mesons emitted. For example, in the case of nucleon-antinucleon annihilation at rest, the initial chirality is zero, and, since the final state contains no nucleons, we would have $\langle\chi_\pi\rangle_f=0$, i.e., the amplitude for the emission of very low energy pions would be unusually small.

The notion of γ_5 invariance or chirality conservation can be extended to composite systems and strange particles.[4] There is also a possibility that the K meson plays a role similar to the pion in the conservation of strangeness-changing chirality current. It is likely, however, that even if such a symmetry existed in essence, the large mass of the K meson would tend to make it more approximate in nature than for the case involving pions, except perhaps at sufficiently high energies.

ACKNOWLEDGMENTS

One of the authors (Y.N.) thanks Dr. J. W. Calkin for his hospitality at the Brookhaven National Laboratory where part of the work was done.

PHYSICAL REVIEW VOLUME 125, NUMBER 4 FEBRUARY 15, 1962

Perturbation Theory of Pion-Pion Interaction. I. Renormalization

Tai Tsun Wu*
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts and Institute for Advanced Study, Princeton, New Jersey

(Received October 9, 1961)

The problem of pion-pion scattering is studied on the basis of the model of a four-particle direct interaction without derivative coupling. Renormalization is carried out for this model with a detailed analysis of overlap insertions. To every finite order in the renormalized coupling constant, it is shown that the unitarity relation holds and that the Feynman integral representation is still valid, and hence renormalization has no effect on analytic properties.

1. INTRODUCTION

SINCE the pion is a pseudoscalar boson, the simplest coupling among pions is a local ϕ^4 coupling. Furthermore, this leads to a dimensionless coupling constant. If this coupling is taken to be correct, then the problem of pion-pion interaction is the simplest among all problems involving strongly-interacting particles. It is the purpose here to study the pion-pion interaction under this coupling using perturbation theory.

In order that the perturbation theory be meaningful, it is necessary to have a consistent procedure to remove the infinities due to integrations over large momenta and to interpret this removal as mass renormalization and coupling-constant renormalization.[1] In the much more familiar case of electrodynamics, the procedure of Ward[2] seems simpler than that of Salam[3]; hence, in the present case, differentiation with respect to external momenta is to be used for the purpose of treating overlap divergences, which are of main concern here. However, the problem of which path to use in carrying out the differentiation is quite complicated in the present case. In quantum electrodynamics, the treatment of the photon self-energy has been carried out by Mills and Yang,[4] and their treatment is the starting point for the present consideration. Thus, this case of electrodynamics is considered first in Sec. 4 after a preliminary study of the case of the ϕ^4 coupling. Renormalization is completed in Sec. 7, and some properties of this procedure are discussed in Secs. 8–11. In particular, the validity of the Feynman integral representation implies that renormalization does not change the domains of analyticity to every order of the coupling constant.

This paper is concerned mainly with the formal question of renormalization within the framework of perturbation theory. Thus, on the basis of the particular Lagrangian under consideration, all the equations here are exact in the sense of being true to every finite order of the coupling constant. In a later paper, the problem is considered concerning the derivation of a closed system of equations for the approximate description of the pion-pion system at low energies.

2. STATEMENT OF THE PROBLEM

Let ϕ_+, ϕ_0, and ϕ_-, respectively, be the field operators for the creation of the pions π^+, π^0, and π^-. Let $\phi_3=\phi_0$, and the triplet of operators (ϕ_1,ϕ_2,ϕ_3) transform as a vector in the space of isotopic spin; then with the usual phase conventions

$$\phi_\pm=\mp(\phi_1\pm i\phi_2)/\sqrt{2}. \qquad (1)$$

Since π^0 is its own antiparticle, ϕ_i are Hermitian. Throughout this paper, the Lagrangian density is

* Alfred P. Sloan Foundation Fellow. Work also supported in part by a grant from the National Science Foundation.
[1] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).
[2] J. C. Ward, Proc. Phys. Soc. (London) **A64**, 54 (1951).
[3] A. Salam, Phys. Rev. **82**, 217 (1951).
[4] R. L. Mills and C. N. Yang (private communication from Professor Yang).

assumed to be

$$L=L_0+L_1, \tag{2}$$

where

$$L_0=-\tfrac{1}{2}\sum_{\mu=1}^{4}\sum_{i=1}^{3}(\partial\phi_i/\partial x_\mu)^2-\tfrac{1}{2}m^2\sum_{i=1}^{3}\phi_i^2, \tag{3}$$

and

$$L_1=2\pi^4\lambda\Big(\sum_{i=1}^{3}\phi_i^2\Big)^2+\tfrac{1}{2}\delta m^2\sum_{i=1}^{3}\phi_i^2. \tag{4}$$

Here the metric used is $(-1, 1, 1, 1)$ with $x_4=ix_0$; m is the physical mass of the pion; and the last term in (4) is the counter term for mass renormalization.

For this Lagrangian density, the Feynman rules are those given in Fig. 1, provided that the states are normalized by

$$\langle\mathbf{k}'|\mathbf{k}\rangle=2k_0(2\pi)^3\delta(\mathbf{k}-\mathbf{k}'), \tag{5}$$

where $k_0=(k^2+m^2)^{\frac{1}{2}}$. In other words, to get a matrix element of the S matrix with respect to states satisfying (5), multiply the various factors of Fig. 1 together with the δ-function expressing over-all energy-momentum conservation, carry out the necessary integrations, and divide by the intrinsic symmetry number $\mathcal{S}$ of the graph, i.e., the symmetry number of the graph with the external lines kept fixed. A few examples of $\mathcal{S}$ are shown in Fig. 2. Note that lines are permitted to end on themselves.

It is convenient to use that quantity obtained from a graph without momentum differentiation by the Feynman rules but omitting the over-all δ function and the factors corresponding to external lines. For graphs with four external lines, let all external momenta point toward the graph; $A(k_1,k_2,k_3,k_4)$ be the above-mentioned quantity corresponding to the isotopic spin indices 1, 1, 1, 1; and $B(k_1,k_2;k_3,k_4)$ be that corresponding to the indices 1, 1, 2, 2. More precisely, for $i\neq j$,

$$\langle\mathbf{p},i;\,\mathbf{p}',i|S|\mathbf{k},i;\,\mathbf{k}',i\rangle =-(2\pi)^8A(k,\,k',\,-p,\,-p')\delta^4(k+k'-p-p'),$$

$$\langle\mathbf{p},i;\,\mathbf{p}',i|S|\mathbf{k},j;\,\mathbf{k}',j\rangle =-(2\pi)^8B(k,\,k';\,-p,\,-p')\delta^4(k+k'-p-p'),$$

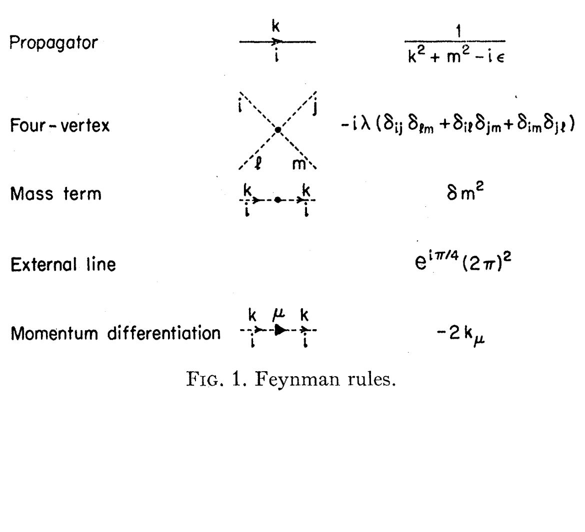


FIG. 1. Feynman rules.

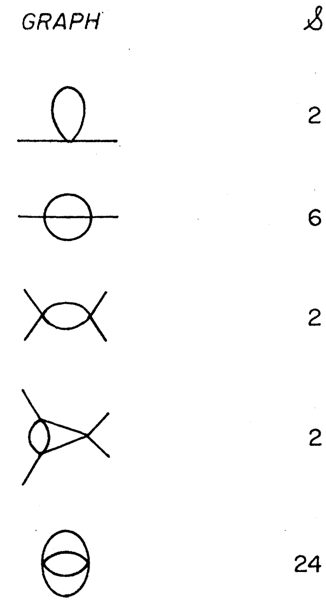


FIG. 2. Examples of the intrinsic symmetry number $\mathcal{S}$.

and

$$\langle\mathbf{p},i;\,\mathbf{p}',j|S|\mathbf{k},i;\,\mathbf{k}',j\rangle =-(2\pi)^8B(k,\,-p;\,k',\,-p')\delta^4(k+k'-p-p'). \tag{6}$$

Equation (1) then gives the value of the S matrix between the various charge states of the pion.

3. CLASSIFICATION OF FEYNMAN GRAPHS

The renormalization program is going to be modeled after that of quantum electrodynamics by Dyson,[1] Ward,[2] and Mills and Yang.[4] Since the absence of the gauge group and the divergence of graphs with four external lines complicate the present problem considerably, this program is to be written down in detail.

The main interest here is to have an adequate treatment of overlap divergences; it is taken for granted that other difficulties do not actually occur. More precisely, Dyson's divergence of the first kind[1] is ignored, displaced poles[1] do not cause additional divergence, and partially renormalized propagators and vertex functions behave in the same way as the corresponding bare propagators and vertex functions in the limit of large momenta except for logarithmic factors.[5]

This renormalization program occupies the following sections. In this section, the Feynman graphs are classified according to Dyson; in Sec. 4, the procedure of Mills and Yang is reviewed and modified for adaptation to the present case; in Secs. 5–6, a possible rule of differentiating the self-energy and four-vertex graphs is given. The integral equations are finally written down in Sec. 7.

