\mathbf{s}',\\ \epsilon_\mathbf{p}' &\equiv \epsilon_\mathbf{p}-\sum\nolimits_\mathbf{k} v(\mathbf{k}) n_{\mathbf{k}+\mathbf{p}},\\ \chi(\mathbf{k},\mathbf{q}) &\equiv \sum\nolimits_{\mathbf{k}'} (n_{\mathbf{k}'+\mathbf{q}}-n_{\mathbf{k}'})/\epsilon(\mathbf{k}'+\mathbf{q},\mathbf{k};\mathbf{k}+\mathbf{q},\mathbf{k}'). \end{aligned} \tag{17}$$

In deriving the second RPA, Eqs. (15) and (16), the further approximation has been made of retaining only such terms in the commutator of trilinear operators as contribute to the shielding of the potential (the shielding factor χ); the remaining terms in this commutator affect X and ω only to one higher order in the interaction and are unnecessary for present purposes. Also in Eqs. (15) and (16), the renormalization property under scrutiny has in fact already been assumed, in that $C_\mathbf{k}^* C_\mathbf{k}$ has been set equal to its Fermi state expectation $n_\mathbf{k}$. Again because only a low order of coupling is needed for the moment, the replacement is valid; in a later section we investigate the consistency of this assumption. Finally, since it is easily verified that $X_\mathbf{k}^{(2)}$ defined by Eq. (15) satisfies the anticommutation relations, Eq. (14), within the appropriately restricted manifold of states, no multiplicative normalization factor is needed.

Expression (15) may now be applied to the other second RPA calculation of (I), that of the dynamic dielectric constant of an electron gas. In this calculation, an external density fluctuation with frequency Ω and wave number $\mathbf{Q}$ is impressed on the system, resulting in an additional interaction energy

$$H'=v(\mathbf{Q})A\sum\nolimits_\mathbf{k} C(\mathbf{k}-\mathbf{Q};\mathbf{k})+\text{c.c.}, \tag{19}$$

with $A\equiv a(\mathbf{Q})e^{-i\Omega t}$; the dielectric constant depends on the linear response of density elements with wave number $\mathbf{Q}$, $C(\mathbf{k}+\mathbf{Q};\mathbf{k})$. The equation of motion for this quantity is

$$\begin{aligned} &[H+H', C(\mathbf{k}+\mathbf{Q};\mathbf{k})]\\ &\quad=\{\epsilon(\mathbf{k}+\mathbf{Q};\mathbf{k})+\sum\nolimits_{\mathbf{k}'} C(\mathbf{k}';\mathbf{k}')\\ &\qquad\times[v(\mathbf{k}'-\mathbf{k})-v(\mathbf{k}'-\mathbf{k}-\mathbf{Q})]\}C(\mathbf{k}+\mathbf{Q};\mathbf{k})\\ &\qquad-[C(\mathbf{k}+\mathbf{Q};\mathbf{k}+\mathbf{Q})-C(\mathbf{k};\mathbf{k})]\\ &\qquad\cdot\{\sum\nolimits_{\mathbf{k}'}[v(\mathbf{Q})-v(\mathbf{k}'-\mathbf{k})]C(\mathbf{k}'+\mathbf{Q};\mathbf{k}')+v(\mathbf{Q})A\}\\ &\qquad+\sum\nolimits_{\mathbf{k}'\mathbf{q}} v(\mathbf{q})[C(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}';\mathbf{k},\mathbf{k}'+\mathbf{q})\\ &\qquad\qquad-C(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q},\mathbf{k}+\mathbf{q},\mathbf{k}')]. \end{aligned} \tag{20}$$

To obtain the dielectric constant to the next RPA order, Eq. (20) must be augmented with equations of motion for the quadrilinear operator products appearing in its last term. However, since the response is being calculated only to second order in the shielded potential, it is permissible to replace the diagonal operators $C(\mathbf{k};\mathbf{k})$ by $X_\mathbf{k}^* X_\mathbf{k}=n_\mathbf{k}$ everywhere, except in the term of Eq. (20) with factor $v(\mathbf{Q})A$; and also, since it is this term only whose renormalization is being studied in detail, all terms in the quadrilinear equations not relevant for this purpose may be ignored here. Thus the

