existence of standing waves within the plasma layer. The power density at the hth harmonic varies as the input power density to the hth power for fixed values of the normalized plasma parameters. The peak values of the third- and fourth-harmonic powers, $Q_{3\,\max}$ and $Q_{4\,\max}$, have been discussed, indicating that they are independent of ω_p/ω in the range $0.1<\omega_p/\omega<0.8$. Outside this range both decrease rapidly. The peak values also vary inversely with ν/ω. Similar statements hold for the harmonic powers reflected from the plasma layer.

The question of the convergence of the series used to solve the equations has been examined. A condition on P_0, the incident power density at frequency ω, is derived such that the small signal analysis is valid.

An analysis of the effects of the nonlinear terms on propagation at the incident frequency ω is then discussed. This is accomplished through a reiteration procedure, including only the effects of the second harmonic, which yields a correction to the equation for the power transmitted through the layer calculated from the linearized equations. The results indicate that the correction can be as much as 50% for $\omega_p/\omega<0.2$ and $\nu/\omega<0.005$. This may be of importance when using an electromagnetic wave to measure the properties of a plasma and also when considering ionospheric propagation phenomena.

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Thermodynamic Behavior of Liquid Helium-Three in Its Possible Superfluid Phase. I*

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The thermodynamic behavior of liquid He^3 in its possible superfluid phase is investigated by extending the methods of Brueckner *et al.* They suggest that such a correlated phase can exist at very low temperatures due to the fact that there exist attractive D-state interactions near the Fermi surface. The free energy and the energy gap of the system for D-state interactions corresponding to different pure azimuthal modes are calculated at different temperatures. It is found that $l=2$, $m=2$ and $l=2$, $m=1$ modes correspond to the lowest free energy of the system near the critical temperature. In the intermediate range of temperatures the free-energy curves for the two modes, when the computations are made numerically, come out to be very nearly the same. But actually it can be shown by an analytical method that they are identical. The $l=2$, $m=0$ mode yields a higher free energy for all temperatures less than the critical temperature. The mixing of modes is investigated near the critical temperature. Any linear combination of all the modes $l=2$, $m=0$, 1, -1, 2, and -2 does not seem to lead to a lower free energy than that of the $l=2$, $m=\pm2$, and $m=\pm1$ modes. The correlation lengths at different temperatures are also analyzed. The specific heat and entropy curves for the $l=2$, $m=2$ mode are given.

I. INTRODUCTION

RECENTLY, it has been suggested by Brueckner, Anderson, Morel, and Soda[1,2] and Emery and Sessler[3] that liquid He^3 may have a superfluid phase at very low temperatures. They extended the method of Bardeen *et al.*[4] to a system in which the interactions are represented by non-spherically-symmetric potentials and found that a fermion system such as He^3 can become superfluid due to the attractive interaction in the $l=2$ state very close to the Fermi surface.

In the above-mentioned papers, the total energy and the energy gap of the system for the ground state have been calculated. The transition temperature T_c as well as the discontinuity of the specific heat at T_c have been

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determined. It is the purpose of this paper to explore the thermodynamic behavior of the system for temperatures below the transition temperature.

In Sec. II, we outline our method and list the various quantum-statistical formulas and other necessary expressions in a form convenient for use in later sections. In Sec. III we use numerical methods to calculate the energy gap and free energy of the system corresponding to different pure modes. In Sec. IV we discuss the problem of whether the system favors a mixing of the different modes. Near the transition temperature, we are able to show that some types of mixing of modes do lead to a lower free-energy state, corresponding to the pure $l=2$, $m=2$ or $m=1$ mode. In Sec. V, we investigate the correlation length. Finally, in Sec. VI, we obtain the specific heat curve of the system for $l=2$; $m=2$ mode below the transition temperature.

II. FORMULAS FOR THE ENERGY GAP AND THERMODYNAMIC QUANTITIES

In the earlier papers[1,2] the energy gap, the distribution function, and entropy of the system are calculated by a variational procedure which minimizes the free energy. Following the paper of Brueckner *et al.*[1] and using their notation, we take the interaction Hamiltonian as

$$H_i=-\sum_l (2l+1)V_l \sum_{\mathbf{k},\mathbf{k}'} c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger c_{-\mathbf{k}'} c_{\mathbf{k}'} P_l(\hat{k},\hat{k}'), \quad (2.1)$$

with

$$V_l=-(k_F|K_l|k_F). \quad (2.2)$$

The wave function of the system at a finite temperature T can be written as

$$\Psi=\underbrace{\prod_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger}_{\text{Single particles}} \underbrace{\prod_{\mathbf{k}'} \frac{b_{\mathbf{k}'}^\dagger - A_{\mathbf{k}'}}{(1+|A_{\mathbf{k}'}|^2)^{\frac{1}{2}}}}_{\text{Excited pairs}} \underbrace{\prod_{\mathbf{k}''} \frac{1+A_{\mathbf{k}''} b_{\mathbf{k}''}^\dagger}{(1+|A_{\mathbf{k}''}|^2)^{\frac{1}{2}}}}_{\text{Ground pairs}} \Phi_0, \quad (2.3)$$