Since momentum differentiation is to be used, the

[5] S. Weinberg, Phys. Rev. **118**, 838 (1960).

trivial equation

$$(\partial/\partial k_\mu)(k^2+m^2-i\epsilon)^{-1}=-2k_\mu(k^2+m^2-i\epsilon)^{-2}$$

completes the Feynman rules of Fig. 1. It may be emphasized that the alternative procedure of differentiation with respect to mass[6] leads to very serious difficulties because the renormalization constants all depend on the physical mass, although independent of momenta.

Analogous to, but not quite the same as, quantum electrodynamics, the following definitions are to be used:

1. Vacuum graph—graph with no external lines.
2. Self-energy graph (or simply SE)—connected graph with two external lines and no momentum differentiation.
3. Self-energy prime graph (or simply SE′)—connected graph with two external lines and one momentum differentiation.
4. Admissible graph—connected graph with at most two momentum differentiations such that, if there are two differentiations, they *are not on the same self-energy insertion*.
5. Self-energy double prime graph (or simply SE″)—*admissible* graph with two external lines and two momentum differentiations.
6. Proper graph—admissible graph that cannot be made disconnected by cutting one internal line.
7. Superproper graph—admissible graph that cannot be made disconnected by cutting two internal lines.
8. Vertex graph (or simply V)—proper graph with four external lines and no momentum differentiation.
9. Vertex prime graph (or simply V′)—proper graph with four external lines and one momentum differentiation.
10. Irreducible graph—admissible graph that does not contain any SE, SE′, or V insertion.
11. Reducible graph—admissible graph that is not irreducible.
12. Primitive divergent graph—divergent admissible graph which becomes convergent if *any* one of the variable momenta is held fixed.

For the purpose of classifying admissible graphs, the following two statements are useful.

Lemma 1. If a proper graph is primitive divergent, then it is either SE, SE′, SE″, or V. The divergence is quadratic in the first case, and logarithmic in the others under symmetric integration.

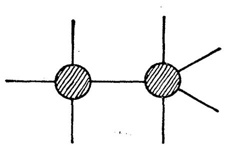


FIG. 3. An improper graph.

[6] J. C. Ward, Phys. Rev. 84, 897 (1951).

Lemma 2. A divergent irreducible graph is primitive divergent and proper.

Lemma 2 may be proved as follows, again in a manner similar to that of quantum electrodynamics. Consider a primitive divergent irreducible graph. If it is not proper, then it must be of the form of two bubbles connected by one internal line, as shown in Fig. 3. By energy-momentum conservation, the internal line shown must carry a fixed energy-momentum. Thus at least one of the bubbles, considered as a graph by itself, is divergent. If the other bubble contains a line of variable momentum, then that can be held fixed and the entire graph is still divergent. Since this contradicts the definition of a primitive divergent graph, the second bubble can contain no line with variable momentum. Thus, the first bubble, considered as a graph by itself, must be primitive divergent. This process of splitting into two bubbles can be repeated, and finally a proper primitive divergent graph is obtained. By Lemma 1, this is either SE, SE′, SE″, or V. Since the original graph is assumed to be irreducible, this must be SE″. However, by definition of admissible graphs, this is not possible. The conclusion is therefore reached that a primitive divergent irreducible graph must be proper. Next consider a divergent irreducible graph with F internal lines, of which F_1 internal lines carry variable momenta. Each of these F_1 momenta can vary or may be held fixed; in this way there are no more than 2^{F_1} different choices. Since of the various choices some still give divergent integrals, let F_2 be the minimum number of variable momenta not kept fixed such that the corresponding integral is still divergent. Let G' be a graph obtained from the original graph by cutting some internal lines to make two external lines out of each such that G' has F_2 internal lines and is divergent. Since the original graph is irreducible, G' is irreducible. Furthermore, since G' is primitive divergent by construction, it is proper, and is indeed either SE, SE′, SE″, or V. Since the original graph is assumed to be irreducible and thus in particular admissible, G' must be the same as the original graph. This proves Lemma 2.

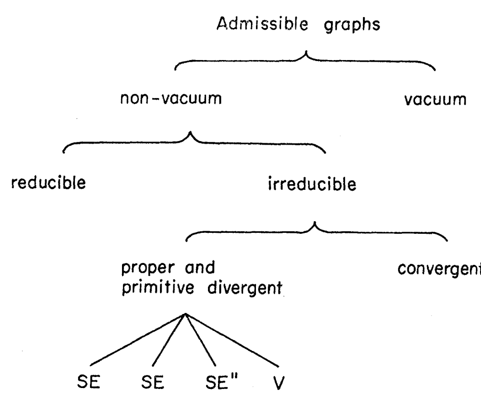
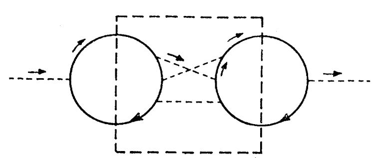


FIG. 4. Classification of admissible graphs.

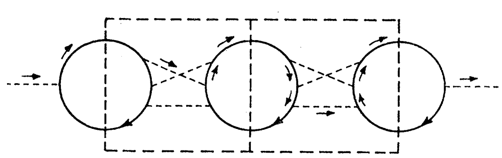
These two statements taken together give the classification of admissible graph shown in Fig. 4.

4. PHOTON SELF-ENERGY GRAPHS IN QUANTUM ELECTRODYNAMICS

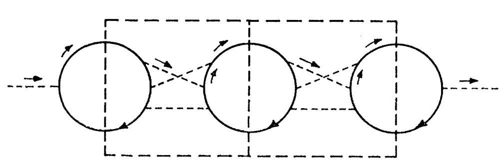
The prescription of Mills and Yang[4] for the differentiation of a photon self-energy graph without self-energy insertion is as follows: 1. Split the graph into two pieces by cutting two internal electron lines. 2. Repeat this process for each piece until no further splitting in this manner is possible; hence the result is an *ordered* sequence of pieces where the first and last pieces are vertex graphs and each of the middle pieces has two electron lines to the right and two electron lines to the left. 3. Differentiate all electron lines that point to the right. 4. Differentiate each of the middle pieces in the ordered sequence in an arbitrary manner, the only limitation being that topologically identical pieces must be differentiated in identical manner. The main point is that if each irreducible photon self-energy graph is differentiated in a completely arbitrary and independent manner, then in the above language identical pieces may be differentiated differently and then overlap divergences in the photon self-energy cannot be properly disentangled. To illustrate this difficulty, consider the prescription for differentiation as shown in Figs. 5(a) and 5(b), where the boxes split the original graphs into pieces as stated above. Then



(a)

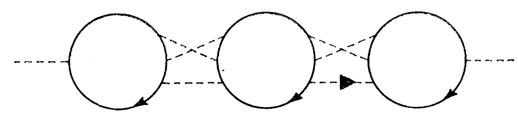


(b)

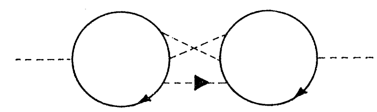


(c)

FIG. 5. Examples of the path of differentiation, as indicated by the arrow.



(a)



(b)

FIG. 6. Graphs obtained from that of Fig. 5(b).

the differentiation in Fig. 5(b) leads in particular to the graph shown in Fig. 6(a), whose skeleton is shown in Fig. 6(b) but is not a graph that can be obtained from differentiating some photon self-energy graph. The prescription of Mills and Yang is designed to avoid this kind of situation; to be consistent with the prescription shown in Fig. 5(a), the graph of Fig. 5(b) must be differentiated as shown in Fig. 5(c). The particular example discussed is one of the simplest; since this is already a twelfth-order diagram, the prescription has no appreciable effect on actual explicit computations by perturbation theory in quantum electrodynamics. Since the coupling constant is no longer so small in the problem of pion interaction, the prescription for differentiation is of more practical importance here.

It is therefore desired to adapt the Mills-Yang method to the pion case. The first step is to modify the prescription so that it does not depend on the direction of the electron line. So far as the original rules are concerned, pieces that differ only in the direction of the electron lines are considered to be different and hence may be differentiated independently. For example, the method of differentiation as shown in Fig. 7 is acceptable. For the sake of definiteness, consider the eight graphs shown in Fig. 8, where each graph contains two pieces without any symmetry property besides the two vertex parts, and the eight graphs differ from one another only in the direction of the electron lines. By Furry's theorem for an even number of photon lines, the directions of these electron lines are actually irrelevant and may be omitted from the graphs. By adding these graphs, these eight graphs may be replaced by the one shown in Fig. 9(a), where the number 4 merely means that an extra factor of 4 is needed. It is entirely equivalent if each of the eight graphs is replaced by that shown in Fig. 9(b). Insofar as each piece is concerned, the above argument also implies that an average should be used; more precisely, in the case shown in Fig. 7, the corresponding "average" pre-

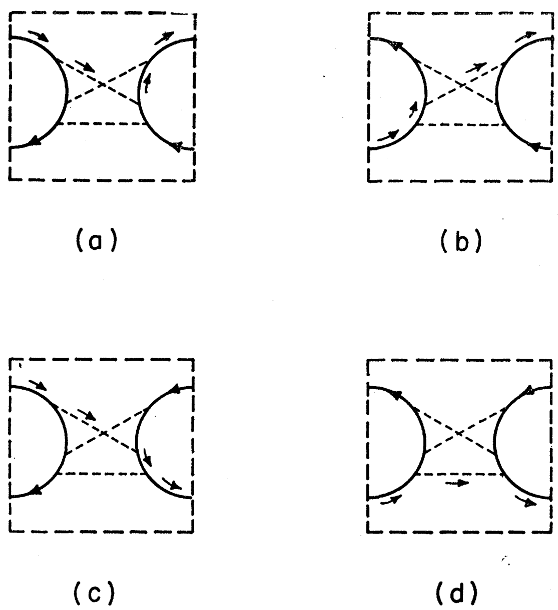


FIG. 7. An example of the paths of differentiation for related graphs.