they are only defined within a manifold consisting of the single state $|0;N\rangle$. The defining relation (1) has no meaning as an operator relation $[H,X_{[\mathbf{k}]}^*]=\omega_{[\mathbf{k}]}X_{[\mathbf{k}]}^*$, independent of the state on which it acts. Thus the states $X_{[\mathbf{k}]}^* X_{[\mathbf{k}']}|0;N\rangle$ are not in general exact eigenstates of the N-particle system, nor are the excitations factorable, i.e., $X_{\mathbf{k}\mathbf{k}'\mathbf{k}''}\neq X_\mathbf{k}^* X_{\mathbf{k}'} X_{\mathbf{k}''}$; the situation is quite similar to the nonsuperposability of spin-wave excitations of ferromagnets. As a consequence, it is meaningless to form the state $|0;N\rangle$ as a multiple product of X^* operators on the fermion vacuum, $|0;N\rangle\neq\prod_{\omega_\mathbf{k}\leq\mu(N)} X_\mathbf{k}^*|0;0\rangle$, as is true for the Fermi sea of occupied plane-wave states; even though in the ordinary first RPA this product state does formally satisfy Eqs. (10) and (12), $|0;N\rangle$ when written in this way is not an energy eigenstate.

[6] This intuitive picture is strictly true only for spherically symmetric interparticle potentials (for which the *shape* of the Fermi surface does not change) as shown rigorously by J. M. Luttinger, Phys. Rev. **119**, 1153 (1960).

remainder pertinent to the renormalization is just

$$\begin{aligned}\Omega C(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4) &= [H+H', C(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4)] \rightarrow \epsilon(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4) \\ &\times C(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4) - v(\mathbf{Q})A[C(\mathbf{k}_1-\mathbf{Q},\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4) \\ &+ C(\mathbf{k}_1,\mathbf{k}_2-\mathbf{Q};\mathbf{k}_3,\mathbf{k}_4) - C(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3+\mathbf{Q},\mathbf{k}_4) \\ &- C(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4+\mathbf{Q})]. \qquad (21)\end{aligned}$$

The terms with factor $v(\mathbf{Q})A$, although they have finite true ground-state expectation value, vanish with respect to the Fermi state and were previously discarded.

After suitable adjustment of momentum indices, Eq. (21) may be substituted in Eq. (20); the relevant portion of the latter, upon rearrangement, reads

$$\begin{aligned}D &\equiv v(\mathbf{Q})A[C(\mathbf{k}+\mathbf{Q};\mathbf{k}+\mathbf{Q}) - C(\mathbf{k};\mathbf{k})] - \sum_{\mathbf{k}'\mathbf{q}} v(\mathbf{q})[C(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}';\mathbf{k},\mathbf{k}'+\mathbf{q}) - C(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}')] \\ &= v(\mathbf{Q})A\Bigg\{C(\mathbf{k}+\mathbf{Q};\mathbf{k}+\mathbf{Q}) - C(\mathbf{k};\mathbf{k}) - \sum_{\mathbf{k}'\mathbf{q}} v(\mathbf{q})\Bigg[-\frac{C(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}';\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q})}{\epsilon(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}';\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q})} \\ &\quad + \frac{C(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}')}{\epsilon(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}')} + \frac{C(\mathbf{k}+\mathbf{q},\mathbf{k}';\mathbf{k},\mathbf{k}'+\mathbf{q})}{\epsilon(\mathbf{k}+\mathbf{q},\mathbf{k}';\mathbf{k},\mathbf{k}'+\mathbf{q})} - \frac{C(\mathbf{k},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}')}{\epsilon(\mathbf{k},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}')}\Bigg] + \Delta_{\mathbf{k}}(\mathbf{Q},\Omega)\Bigg\}. \qquad (22)\end{aligned}$$

This particular form has been selected because, by referring to Eq. (15), it is seen that the quantity in square brackets renormalizes the plane-wave occupation operators to the proper order in the interparticle coupling, such that

$$D \cong v(\mathbf{Q})A[X_{\mathbf{k}+\mathbf{Q}}{}^*X_{\mathbf{k}+\mathbf{Q}} - X_{\mathbf{k}}{}^*X_{\mathbf{k}} + \Delta_{\mathbf{k}}(\mathbf{Q},\Omega)]. \qquad (23)$$

The remainder $\Delta_{\mathbf{k}}(\mathbf{Q},\Omega)$ is the rather lengthy expression