(Φ_0 being the vacuum state). Ψ represents the existence of excited pairs, ground pairs, and single-particle excited states. $c_{\mathbf{k}\sigma}^\dagger$ is the creation operator for a single particle of momentum $\mathbf{k}$ and spin σ, and $b_{\mathbf{k}}^\dagger$ is the pair creation operator

$$b_{\mathbf{k}}^\dagger = c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger \quad \text{(for even values of } l). \quad (2.4)$$

The free energy of the system is

$$\begin{aligned} F&=E-TS \\ &=2\sum_{\mathbf{k}} |\epsilon_k| \left(f_{\mathbf{k}} + (1-2f_{\mathbf{k}}) \frac{A_{\mathbf{k}}^2}{1+|A_{\mathbf{k}}|^2} \right) \\ &\quad -4\pi \sum_{lm} V_l \sum_{\mathbf{k}} Y_{lm}(\hat{k},\hat{k}_0) \frac{A_{\mathbf{k}}^*}{1+|A_{\mathbf{k}}|^2}(1-2f_{\mathbf{k}}) \\ &\quad \times \sum_{\mathbf{k}'} Y_{lm}^*(\hat{k}',\hat{k}_0) \frac{A_{\mathbf{k}'}}{1+|A_{\mathbf{k}'}|^2}(1-2f_{\mathbf{k}'}) \\ &\quad +2\sum_{\mathbf{k}} \{ f_{\mathbf{k}} \ln f_{\mathbf{k}} + (1-f_{\mathbf{k}}) \ln(1-f_{\mathbf{k}}) \} kT. \end{aligned} \quad (2.5)$$

In the above, $f_{\mathbf{k}}$ is the distribution function for occupancy of state of momentum $\mathbf{k}$, the $A_{\mathbf{k}}$'s are the variational parameters introduced to represent the probabilities of the pair states, $\mathbf{k}$ being occupied. As in reference 1, we take the K matrix for the effective interactions. Since the interaction in the $l=2$ state seems to be predominant near the Fermi surface, we use only V_2. Introducing for simplicity the quantity $B_{\mathbf{k}}$ defined as

$$B_{\mathbf{k}}=A_{\mathbf{k}}/(1+|A_{\mathbf{k}}|^2), \quad (2.6)$$

and minimizing the free energy with respect to B_k^*, we obtain

$$\frac{2\epsilon_k B_{\mathbf{k}}}{(1-4|B_{\mathbf{k}}|^2)^{\frac{1}{2}}} = 4\pi V_2 \times \sum_m \sum_{\mathbf{k}'} [B_{\mathbf{k}'} Y_{2m}^*(\hat{k}',\hat{k}_0)(1-2f_{\mathbf{k}'}) Y_{2m}(\hat{k},\hat{k}_0)]. \quad (2.7)$$

With the definition

$$\epsilon_{02m}=4\pi V_2 \sum\nolimits_{\mathbf{k}'} B_{\mathbf{k}'} Y_{2m}^*(\hat{k}',\hat{k}_0)(1-2f_{\mathbf{k}'}), \quad (2.8)$$

Eq. (2.7) becomes

$$\frac{2\epsilon_k B_{\mathbf{k}}}{(1-4|B_{\mathbf{k}}|^2)^{\frac{1}{2}}} = \sum_m \epsilon_{02m} Y_{2m}(\hat{k},\hat{k}_0). \quad (2.9)$$

Let us introduce a quantity $c(\theta,\varphi)$ by defining

$$\frac{B_{\mathbf{k}}}{(1-4|B_{\mathbf{k}}|^2)^{\frac{1}{2}}} = \frac{c(\theta,\varphi)}{2\epsilon_k}. \quad (2.10)$$

Expanding $c(\theta,\varphi)$ in terms of spherical harmonics, we have

$$c(\theta,\varphi)=\sum\nolimits_{lm} c_{lm} Y_{lm}(\theta,\varphi). \quad (2.11)$$