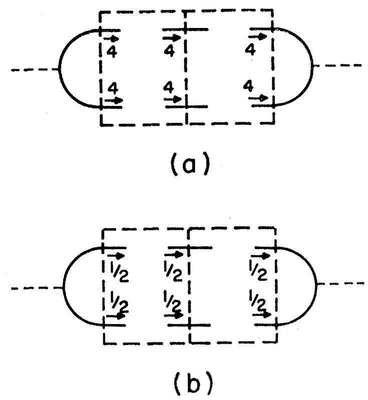


FIG. 9. Average differentiation.

scription is given in Fig. 10(a), which in particular implies that the graphs in Fig. 10(b) need be included. Once Fig. 10(a) is found, the original prescription shown in Fig. 7 may be forgotten, i.e., there is now an alternative prescription to that given by Mills and Yang.

As stated above, the advantage of this alternative prescription is to be free of the arrows carried by electron lines, which have no analog in the pion case. So far as symmetry is concerned, it is even advantageous to go one step further. By taking the average of the prescription of Fig. 10(a) and its right-left image as shown in Fig. 11(a), the result of Fig. 11(b) is obtained. In the next two sections, the present problem of the pion interaction is to be treated in an analogous but somewhat more complicated manner.

5. MOMENTUM DIFFERENTIATION FOR VERTEX GRAPHS

Since the operators ϕ_1, ϕ_2, and ϕ_3 are Hermitian, the pion lines do not carry an intrinsic arrow, unlike the

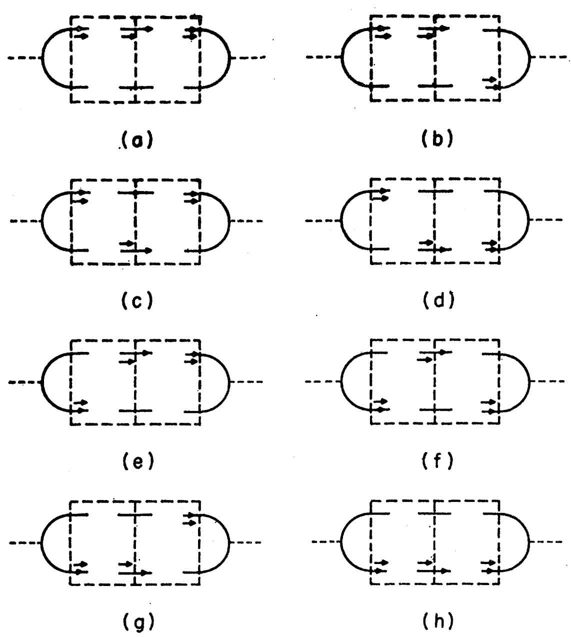


FIG. 8. Differentiation of eight graphs that differ only in the directions of the election lines.

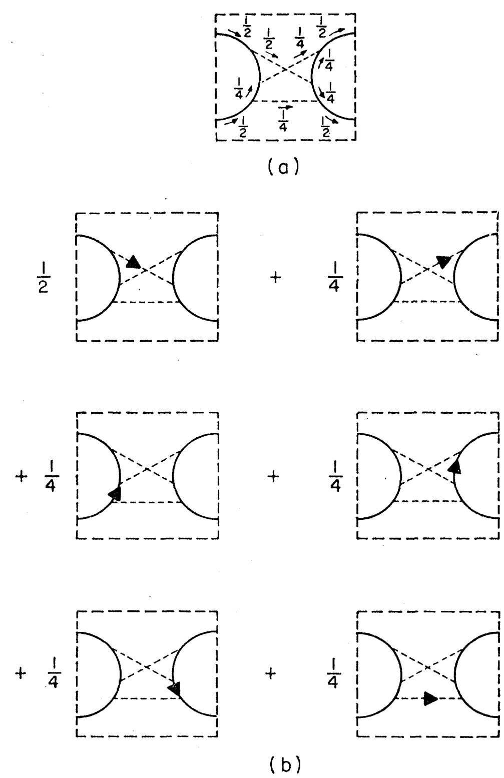


FIG. 10. An example of the average differentiation of a piece of a graph.

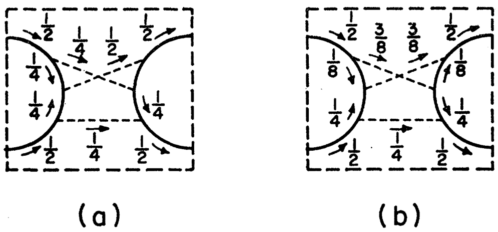


FIG. 11. Further averaging.

case of quantum electrodynamics. Accordingly, it is easier to adapt the modified prescription as discussed above than the original prescription of Mills and Yang. For the case of the vertex graph, the prescription is as follows.

1. *Remove all* SE *insertions.*—This is unambiguous. Also note that V is by definition proper.

2. *Split the vertex graph into two pieces by cutting two internal lines. Repeat this process until no further splitting in this manner is possible.*—First note that each internal line can be cut at most once and that each piece obtained by this splitting is a vertex graph when taken by itself. A moment's reflection indicates that, if a graph can be split by cutting two internal lines such that the external lines 1 and 2 are attached to one piece while the external lines 3 and 4 are attached to the other piece, then it is impossible to split in this manner such that the lines 1 and 3 are attached to one piece while the lines 2 and 4 to the other piece. This fact in particular implies that the splitting can be carried out in the following way: First split the original vertex graph into an ordered sequence of vertex graphs similar to the case of quantum electrodynamics, then split each piece into an ordered sequence, repeat this process until no further splitting in this way is possible. The final pieces are of course superproper. An example of such a sequence of splitting is shown in Fig. 12(a).

3. *After the repeated splitting, differentiate in accordance with the rules shown in Fig. 13.*—For example, the differentiation of the graph in Fig. 12(a) from right to left leads to the sum of the graphs shown in Fig. 12(b) to (d). It remains to specify how each superproper V should be differentiated.

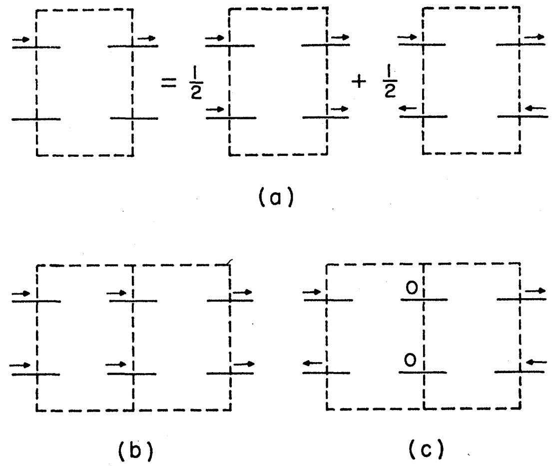


FIG. 13. Basic rules for differentiating a vertex graph.

4. *For each superproper* V, *remove all* V *insertions.*—To show that overlap insertions do not cause any trouble, note that all overlap insertions must be of one of the forms shown in Fig. 14. The cases of Fig. 14(b) to (d) cannot occur since all SE insertions have been removed at the beginning. The case of Fig. 14(a) just represents a way of drawing three successive V insertions.

5. *For each superproper irreducible* V, *differentiate in any manner consistent with the symmetry of the graph.*—See, however, the discussion in Sec. 11 in connection with the Schwinger-Edwards identity.

6. *Use the same rule of differentiation for each successive* V *insertion. For the* SE *insertions, use the rule to be given in the next section.*

This completes the prescription for the differentiation of a vertex graph. Note that the rules imply that all differentiated vertex graphs can be obtained from the irreducible ones by *independent* SE, SE′, and V insertions. Indeed, this is the main purpose of the prescription.

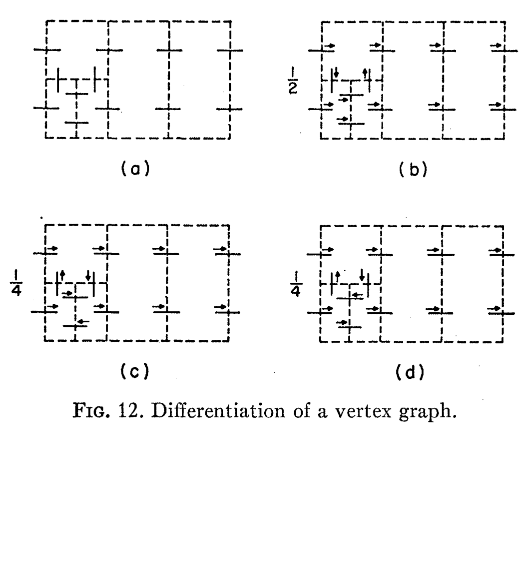


FIG. 12. Differentiation of a vertex graph.

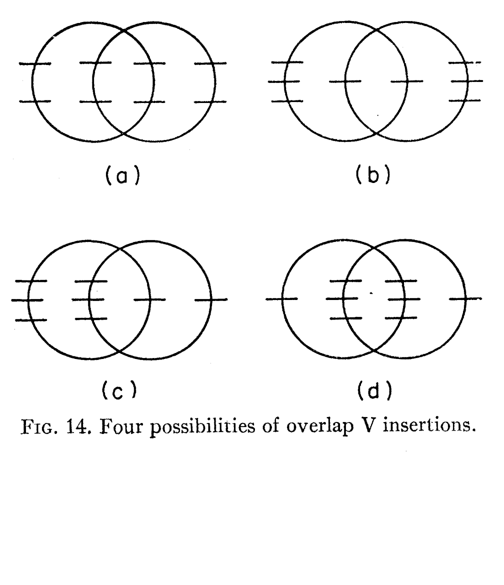


FIG. 14. Four possibilities of overlap V insertions.