$$\begin{aligned}\Delta_{\mathbf{k}}(\mathbf{Q},\Omega) &\equiv \sum_{\mathbf{k}'\mathbf{q}} v(\mathbf{q})\Bigg\{C(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}';\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q})\Bigg[-\frac{1}{\epsilon(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}';\mathbf{k},\mathbf{k}'+\mathbf{q})-\Omega} + \frac{1}{\epsilon(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}';\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q})}\Bigg] \\ &+ C(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}')\Bigg[\frac{1}{\epsilon(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}')-\Omega} - \frac{1}{\epsilon(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}')}\Bigg] \\ &+ C(\mathbf{k}+\mathbf{q},\mathbf{k}';\mathbf{k},\mathbf{k}'+\mathbf{q})\Bigg[\frac{1}{\epsilon(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}';\mathbf{k},\mathbf{k}'+\mathbf{q})-\Omega} - \frac{1}{\epsilon(\mathbf{k}+\mathbf{q},\mathbf{k}';\mathbf{k},\mathbf{k}'+\mathbf{q})}\Bigg] \\ &+ C(\mathbf{k},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}')\Bigg[-\frac{1}{\epsilon(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}')-\Omega} + \frac{1}{\epsilon(\mathbf{k},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}')}\Bigg] \\ &+ C(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}';\mathbf{k},\mathbf{k}'+\mathbf{Q}+\mathbf{q})\Bigg[\frac{1}{\epsilon(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}'+\mathbf{Q};\mathbf{k},\mathbf{k}'+\mathbf{Q}+\mathbf{q})-\Omega} \\ &- \frac{1}{\epsilon(\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}';\mathbf{k},\mathbf{k}'+\mathbf{q})-\Omega}\Bigg] + C(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}'+\mathbf{Q}) \\ &\times\Bigg[-\frac{1}{\epsilon(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{Q}+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}'+\mathbf{Q})-\Omega} + \frac{1}{\epsilon(\mathbf{k}+\mathbf{Q},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}')-\Omega}\Bigg]. \qquad (24)\end{aligned}$$

Two different arguments may now be put forward for regarding Δ as negligible. The first proceeds by noting that $\Delta(0,\Omega)$ vanishes, and in fact $\Delta(\mathbf{Q},\Omega)$ has the fortunate property that its Taylor expansion for small arguments begins with *quadratic* terms,

$$\begin{aligned}\Delta(\mathbf{Q},\Omega) &= \tfrac{1}{2}\Delta_{20}(0,0)(Q/k_F)^2 \\ &+ \Delta_{11}(0,0)(\hbar\Omega Q/k_F\epsilon_F) + O(\text{cubic}). \qquad (25)\end{aligned}$$

In the calculation of the dielectric constant, furthermore, Δ enters as an additive contribution to the susceptibility (or shielding function), of second order in the interaction:

$$\Delta\chi(\mathbf{Q},\Omega) = \sum_{\mathbf{k}} \Delta_{\mathbf{k}}(\mathbf{Q},\Omega)/[\epsilon(\mathbf{k}+\mathbf{Q};\mathbf{k})-\Omega], \qquad (26)$$

so that Eq. (25) yields $\Delta\chi(0,0)=0$. This is in contrast to the other second RPA contributions to the susceptibility previously calculated in (I), which are all finite in the static, long wavelength limit. Since for all experimentally realizable applied density disturbances, the wavelength will be large compared to a reciprocal Fermi momentum and the frequency will be small compared to $\epsilon_F/\hbar$, Δ may justifiably be neglected in the response for

actual physical situations. More importantly, the formal expression for the dielectric constant at arbitrary frequency and wavelength is often used to calculate other quantities of interest, such as the ground state energy; nevertheless, when expanding these latter in powers of the coupling (and its logarithm), only the $\mathbf{Q}$, $\Omega=0$ limit need be considered for the second order part of the susceptibility.[7] From this point of view, then, Δ is certainly negligible, and the desired result

$$D=v(\mathbf{Q})A(n_{\mathbf{k}+\mathbf{Q}}-n_{\mathbf{k}}) \tag{27}$$

is obtained.

The above arguments, involving the $\mathbf{Q}$, $\Omega\rightarrow 0$ limit, are sufficient for the second RPA. When considering the occupation renormalization in higher RPA's, however, the demonstration is no longer valid, and a more fundamental line of reasoning must be adopted. This is provided by including more of the complete set of true excitation operators $X_{[\mathbf{k}]}$, that is, the $\nu=1$ and higher classes together with the $\nu=0$ class as above. In particular, a quadrilinear operator product such as $C(\mathbf{k},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q},\mathbf{k}')$ which comprises the remainder $\Delta_{\mathbf{k}}(\mathbf{Q},\Omega)$, Eq. (24), may be regarded as a zeroth approximation to a similar product of exact excitation operators, for example,

$$C(\mathbf{k},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q})C_{\mathbf{k}'}+\cdots=X_{\mathbf{k}+\mathbf{q},\mathbf{k}'+\mathbf{q},\mathbf{k}}{}^{*}X_{\mathbf{k}'}. \tag{28}$$

In the sense of an expectation value in the state $|0;N\rangle$, the right-hand-side vanishes by virtue of Eqs. (13), and thus $\langle\Delta_{\mathbf{k}}(\mathbf{Q},\Omega)\rangle_0$ is zero in this approximation for all values of its argument.