From (2.9) and (2.10), we find that

$$c_{lm}=0 \text{ for } l\neq 2, \quad c_{lm}=\epsilon_{02m} \text{ for } l=2. \quad (2.12)$$

Hence

$$B_{\mathbf{k}}=\frac{\sum_{\mathbf{k}} [\sum_m \epsilon_{02m} Y_{2m}(\hat{k},\hat{k}_0)]}{2[\epsilon_k^2 + |\sum_m \epsilon_{02m} Y_{2m}(\hat{k},\hat{k}_0)|^2]^{\frac{1}{2}}}. \quad (2.13)$$

From Eq. (2.8) we see that

$$\epsilon_{02m}=4\pi V_2 \sum_{\mathbf{k}} \frac{Y_{2m}^*(\hat{k},\hat{k}_0)[\sum_{m'} \epsilon_{02m'} Y_{2m'}(\hat{k},\hat{k}_0)]}{2[\epsilon_k^2 + (|\sum_{m''} \epsilon_{02m''} Y_{2m''}(\hat{k},\hat{k}_0)|^2)]^{\frac{1}{2}}} \times (1-2f_{\mathbf{k}}). \quad (2.14)$$

By minimizing the free energy with respect to $f_{\mathbf{k}}$, we obtain

$$\ln \frac{f_{\mathbf{k}}}{1-f_{\mathbf{k}}} = E_{\mathbf{k}} = [\epsilon_k^2 + (|\sum_m \epsilon_{02m} Y_{2m}(\hat{k},\hat{k}_0)|^2)]^{\frac{1}{2}}, \quad (2.15)$$

and

$$f_{\mathbf{k}}=f(\beta E_{\mathbf{k}})=\frac{1}{e^{\beta E_{\mathbf{k}}}+1}, \quad (2.16)$$

where $E_{\mathbf{k}}$ represents the "single-particle-like" excitation energy. Equations (2.15) and (2.16) lead us to the integral equation for the energy gap,

$$\epsilon_{02m}(T)=\frac{4\pi V_2\sum_{\mathbf{k}} Y_{2m}^*(\hat{k},\hat{k}_0)[\sum_{m'}\epsilon_{02m'}Y_{2m'}(\hat{k},\hat{k}_0)]}{2[\epsilon_k^2+|\sum_{m'}\epsilon_{02m'}Y_{2m'}(\hat{k},\hat{k}_0)|^2]^{\frac{1}{2}}}\times\tanh(\beta E_{\mathbf{k}}/2). \quad (2.17)$$

The free energy and entropy of the system are

$$F=\sum_{\mathbf{k}}|\epsilon_k|-\sum_{\mathbf{k}}\{\epsilon_k^2+|[\sum_m \epsilon_{02m}Y_{2m}(\hat{k},\hat{k}_0)]|^2\}^{\frac{1}{2}}\tanh\frac{\beta E_{\mathbf{k}}}{2}-\sum_{\mathbf{k}}\frac{(\sum_m \epsilon_{02m}Y_{2m}(\hat{k},\hat{k}_0))\tanh(\beta E_{\mathbf{k}}/2)}{2[\epsilon_k^2+|(\sum_m \epsilon_{02m}Y_{2m}(\hat{k},\hat{k}_0)|^2]^{\frac{1}{2}}}-TS, \quad (2.18)$$

and

$$S=2k\sum_{\mathbf{k}}\{\beta E_{\mathbf{k}}f_{\mathbf{k}}+\ln(1+e^{-\beta E_{\mathbf{k}}})\}. \quad (2.19)$$

The specific heat $C_s=T(dS/dT)$ can be calculated from the entropy curve of the system.

III. CALCULATION OF THE ENERGY GAP AND FREE ENERGY OF THE SYSTEM

We shall now calculate the energy gap and the free energy of the liquid corresponding to different pure azimuthal modes for the D-state interaction in the liquid. The problem of mixing of the different modes will be dealt with in the next chapter.