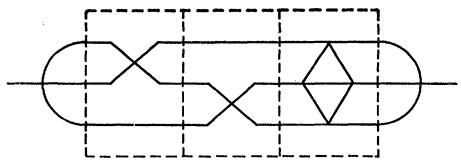


FIG. 15. An example of splitting a self-energy graph.

With these prescriptions, it is possible to find the "weight" to be attached to each internal line when the corresponding values for the external lines are given. Unlike the case of the self-energy in quantum electrodynamics, a vertex graph here has four external lines, and hence there are many independent derivatives. Therefore, it remains to specify the weight attached to each external line; i.e., to specify which linear combination of derivatives is to be used. For the sake of maintaining the symmetry properties possessed by the vertex graphs, it is convenient to use the momentum carried by the external line as its weight. Examples of this rule are to be found in Fig. 17.

6. MOMENTUM DIFFERENTIATION FOR SELF-ENERGY GRAPHS

Similar to, but somewhat more complicated than, the case of the vertex graphs, the prescription for the self-energy graphs may be stated as follows.

1. *Split the self-energy graph into proper parts.*—This is unambiguous, and it is thus sufficient to specify how proper SE should be differentiated.
2. *For each proper* SE, *remove all* SE *insertions.*—This is again unambiguous for a proper SE.
3. *Without using the external lines, remove all* V *insertions.*—In other words, all V insertions whose external lines are internal lines of the original proper SE are removed. Insofar as overlap insertions are concerned, the situation is identical to that of step 4 in the last section.
4. *Split the proper* SE *into an ordered sequence of pieces where the first and the last pieces are vertex graphs and each of the middle pieces has three lines to the right and three lines to the left.*—This splitting is supposed to be maximal in the sense that no longer sequence of this nature is possible. The situation here resembles rather more closely that for quantum electrodynamics than that for the vertex graph. The splitting is unambiguous, but it is no longer true that each internal line can be cut only once. Instead, it is required that two different systems of three cuts each have at most one internal line in common. An example of this splitting is shown in Fig. 15.

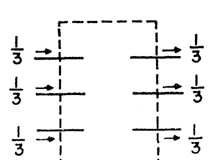


FIG. 16. Rule for differentiating each piece of a self-energy graph.

5. *After this splitting, differentiate in accordance with the rule shown in Fig. 16.*
6. *For each piece with three lines to the right and three lines to the left, differentiate in any manner consistent with the symmetry of the piece.*—Again however, see the discussion in Sec. 11.
7. *Use the prescription of last section to differentiate successive* V *insertions.*—This gives a definitive prescription as to how SE may be differentiated once, provided that there is no SE insertion.
8. *At the place where the first differentiation is executed, convert the internal line into two external lines by cutting.*—This process transforms a proper SE′ into a V, since SE insertions have been removed at step 2.
9. *A proper* SE *is differentiated a second time by applying the prescription of Sec. 5 to the* V *obtained in step 8.*—It only remains to consider SE insertions.
10. *For* SE *insertions, repeat the prescriptions of this section.*—Note that the steps are arranged so that differentiating a SE once gives only SE′ and differentiating SE twice gives only SE″; in particular, the two momentum differentiations never appear on the same SE insertion.

This completes the prescription for differentiating a self-energy graph twice. Again note that all *twice* differentiated self-energy graphs can be obtained from the irreducible ones by *independent* SE, SE′, and V insertions. The prescriptions of these two sections therefore completely disentangle the problem of overlap divergences.

The results of applying the above prescriptions to a few graphs of low orders are explicitly shown in Figs. 17–18. In Fig. 17, the pair of numbers that appear below the vertex graphs indicate the symmetry of the graph; the first number is just $\mathcal{S}$ mentioned above, while the second symmetry number $\mathcal{S}'$ indicates the symmetry with respect to the attaching of external lines, i.e., $24/\mathcal{S}'$ different graphs may be obtained from the one shown by permuting the external lines. The meaning of symbols that appear in Fig. 18 is as follows. A dot near a symbol for momentum differentiation indicates that this particular differentiation is obtained by step 9 above. The number in the upper right corner of a graph indicates whether their skeletons are identical or not; for example, 1 means the first irreducible graph, while 2 and 2′ differ only in the position of the dot. The presence of a letter indicates a reducible graph; for example, 1*a* means that this graph may be obtained from graph 1 by an insertion at a vertex designated by *a*, while 1*ab* means that it may be obtained from 1 by insertions at the two vertices *a* and *b*. It is interesting to note that graphs that differ only in the position of the dot may carry different numerical coefficients, which include the symmetry number $\mathcal{S}$; for example, the graph 3′ does not appear at all in Fig. 18. The graph shown in Fig. 19 is the simplest one where the prescriptions so far does not determine uniquely the method of differentiation.

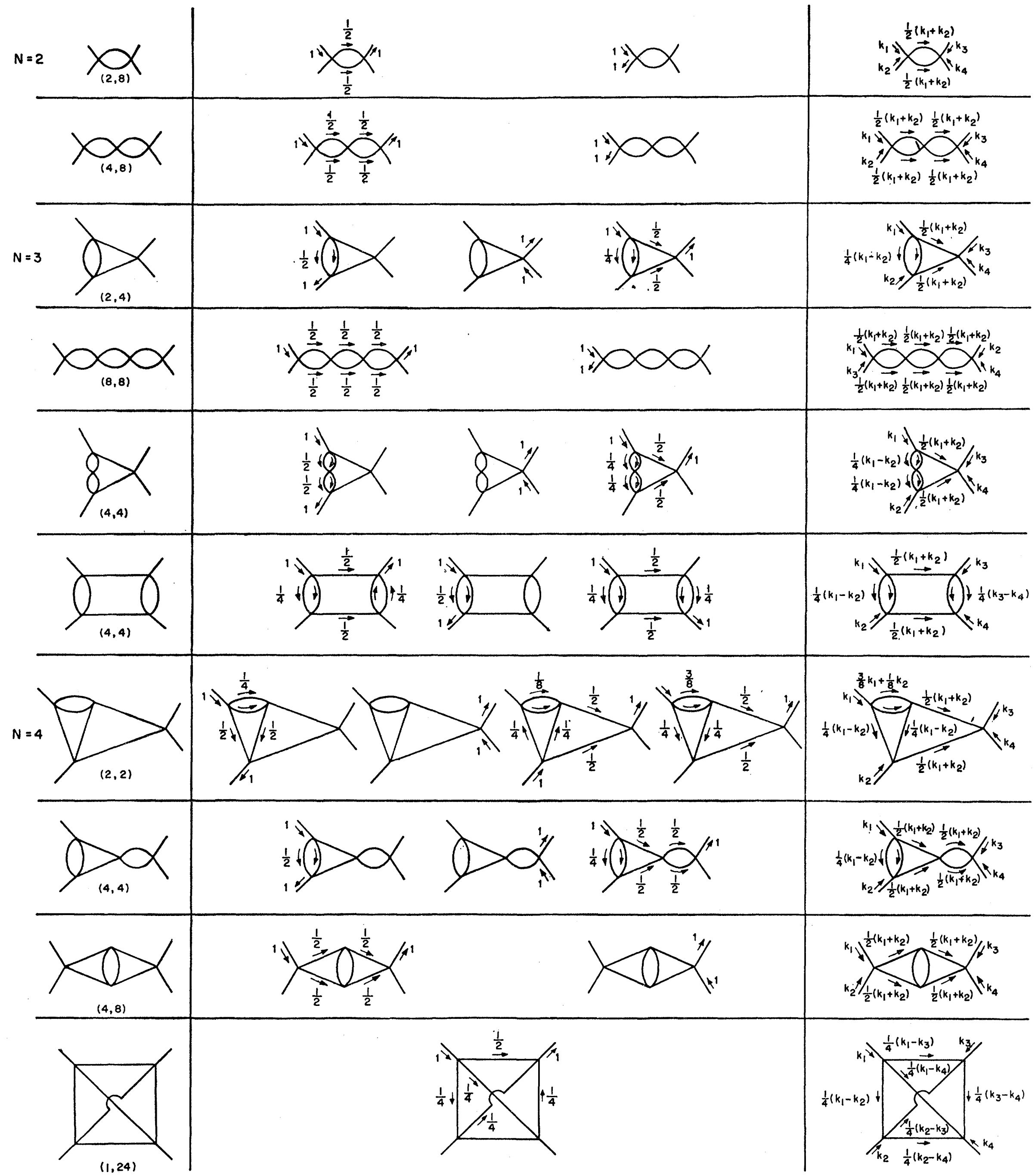


FIG. 17. Examples of the momentum differentiations of vertex graphs.

7. RENORMALIZATION

As seen from the examples in Fig. 18, each SE′ or SE″ carries a numerical coefficient, to be designated by $\mathcal{T}$. For graphs without momentum differentiation, this coefficient is simply the inverse of the intrinsic symmetry number $\mathcal{S}$. Furthermore, this coefficient can also be defined for V′, being just the number shown in the right column of Fig. 17 divided by the $\mathcal{S}$ for the corresponding V. For those admissible graphs that contain at least one momentum differentiation and do not appear in the process of carrying out the differentiation as prescribed, $\mathcal{T}$ is defined to be zero.