The "dots" in Eq. (28) presumably would start with combinations of sextuple products of plane-wave operators, and one potential factor. Although not affecting the conclusions reached above, different factorizations of the products other than Eq. (28) are possible at this stage, including use of excited states of the $N\pm 2$ particle systems; the ambiguity cannot be resolved without detailed treatment of the third RPA, in order to determine explicitly the coefficients of the sextuple terms, and the form of the $\nu=1$ class of X operators. Such a program is well beyond the scope of the present paper.

Nevertheless, it is intuitively reasonable that the validity of a momentum occupation renormalization should be related both to the $\mathbf{Q}$, $\Omega\rightarrow 0$ limit of the susceptibility, and to the role of more complicated eigenmodes of higher RPA's. An externally applied disturbance of very low frequency and wavenumber is only capable of exciting the lowest lying available eigenstates, which are predominantly of a single particle-hole pair character. Thus for a disturbance of this type, the ordinary first RPA contains most of the physically relevant modes. At higher frequencies and wavenumbers, however, states of multi-pair character ($\nu\geqq 1$) may be excited, and the quasi-particles of the ordinary RPA become more and more strongly damped. In this regime of disturbance, it is necessary to go to a higher RPA of larger manifold in order to include the new types of modes; only when the correct quasi-particles have been obtained will the momentum occupation be a Fermi step distribution.

Finally, there remains the question of the renormalization in the second RPA excitation operator, Eqs. (15) and (16). As mentioned before, the product $C_{\mathbf{k}+\mathbf{q}}{}^{*}C_{\mathbf{k}+\mathbf{q}}$ had been set equal to $n_{\mathbf{k}+\mathbf{q}}$ here since the operator was needed to only first order in the shielded interaction. However, a more careful investigation retaining quadrilinear products with finite expectation value leads to

$$\omega_{\mathbf{k}}{}^{(2)}=-\epsilon_{\mathbf{k}}+\textstyle\sum_{\mathbf{q}}v(\mathbf{q})n_{\mathbf{k}+\mathbf{q}}/[1-v(\mathbf{q})\chi(\mathbf{k},\mathbf{q})]+\Delta\omega_{\mathbf{k}}, \tag{16$'$}$$

where

$$\begin{aligned}\Delta\omega_{\mathbf{k}}=\sum_{\mathbf{k}'\mathbf{q}\mathbf{q}'}v(\mathbf{q})v(\mathbf{q}')\Big\langle&\frac{C(\mathbf{k}+\mathbf{q}+\mathbf{q}',\mathbf{k}';\mathbf{k}+\mathbf{q},\mathbf{k}'+\mathbf{q})+C(\mathbf{k}+\mathbf{q},\mathbf{k}'+\mathbf{q};\mathbf{k}+\mathbf{q}+\mathbf{q}',\mathbf{k}')}{\epsilon(\mathbf{k}'+\mathbf{q}',\mathbf{k}+\mathbf{q};\mathbf{k}+\mathbf{q}+\mathbf{q}',\mathbf{k}')}\\&+\frac{C(\mathbf{k}'+\mathbf{q}-\mathbf{q}',\mathbf{k}+\mathbf{q}';\mathbf{k}+\mathbf{q},\mathbf{k}')-C(\mathbf{k}'+\mathbf{q},\mathbf{k}+\mathbf{q}';\mathbf{k}+\mathbf{q}+\mathbf{q}',\mathbf{k}')-C(\mathbf{k}'+\mathbf{q},\mathbf{k}+\mathbf{q}';\mathbf{k}+\mathbf{q},\mathbf{k}'+\mathbf{q}')}{\epsilon(\mathbf{k}'+\mathbf{q},\mathbf{k};\mathbf{k}+\mathbf{q},\mathbf{k}')}\\&-\frac{C(\mathbf{k}+\mathbf{q}+\mathbf{q}',\mathbf{k}';\mathbf{k}+\mathbf{q},\mathbf{k}'+\mathbf{q}')}{\epsilon(\mathbf{k}+\mathbf{q}',\mathbf{k}+\mathbf{q};\mathbf{k}+\mathbf{q}+\mathbf{q}',\mathbf{k})}\Big\rangle.\end{aligned} \tag{29}$$

Arguments similar to those used above for $\Delta\chi$, involving $X_{[\mathbf{k}]}$ operators with $\nu\geqq 1$ as in Eq. (28), now show that $\Delta\omega_{\mathbf{k}}$ is zero within the second RPA.

[7] The point is illustrated in the evaluation of the ground state energy by D. F. DuBois, Ann. Phys. 7, 174 (1959), Appendix C.