According to Eq. (2.18), the energy gap for a particular pure mode m is

$$\epsilon_{02m}(T)=4\pi N^*(0)V_2\int d\Omega\times\int_0^{\Delta E}d\epsilon_k\frac{|Y_{2m}(\hat{k},\hat{k}_0)|^2\tanh(\beta E_{\mathbf{k}}/2)}{[\epsilon_k^2+|\epsilon_{02m}Y_{2m}(\mathbf{k},\mathbf{k}_0)|^2]^{\frac{1}{2}}}. \quad (3.1)$$

Here $N^*(0)$ is equal to $N(0)/4\pi$, where $N(0)$ is Bloch density of states at the Fermi surface.[1] We replace the function $\tanh(\beta E_{\mathbf{k}}/2)$ in the above by the expansion

$$\tanh(\beta E_{\mathbf{k}}/2)=1-2\sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\beta E_{\mathbf{k}}}. \quad (3.2)$$

Using the analogous expression for the energy gap at $T=0$, we can find the following equation for $\epsilon_{02m}(0)$

$$\frac{1}{N(0)V_2}=\int d\Omega\int_0^{\Delta E}\frac{|Y_{2m}|^2}{[\epsilon_k^2+|(\epsilon_{02m}(0)Y_{2m})^2|]^{\frac{1}{2}}}d\epsilon_k. \quad (3.3)$$

Hence we easily see that

$$\int d\Omega\int_0^{\Delta E}d\epsilon_k\frac{|Y_{2m}|^2}{[\epsilon_k^2+|\epsilon_{02m}(0)Y_{2m}|^2]^{\frac{1}{2}}}=\int d\Omega\int_0^{\Delta E}\frac{d\epsilon_k|Y_{2m}|^2}{\{\epsilon_k^2+|[\epsilon_{02m}(T)Y_{2m}]^2|\}^{\frac{1}{2}}}-2\sum_{n=1}^{\infty}(-1)^{n+1}\times\int d\Omega\int_0^{\Delta E}d\epsilon_k\frac{|Y_{2m}|^2e^{-n\beta E_{\mathbf{k}}}}{[\epsilon_k^2+|\epsilon_{02m}(T)Y_{2m}|^2]^{\frac{1}{2}}}. \quad (3.4)$$

Making change of variable

$$\epsilon_k=|\epsilon_{02m}(T)Y_{2m}|\sinh s, \quad (3.5)$$

we obtain

$$\ln\frac{\epsilon_{02m}(0)}{\epsilon_{02m}(T)}=2\sum_{n=1}^{\infty}(-1)^{n+1}\int d\Omega|Y_{2m}|^2\int_0^L ds\times\exp(-n\beta|\epsilon_{02m}(T)Y_{2m}|\cosh s). \quad (3.6)$$

where

$$L\equiv\sinh^{-1}(\Delta E/|\epsilon_{02m}(T)Y_{2m}|).$$

Taking as an approximation the upper limit L as infinity, which is very good near T_c and $T=0$, we find

$$\ln\frac{\epsilon_{02m}(0)}{\epsilon_{02m}(T)}=2\sum_n(-1)^{n+1}\int d\Omega|Y_{2m}|^2\times K_0(n\beta|\epsilon_{20m}(T)Y_{2m}|), \quad (3.7)$$

where K_0 is the zero-order modified Bessel function. We then do the integration on the right-hand side numerically using Weddle's rule, taking the integrand as a function of $x_{2m}=\beta\epsilon_{02m}(T)$ and varying this value x_{2m} over the desired temperature range. The results of these integrations and the subsequent sums over n are shown in Figs. 1(a), 1(b), and 1(c). To solve for the $\epsilon_{02m}(T)$ at different temperatures, we plot the function

$$F_{2m}=2\sum_{n=1}^{\infty}(-1)^{n+1}\int d\Omega|Y_{2m}|^2K_0(n\beta|\epsilon_{02m}Y_{2m}|), \quad (3.8)$$

as a function of x_{2m} on the semilog graph paper. The left-hand side of (3.7) can also be expressed as a function of x_{2m}, i.e.,

$$y_{2m}(x_{2m})=\ln[\beta_c\epsilon_{02m}(0)]-\ln x_{2m}-\ln(T/T_c) \quad (3.9)$$

and

$$y_{2m}(x_{2m})=F_{2m}(x_{2m}). \quad (3.10)$$

On the same semilog paper, y_{2m}, plotted as a function of x_{2m}, according to Eq. (3.9) will be a set of parallel lines. Different lines will correspond to different temperatures. The intersections of these lines with the curve drawn for $F_{2m}(x_{2m})$ according to Eq. (3.8) will fix the values of $x_{2m}=\beta\epsilon_{02m}(T)$ for that temperature. Figures 2(a), 2(b), and 2(c) indicate the procedure adopted. In these calculations we use the values of $\beta_c\epsilon_{02m}(0)$ determined in reference 1. Hence the energy gap at different temperatures is determined and the curves of Fig. 3 are obtained for different modes.