Given an admissible graph G, let $\mathfrak{F}_0(k_1,I_1;\cdots k_n,I_n;\mathcal{A},\mathcal{B},\mathcal{C}_0;G)$ be the value obtained by applying the

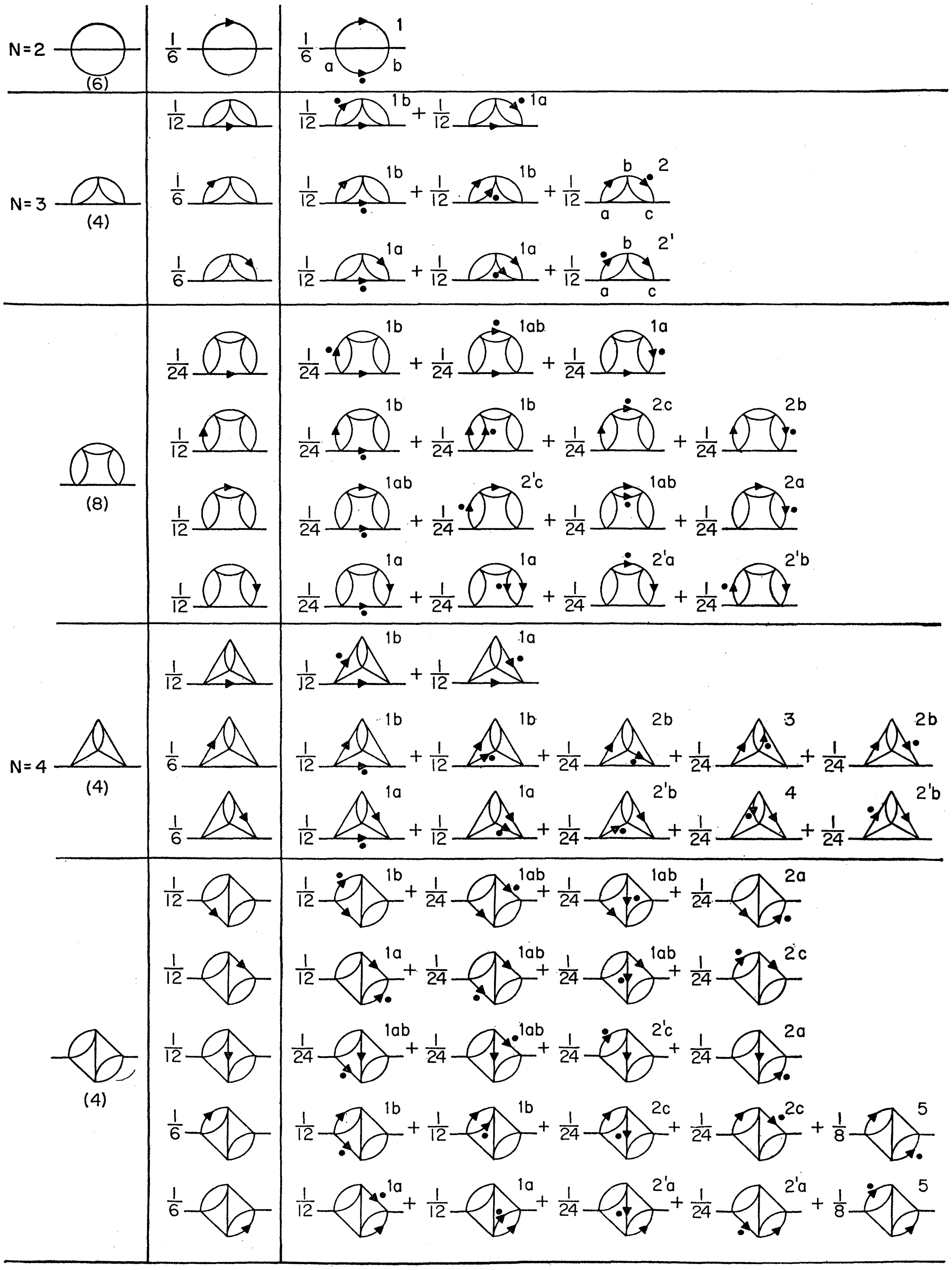


FIG. 18. Examples of the momentum differentiations of self-energy graphs.

generalized Feynman rules shown in Fig. 20 to G, carrying out the necessary integrations and multiplying by $\mathcal{T}$. If G does not contain a momentum differentiation, and the functions $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}_0$ are taken so that the rules of Fig. 1 are used, then $\mathfrak{F}_0$ differs from a partial matrix element of the S matrix only in the absence of

the factors that correspond to external lines and the δ-function expressing over-all energy-momentum conservation. For invariance under a rotation in isotopic-spin space and crossing symmetry, the function $\mathcal{C}_0$ may be taken to have the form

$$\begin{aligned}\mathcal{C}_0(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4)\\ =\mathcal{C}(k_1,k_2;k_3,k_4)\delta(I_1,I_2)\delta(I_3,I_4)\\ +\mathcal{C}(k_1,k_3;k_2,k_4)\delta(I_1,I_3)\delta(I_2,I_4)\\ +\mathcal{C}(k_1,k_4;k_2,k_3)\delta(I_1,I_4)\delta(I_2,I_3),\end{aligned} \quad (7)$$

where δ denotes the Kronecker delta, and $\mathcal{C}$ is a Lorentz-invariant function with the symmetry

$$\mathcal{C}(k_1,k_2;k_3,k_4)=\mathcal{C}(k_2,k_1;k_3,k_4)=\mathcal{C}(k_3,k_4;k_1,k_2). \quad (8)$$

Since (7) is always satisfied, define

$$\begin{aligned}\mathfrak{F}_0(k_1,I_1;\cdots k_n,I_n;\mathcal{A},\mathcal{B},\mathcal{C}_0;G)\\ =\mathfrak{F}(k_1,I_1;\cdots k_n,I_n;\mathcal{A},\mathcal{B},\mathcal{C};G).\end{aligned} \quad (9)$$

If

$$\Delta_F{}^0(k^2)=(k^2+m^2-i\epsilon)^{-1}$$

is the Feynman propagator for the free field, then the generalized rules reduce to the original ones when

$$\mathcal{A}=\Delta_F{}^0,\quad \mathcal{B}=-2,\quad \text{and}\quad \mathcal{C}=-i\lambda. \quad (10)$$

With the notation of Dyson,[1] the following sums over graphs are defined:

$$\Sigma(k^2)=\sum_{\text{SE}}\mathfrak{F}(k,I;\Delta_F{}^0,-2,-i\lambda;G); \quad (11)$$

$$\Sigma^*(k^2)=\sum_{\text{proper SE}}\mathfrak{F}(k,I;\Delta_F{}^0,-2,-i\lambda;G); \quad (12)$$

$$\Sigma_1{}^*(k^2)k_\mu =-2+\sum_{\text{proper SE}'}\mathfrak{F}(k,I;\Delta_F{}^0,-2,-i\lambda;G); \quad (13)$$

$$\Sigma_2{}^*(k^2)\delta_{\mu\nu}+\Sigma_3{}^*(k^2)k_\mu k_\nu =\sum_{\text{proper SE}''}\mathfrak{F}(k,I;\Delta_F{}^0,-2,-i\lambda;G); \quad (14)$$

$$\begin{aligned}\Gamma_0(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4)=\sum_{\text{V}}\mathfrak{F}(k_1,I_1;k_2,I_2;\\ k_3,I_3;k_4,I_4;\Delta_F{}^0,-2,-i\lambda;G);\end{aligned} \quad (15)$$

$$\begin{aligned}\Lambda_0(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4)\\ =\Gamma_0(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4)+i\lambda[\delta(I_1,I_2)\delta(I_3,I_4)\\ +\delta(I_1,I_3)\delta(I_2,I_4)+\delta(I_1,I_4)\delta(I_2,I_3)];\end{aligned} \quad (16)$$

$$\Delta_F(k^2)=\Delta_F{}^0(k^2)+\Delta_F{}^0(k^2)\Sigma(k^2)\Delta_F{}^0(k^2); \quad (17)$$

and

$$\begin{aligned}\Lambda_{10}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4)=\sum_{\text{V}'}\mathfrak{F}(k_1,I_1;k_2,I_2;\\ k_3,I_3;k_4,I_4;\Delta_F{}^0,-2,-i\lambda;G).\end{aligned} \quad (18)$$

Both Γ_0, Λ_0, and Λ_{10} are admissible as $\mathcal{C}_0$; i.e., they are expressible in the form (7). Call the corresponding functions Γ, Λ, and Λ_1. Then it follows from (16) that

$$\Lambda(k_1,k_2;k_3,k_4)=\Gamma(k_1,k_2;k_3,k_4)+i\lambda. \quad (19)$$

Other relations between the functions defined above are

$$\Sigma=\Sigma^*(1+\Delta_F{}^0\Sigma)=\Sigma^*+\Sigma^*\Delta_F\Sigma^*, \quad (20)$$

$$\Delta_F=\Delta_F{}^0+\Delta_F{}^0\Sigma^*\Delta_F, \quad (21)$$

$$\Sigma_2{}^*=\Sigma_1{}^*+2=2\Sigma^{*\prime}, \quad (22)$$

$$\Sigma_3{}^*=4\Sigma^{*\prime\prime}, \quad (23)$$

$$(1-\Delta_F{}^0\Sigma^*)(1+\Delta_F\Sigma^*)=1, \quad (24)$$

$$\Delta_F{}'=\Delta_F{}^2(\Sigma^{*\prime}-1), \quad (25)$$

$$\Lambda_{10}=\mathfrak{D}\Lambda_0, \quad (26)$$

and

$$\Lambda_1=\mathfrak{D}\Lambda, \quad (27)$$

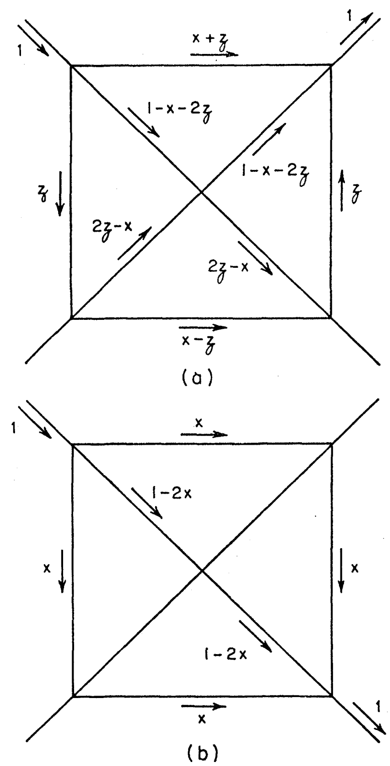


FIG. 19. Simplest vertex graph with arbitrariness in differentiation.

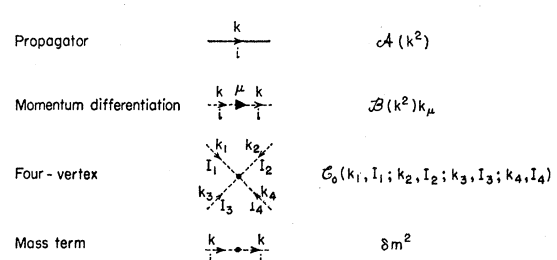


FIG. 20. Generalized Feynman rules.

where

$$\mathfrak{D}=\sum_\mu(k_{1\mu}\partial/\partial k_{1\mu}+k_{2\mu}\partial/\partial k_{2\mu}+k_{3\mu}\partial/\partial k_{3\mu}+k_{4\mu}\partial/\partial k_{4\mu}). \quad (28)$$

Since all proper SE″ and V′ can be obtained from the irreducible ones by independent SE, SE′, and V insertions, the following equations are results of (14) and (18):

$$\Sigma_2^*(k^2)\delta_{\mu\nu}+\Sigma_3^*(k^2)k_\mu k_\nu=\sum_{\text{irred. SE}''}\mathfrak{F}(k,I;\Delta_F,\Sigma_1^*,\Gamma;G), \quad (29)$$

and

$$\Lambda_{10}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4)=\sum_{\text{irred. V}'}\mathfrak{F}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4;\Delta_F,\Sigma_1^*,\Gamma;G). \quad (30)$$

Note that in irreducible SE″ and V′, the mass term δm^2 cannot appear. Equations (29) and (30) are the Dyson integral equations, when (22), (20), (21), (26), (27), and (19) are used. Their iterative solution reproduces all the original graphs. This is true even though (23) is ignored.

Besides λ, this system of equations contains two constants due to (22) and (27). The constant from (22) may be chosen at will by suitably selecting δm^2. Thus the system of equations may be rewritten as follows:

$$\Sigma_1^*(k^2)=-2+L_\Sigma+\tfrac{1}{3}\sum_{\mu,\nu}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\times\sum_{\text{irred. SE}''}\mathfrak{F}(k',I;\Delta_F,\Sigma_1^*,\Gamma;G)\Big|_{k'^2=-m^2}^{k'=k},$$

$$\Delta_F(k^2)=\left\{\Delta_F{}^0(k^2)^{-1}-\tfrac{1}{2}\int_{-m^2}^{k^2}dk'^2[\Sigma_1^*(k'^2)+2]\right\}^{-1},$$

and

$$\Gamma(k_1,k_2;k_3,k_4)=-i\lambda+L_\Lambda+\int_0^1\frac{d\alpha}{\alpha}\sum_{\text{irred. V}'}\mathfrak{F}(\alpha k_1',1;\alpha k_2',1;\alpha k_3',2;\alpha k_4',2;\Delta_F,\Sigma_1^*,\Gamma;G)\Big|_0^1, \quad (31)$$

where 1 denotes evaluation at the point $k_i'=k_i$ for all i, and 0 means evaluation at the "symmetry point" $k_i^2=-m^2$, $(k_i+k_j)^2=-\frac{4}{3}m^2$ for all $i\neq j$. Formally, the numbers L_Σ and L_Λ are given by $L_\Sigma=\Sigma_1^*(-m^2)$ and $L_\Lambda=\Lambda(k_1,k_2;k_3,k_4)|_0$.

If the solution of (31) is designated by $\Sigma_1^*(\lambda,L_\Sigma,L_\Lambda)$, $\Delta_F(\lambda,L_\Sigma,L_\Lambda)$, and $\Gamma(\lambda,L_\Sigma,L_\Lambda)$, then it may be verified that

$$\Sigma_1^*(\lambda,L_\Sigma,L_\Lambda)=Z\Sigma_1^*(\bar\lambda,0,0),$$
$$\Delta_F(\lambda,L_\Sigma,L_\Lambda)=Z^{-1}\Delta_F(\bar\lambda,0,0), \quad (32)$$

and

$$\Gamma(\lambda,L_\Sigma,L_\Lambda)=Z^2\Gamma(\bar\lambda,0,0),$$

provided that

$$Z=1-\tfrac{1}{2}L_\Sigma,$$

and the renormalized coupling constant $\bar\lambda$ is given by

$$\bar\lambda=Z^{-2}(\lambda+iL_\Lambda).$$

Furthermore, for any graph G, formally

$$\mathfrak{F}[k_1,I_1;\cdots k_n,I_n;\Delta_F(\lambda,L_\Sigma,L_\Lambda);\Sigma_1^*(\lambda,L_\Sigma,L_\Lambda);\Gamma(\lambda,L_\Sigma,L_\Lambda);G]=Z^{n/2}\mathfrak{F}[k_1,I_1;\cdots k_n,I_n;\Delta_F(\bar\lambda,0,0);\Sigma_1^*(\bar\lambda,0,0);\Gamma(\bar\lambda,0,0);G]. \quad (33)$$

This gives the wave-function renormalization.

It may be of interest to notice a difference between this procedure and that for quantum electrodynamics. In carrying out the momentum differentiation in the latter case, all derivatives are considered. Thus in integrating back, certain integrability conditions must be satisfied. In the case of quantum electrodynamics, these conditions are trivial, because a four-vector that depends on only the position four-vector is necessarily the gradient of a scalar. In the present case, these corresponding conditions are extremely complicated and are very difficult to satisfy iteratively. What is done here is to consider only certain combinations of derivatives such that there is no integrability condition. The purpose of momentum differentiation is to disentangle the overlap divergences, and there is no need to consider all possible derivatives.

8. FEYNMAN INTEGRALS

In this section, renormalized quantities are of main concern. Like the coupling constant, let a bar denote the renormalized quantity; for example,

$$\bar\Sigma_1^*=\Sigma_1^*(\bar\lambda,0,0).$$

It then follows from (31) that the renormalized system of integral equations is

$$\bar\Sigma_1^*(k^2)=-2+\tfrac{1}{3}\sum_{\mu,\nu}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\times\sum_{\text{irred. SE}''}\mathfrak{F}(k',I;\bar\Delta_F,\bar\Sigma_1^*,\bar\Gamma;G)\Big|_{k'^2=-m^2}^{k'=k}, \quad (34a)$$

$$\bar\Delta_F(k^2)=-2\left[\int_{-m^2}^{k^2}dk'^2\bar\Sigma_1^*(k'^2)\right]^{-1}, \quad (34b)$$

and

$$\bar\Gamma(k_1,k_2;k_3,k_4)=-i\bar\lambda+\int_0^1\frac{d\alpha}{\alpha}\sum_{\text{irred. V}'}\mathfrak{F}(\alpha k_1',1;\alpha k_2',1;\alpha k_3',2;\alpha k_4',2;\bar\Delta_F,\bar\Sigma_1^*,\bar\Gamma;G)\Big|_0^1. \quad (34c)$$

The solution of (34) gives the renormalized propagator and the renormalized vertex function. However, it is also useful to define the contributions $\mathcal{G}(G)$ from individual graphs G inductively as follows. First, δm^2

is ignored. Suppose the contributions from individual graphs with $N-1$ four-vertices minus momentum differentiations are defined. Consider an admissible graph with N four-vertices minus momentum differentiations. If its skeleton is convergent, then its contribution is defined in an obvious manner using insertions, whose contributions are known. Overlap insertions can cause no trouble. If the graph is a SE′, then its contribution is defined to be

$$\mathcal{G}_\mu(k,G)=\tfrac{1}{3}k_\mu \sum_{\nu,\sigma}\left(\delta_{\nu\sigma}-\frac{k_\nu k_\sigma}{k^2}\right)\sum_{G'} \mathcal{G}_{\nu\sigma}(k',G')\Big|_{k'^2=-m^2}^{k'=k}, \tag{35a}$$

where G' are the graphs obtained from G by one further momentum differentiation. If it is a SE, then

$$\mathcal{G}(k,G)=\left(\int_0^k-\int_0^{k'^2=-m^2}\right)\sum_{\mu,G'} dk_\mu'\ \mathcal{G}_\mu(k',G'), \tag{35b}$$

where G' are obtained from G by differentiation. If it is V, then

$$\mathcal{G}(k_1,k_2;k_3,k_4;G)$$
$$=\int_0^1\frac{d\alpha}{\alpha}\sum_{G'} \mathcal{G}(\alpha k_1,\alpha k_2;\alpha k_3,\alpha k_4;G')\Big|_0^1, \tag{35c}$$

where G' are again obtained from G by $\mathfrak{D}$.