The free-energy calculation also follows in a similar manner. From (2.19) the expression for free energy is

$$F=\sum_{\mathbf{k}}|\epsilon_k|-\sum_{\mathbf{k}}[\epsilon_k^2+|\epsilon_{02m}Y_{2m}|^2]^{\frac{1}{2}}\tanh(\beta E_{\mathbf{k}}/2)-\sum_{\mathbf{k}}\frac{|\epsilon_{02m}Y_{2m}|^2\tanh(\beta E_{\mathbf{k}}/2)}{[\epsilon_k^2+|\epsilon_{02m}Y_{2m}|^2]^{\frac{1}{2}}}-4N^*(0)k\beta\int d\Omega\int_0^{\Delta E}d\epsilon_k\frac{\epsilon_k^2+E_{\mathbf{k}}^2}{E_{\mathbf{k}}}\times\tfrac{1}{2}[1-\tanh(\beta E_{\mathbf{k}}/2)]. \quad (3.11)$$

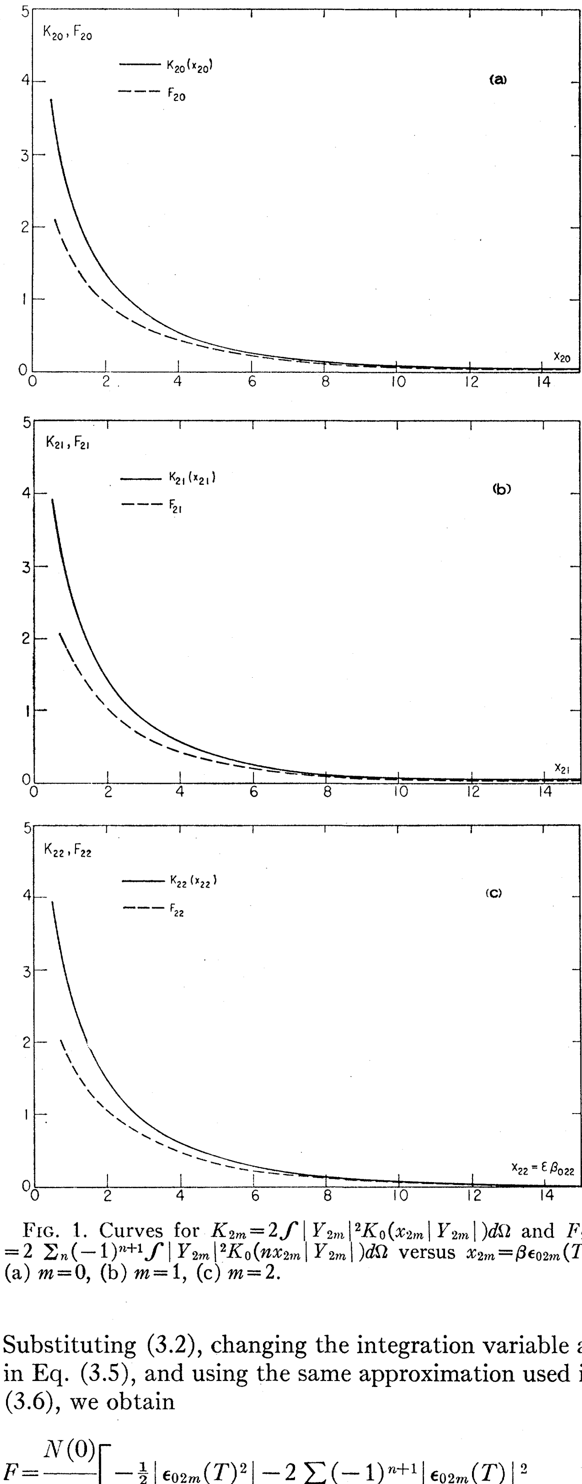


FIG. 1. Curves for $K_{2m}=2\int|Y_{2m}|^2K_0(x_{2m}|Y_{2m}|)d\Omega$ and $F_{2m}=2\sum_n(-1)^{n+1}\int|Y_{2m}|^2K_0(nx_{2m}|Y_{2m}|)d\Omega$ versus $x_{2m}=\beta\epsilon_{02m}(T)$. (a) $m=0$, (b) $m=1$, (c) $m=2$.