It is the purpose of this section to show that Feynman integrals are still applicable to these renormalized quantities. First, consider a convergent graph G with n external lines and N four-vertices but no momentum differentiation. In this case, the number of internal lines is $2N-\frac{1}{2}n$, and the number of independent loops is $N-\frac{1}{2}n+1$. Let α_i be the Feynman parameter associated with the internal line i. Given external momenta $k_1,\cdots,k_n$, let p_i be the momentum associated with the internal line i when a circuit analog is used, i.e., p_i are determined by momentum conservation at each four-vertex and the condition that $\sum(\pm\alpha_i p_i)=0$ when summed over a closed loop. If the "power" dissipated in the graph is defined by

$$Q(\alpha_i)=\sum \alpha_i p_i^2, \tag{36}$$

then it has been shown by Nambu[7] and Symanzik[8] that

$$\mathcal{G}(G)=\text{const}\bar{\lambda}^N i^{\frac{1}{2}n+1}\Gamma(\tfrac{1}{2}n-2)\int_0^\infty\cdots\int_0^\infty d\alpha_1\cdots$$
$$\times d\alpha_{2N-\frac{1}{2}n}\delta(1-\textstyle\sum_i\alpha_i)d(\alpha_i)^{-2}[Q(\alpha_i)+m^2-i\epsilon]^{-\frac{1}{2}n+2}, \tag{37}$$

where d is a polynomial in α_i with non-negative coefficients such that Qd is a polynomial in α_i and the invariants formed from external momenta. The constant in (37) is real; and the factors of i arise as follows: use of the representation

$$(p_i^2+m^2-i\epsilon)^{-1}=i\int_0^\infty d\alpha_i\exp[-i\alpha_i(p_i^2+m^2-i\epsilon)]$$

gives $i^{2N-\frac{1}{2}n}$, the Feynman rule for the four-vertices gives $(-i)^N$, the Gaussian integrals lead to $i^{N-\frac{1}{2}n+1}$ because the metric is $3+1$, and the integration that leads to the δ function also gives $i^{-\frac{1}{2}n}$. Equation (37) requires $n>4$.

Secondly, consider a graph G' obtained from G by a momentum differentiation on the internal line j. Then a modification of the proof of (37) gives

$$\mathcal{G}(G')=\text{const}\bar{\lambda}^N i^{\frac{1}{2}n+1}\Gamma(\tfrac{1}{2}n-1)\int_0^\infty\cdots\int_0^\infty d\alpha_1\cdots$$
$$\times d\alpha_{2N-\frac{1}{2}n}\delta(1-\textstyle\sum_i\alpha_i)\alpha_j\sum_\mu p_{j\mu}k_\mu^{(j)}$$
$$\times d(\alpha_i)^{-2}[Q(\alpha_i)+m^2-i\epsilon]^{-\frac{1}{2}n+1}, \tag{38}$$

where $k_\mu^{(j)}$ is the vector indicating the amount of variation on this internal line; in the case of V, $k_\mu^{(j)}$ is a linear combination of $k_{i\mu}$ as prescribed in Sec. 5. Note that (38) only requires $n>2$ and hence is applicable to a vertex graph. Consider such a case of $n=4$; it follows from (38) that

$$\sum_{G'}\mathcal{G}(\alpha k_1,\alpha k_2;\alpha k_3,\alpha k_4;G')$$
$$=\text{const}\bar{\lambda}^N i^3\int_0^\infty\cdots\int_0^\infty d\alpha_1\cdots d\alpha_{2N-2}\delta(1-\textstyle\sum_i\alpha_i)$$
$$\times\alpha^2 Q(\alpha_i)d(\alpha_i)^{-2}[\alpha^2 Q(\alpha_i)+m^2-i\epsilon]^{-1}, \tag{39}$$

where $Q(\alpha_i)$ pertains to the external momenta k_1, k_2, k_3, and k_4 and is thus independent of α. For such a primitive divergent G, use of (35c) gives

$$\mathcal{G}(k_1,k_2;k_3,k_4;G)=\text{const}\bar{\lambda}^N i^3\int_0^\infty\cdots\int_0^\infty d\alpha_1\cdots$$
$$\times d\alpha_{2N-2}\delta(1-\textstyle\sum_i\alpha_i)d(\alpha_i)^{-2}\{\ln[Q(\alpha_i)+m^2-i\epsilon]$$
$$-\ln[Q_0(\alpha_i)+m^2-i\epsilon]\}, \tag{40}$$

where $Q_0(\alpha_i)$ is the value of $Q(\alpha_i)$ at the symmetry point. The new features are: (1) the appearance of a logarithm, and (2) the presence of an extra constant term. Insofar as analytic properties in perturbation theory are concerned, these features introduce no modification at all.

More generally, consider a convergent graph with the representation

$$\mathcal{G}(G)=\text{const}\,i\bar{\lambda}^N\int_0^\infty\cdots\int_0^\infty d\alpha_1'\cdots d\alpha_{2N-\frac{1}{2}n}'d(\alpha_i')^{-2}$$
$$\times\exp i[Q(\alpha_i')+(m^2-i\epsilon)\textstyle\sum_i\alpha_i'].$$

[7] Y. Nambu, Nuovo cimento **6**, 1064 (1957).
[8] K. Symanzik, Progr. Theoret. Phys. (Kyoto) **20**, 690 (1958).

In fact, (37) can be derived from this by the change of variables $\alpha_i'=\alpha_i s$ and an integration over s. Then, if $\mathbf{D}$ is any differential operator operating on the external momenta,

$$\sum \mathcal{G}(G')=\mathbf{D}\mathcal{G}(G)=\text{const}i\bar{\lambda}\int_0^\infty\cdots\int_0^\infty d\alpha_1'\cdots$$
$$\times d\alpha_{2N-\frac{1}{2}n}'d(\alpha_i')^{-2}\mathbf{D}\exp i[Q(\alpha_i')+(m^2-i\epsilon)\textstyle\sum_i\alpha_i'],$$

where G' is a graph obtained from G by any method of carrying out the differentiation. Moreover, Symanzik's proof[8] of (37) can be used to derive this last equation provided only that all G' are convergent. Accordingly, a further generalization shows that if G is a self-energy graph such that all corresponding double-prime graphs are irreducible, then

$$\mathcal{G}(G)=\text{const}\bar{\lambda}^N\int_{-m^2}^{k^2}dk'^2\int_0^\infty\cdots\int_0^\infty d\alpha_1'\cdots d\alpha_{2N-1}'d(\alpha_i')^{-2}$$
$$\times m^{-2}Q_0(\alpha_i')\{\exp i[Q(k'^2,\alpha_i')+(m^2-i\epsilon)\textstyle\sum_i\alpha_i']$$
$$-\exp i[Q_0(\alpha_i')+(m^2-i\epsilon)\textstyle\sum_i\alpha_i']\}$$
$$=\text{const}\bar{\lambda}^N\int_{-m^2}^{k^2}dk'^2\int_0^\infty\cdots\int_0^\infty d\alpha_1\cdots d\alpha_{2N-1}d(\alpha_i)^{-2}$$
$$\times\delta(1-\textstyle\sum_i\alpha_i)(-m^{-2})Q_0(\alpha_i)\{\ln[Q(k'^2,\alpha_i)$$
$$+m^2-i\epsilon]-\ln[Q_0(\alpha_i)+m^2-i\epsilon]\}$$
$$=\text{const}\bar{\lambda}^N\int_0^\infty\cdots\int_0^\infty d\alpha_1\cdots d\alpha_{2N-1}d(\alpha_i)^{-2}$$
$$\times\delta(1-\textstyle\sum_i\alpha_i)\big([Q(k^2,\alpha_i)+m^2-i\epsilon]$$
$$\times\{\ln[Q(k^2,\alpha_i)+m^2-i\epsilon]-\ln[Q_0(\alpha_i)+m^2-i\epsilon]\}$$
$$-Q(k^2,\alpha_i)+Q_0(\alpha_i)\big).$$

This is the desired Feynman integral for a self-energy graph in the simplest case.

Thirdly, insertions are to be considered. This can easily be done by using for example, the integral representation,

$$\ln[Q+m^2-i\epsilon]-\ln[Q_0+m^2-i\epsilon]$$
$$=-\int_0^\infty\frac{d\alpha}{\alpha}\{\exp[-i\alpha(Q+m^2-i\epsilon)]$$
$$-\exp[-i\alpha(Q_0+m^2-i\epsilon)]\}.$$

The rest of the computation is very similar to that used in the derivation of (37). The conclusion is therefore reached that $\mathcal{G}(G)$ is given by the expected Feynman integral together with certain linear combinations of the Feynman integrals for reduced graphs. Note that logarithms appear when $n\leq 4$ and that the integrations over the Feynman parameters should be carried out after linearly combining the various terms. If the integration is carried out first, the result may be divergent.

9. ANALYTIC PROPERTIES

Since the Landau[9] curves are still applicable after renormalization, it is only necessary to show that the zeros of d do not affect the domains of analyticity. For this purpose, consider an integral

$$I(m^2)=\int d\alpha_1\cdots d\alpha_n\,F(m^2),\tag{41}$$

where both I and F depend on some complex variables k_i, and F also depends on the α's. Let R be a star-shaped domain in the k space where $I(1)$ is analytic; then by dimensional consideration $I(m^2)$ is analytic in mR, where $k\in mR$ if $k/m\in R$.

Next suppose

$$F=(\partial/\partial m^2)\bar{F},$$

and

$$\bar{I}(m^2)=\int d\alpha_1\cdots d\alpha_n\,\bar{F}(m^2).\tag{42}$$

Then for any M

$$\bar{I}(m^2)=\bar{I}(M^2)-\int_{m^2}^{M^2}dm'^2\,I(m'^2).\tag{43}$$

Thus $\bar{I}(m^2)$ is analytic in mR if $\bar{I}(M^2)$ is. Insofar as Feynman integrals are concerned, note that the factors of d can be removed from the denominator by differentiation with respect to m^2, and that when m is chosen sufficiently large the Feynman integral over real α's is analytic in any compact set. Thus the zeros of d are completely irrelevant.

This also shows that the difficulty of Nakanishi[10] can never occur; his integral does not come from a Feynman diagram.

In particular, the representation

$$\bar{\Delta}_F(k^2)=\Delta_F{}^0(k^2)+\int_{9m^2}^\infty dM^2\,\rho(M^2)(k^2+M^2-i\epsilon)^{-1}\tag{44}$$

follows to any finite order in $\bar{\lambda}$.

10. UNITARITY

Without considering the various problems associated with renormalization, Zimmermann[11] has shown that the unitarity relation can be applied to each Feynman graph individually. More precisely by the unitarity relation the following is meant. Let $\mathcal{V}$ be the set of vertices of a given connected Feynman graph G. A "cut" is defined as a way of choosing a subset $\mathcal{V}_1$ of $\mathcal{V}$, with $\mathcal{V}_2=\mathcal{V}-\mathcal{V}_1$. Let $\mathcal{L}$ be the set of lines l of the graph such that one of the ends, l_1, is in $\mathcal{V}_1$ while the

[9] L. D. Landau, Nuclear Phys. **13**, 181 (1959).
[10] N. Nakanishi, Progr. Theoret. Phys. (Kyoto) **25**, 155 (1960).
[11] W. Zimmermann (private communication).

other end, l_2, is in $\mathcal{V}_2$; then the removal of all the lines in $\mathcal{L}$ splits G into G_1 and G_2, corresponding respectively to $\mathcal{V}_1$ and $\mathcal{V}_2$. Let $S(G)$, $S(G_1)$, and $S(G_2)$ be the S-matrix elements obtained from G, G_1, and G_2, respectively; then the unitarity relation is

$$\mathrm{Im}S(G)=\sum\int \mathrm{const}S(G_2)S^*(G_1)\prod_{l\in\mathcal{L}} i\Delta^+(l_2,l_1), \quad (45)$$

where Δ^+ is the usual positive-frequency part of the propagator for the free field; the integral is over all free variables including, for example, the sum over isotopic spin if any; the sum is over all possible cuts; and the constant is a number that depends on the normalization used for the S matrix. Summing (45) over all graphs gives formally the unitarity of the S matrix.

Zimmermann carried out his proof in the coordinate space and the proof is purely combinatorial. The only relevant consequence of the integrations over the internal coordinates of the graph is that many cuts can give no contribution to the right-hand side of (45) because of energy-momentum conservation.

It is seen in Sec. 8 that, for each admissible G, $\mathcal{G}(G)$ can be expressed as a finite sum of Feynman integrals, provided that the sum is taken before integrating over the Feynman parameters. Accordingly, $\mathcal{G}(G)$ can be expressed as a finite sum of integrals of products of $\Delta_F{}^0$, again provided that the sum is taken before integrating over the internal coordinates of the graph. Thus Zimmermann's proof is applicable after noticing for example that the constant in (40) arising from integrating $\ln[Q_0(\alpha_i)+m^2-i\epsilon]$ is purely imaginary. Thus the renormalized S matrix is unitary.

This argument also shows that the point used in renormalizing the coupling constant can be any point satisfying

$$k_1{}^2=k_2{}^2=k_3{}^2=k_4{}^2=-m^2,$$

$$(k_1+k_2)^2>-4m^2, \quad (k_1+k_3)^2>-4m^2,$$

and

$$(k_1+k_4)^2>-4m^2.$$

But of course the symmetry point is most convenient.

In general, unitarity holds when there is a system of subtraction rules to give finite results, but renormalizability further requires that the number of subtraction constants is finite. It is indeed a major advantage of using Feynman graphs that unitarity and crossing symmetry are trivially preserved.

One may ask why is (45) referred to as unitarity, since there is no requirement that on the mass shell the incoming and outgoing particles are separately included in the subsets $\mathcal{V}_1$ and $\mathcal{V}_2$. The answer is that this is compensated by the requirement that the Δ^+ functions in (45) all refer to internal lines. To see this, consider a simple example where $1+2\rightarrow 3+4+5$, and let the set $\mathcal{L}$ consist of two lines called 6 and 7. In what is ordinarily called the unitarity relation, there occurs in particular a term arising from the intermediate states $1+2\rightarrow 3+6+7\rightarrow 3+4+5$, where in the second step 3 does not interact and hence that portion of the S matrix is of the form $(3\rightarrow 3)(6+7\rightarrow 4+5)$. The particular cut under consideration gives precisely the contribution from this intermediate state.

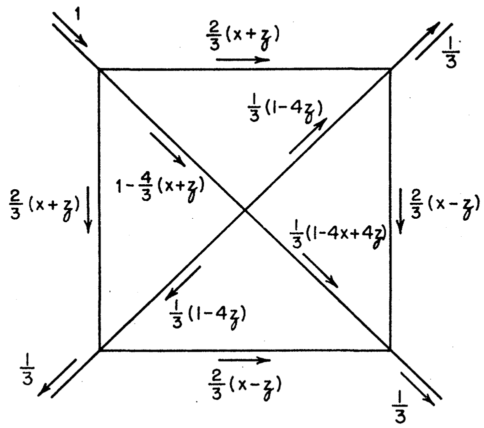


FIG. 21. Rule of differentiation obtained from those of Fig. 19.

11. SCHWINGER-EDWARDS IDENTITY

An analog of a relation used by Schwinger and Edwards[12] is the following formal identity:

$$\Sigma^*(k^2)=\delta m^2-i\lambda\sum_{I_1,I_2,I_3}\int\int\int d^4k_1d^4k_2d^4k_3\delta(k-k_1 -k_2-k_3)[\delta(I,I_1)\delta(I_2,I_3)+\delta(I,I_2)\delta(I_3,I_1) +\delta(I_1,I_3)\delta(I_1,I_2)]\Delta_F(k_1{}^2)\Delta_F(k_2{}^2)\Delta_F(k_3{}^2) \times\Gamma_0(k_1,I_1;k_2,I_2;k_3,I_3;k,I). \quad (46)$$

Although this identity is not relevant to the present renormalization program, it may be of interest to preserve it under momentum differentiation, i.e., to

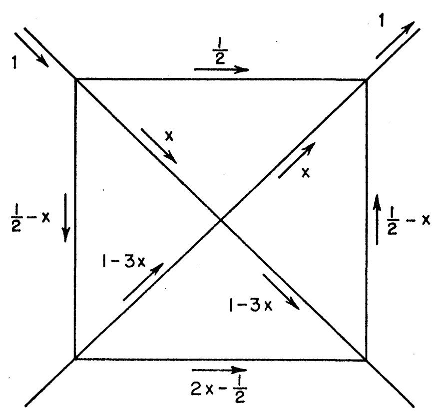


FIG. 22. A rule of differentiating the fifth-order superproper vertex to satisfy the Schwinger-Edwards identity.

[12] S. F. Edwards, Phys. Rev. **90**, 284 (1953).

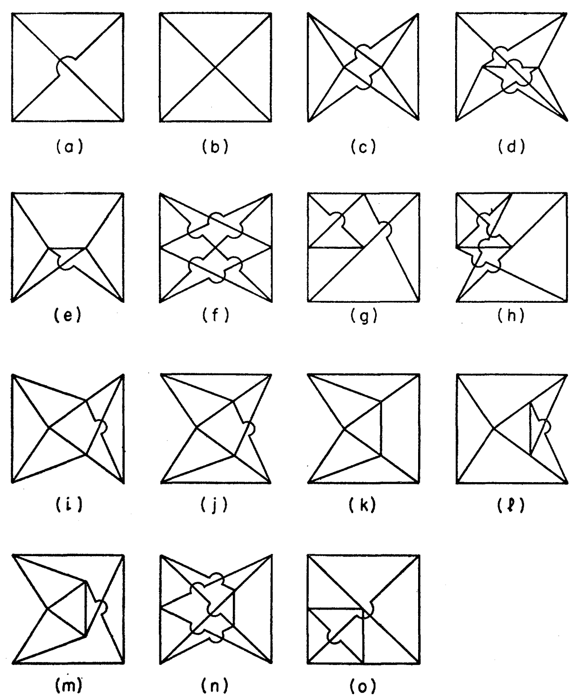


FIG. 23. Examples of superproper vertex graphs.

construct the rules of differentiating graphs such that the result of differentiating a self-energy graph coincides with that of differentiating the vertex graph obtained by deleting one of the two external four-vertices of the SE. For example, consider the graph shown in Fig. 19. A superposition of Fig. 19(a) and Fig. 19(b) is shown in Fig. 21. The three external lines, each carrying a weight $\frac{1}{3}$, may be connected to give a self-energy graph. Then a comparison with the rules of differentiating a self-energy graph shows that the formal fulfillment of the Schwinger-Edwards identity under differentiation requires

$$x+z=\tfrac{1}{2}.$$

When this is used in Fig. 19(a), Fig. 22 results.

Even from this simple example, it is seen that this further restriction does not uniquely determine the rules of differentiating superproper vertex graphs. It has been verified that this restriction can be fulfilled up to seventh order, and the superproper graphs considered are shown in Fig. 23. The numbers of indeterminate constants, like the x in fifth order, are as follows: fourth order, 0; fifth, 1; sixth, 2; seventh, 11. Thus it seems that this present requirement can be satisfied, but the author is unaware of any general proof.

ACKNOWLEDGMENTS

I am greatly indebted to Professor C. N. Yang for numerous discussions on all phases of this work and his correction of many errors, and to Professor T. D. Lee for several enlightening lectures on the case of quantum electrodynamics. Without their guidance this paper could not have been written. I would also like to thank Professor F. J. Dyson, Professor Kerson Huang, Professor R. W. P. King, and Professor W. Zimmerman for helpful discussions, and Professor J. R. Oppenheimer for his hospitality at the Institute.