Substituting (3.2), changing the integration variable as in Eq. (3.5), and using the same approximation used in (3.6), we obtain

$$F=\frac{N(0)}{4\pi}\Big[-\tfrac{1}{2}|\epsilon_{02m}(T)^2|-2\sum_n(-1)^{n+1}|\epsilon_{02m}(T)|^2 \times\int d\Omega|Y_{2m}|^2\Big]K_2[(n\beta|\epsilon_{02m}(T)Y_{2m}|)]. \quad (3.12)$$

Using the parameter x_{2m} defined as above, we have for the free energy

$$\frac{F}{(kT_c)^2}\left(\frac{N(0)}{4\pi}\right)^{-1}=-x_{2m}^2(T/T_c)^2\Big[\tfrac{1}{2}+2\sum_n(-1)^{n+1} \times\int d\Omega|Y_{2m}|^2K_2(nx_{2m}|Y_{2m}|)\Big], \quad (3.13)$$

where K_2 is the modified Bessel function of second order.

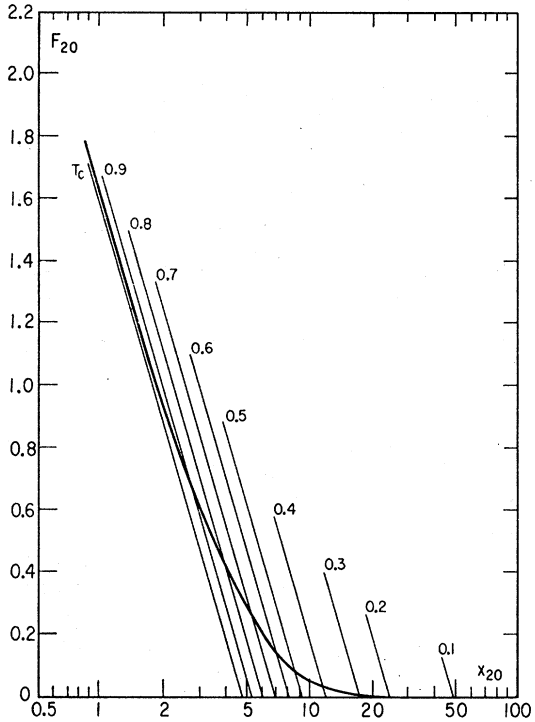
The numerical integration in (3.13) is carried out for different values of x_{2m} over the desired temperature range. After summing over the values of n until the series converges, we draw the curve for that integral as a function of x_{2m}. For different temperatures we know the corresponding values of the parameter x_{2m} from Eq. (3.10). Hence the free energies are evaluated for different temperatures from (3.13). The free-energy curves for the system for different pure azimuthal modes are drawn in Fig. 4. The curves indicate that near T_c, the $l=2$, $m=2$ mode leads to the lowest energy for the system. The $l=2$, $m=1$ mode also yields the same energy as the $l=2$, $m=2$ mode or nearly the same energy throughout the entire temperature range. It can be shown analytically that the $l=2$, $m=2$ mode leads to identical values as the $l=2$, $m=1$ mode. However, the $l=2$, $m=0$ mode always yields a higher free energy. Of course, the mixing of different modes has to be analyzed before determining lowest free-energy state of the system.

IV. MIXING OF AZIMUTHAL MODES

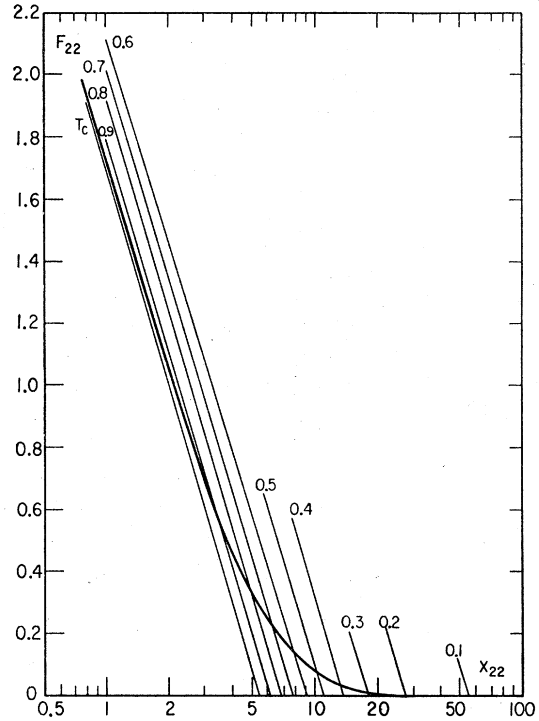
It is very important to find out whether pure modes only exist in the system. If a mixing of different modes leads to a lower free energy, the system will favor that state. But it is rather difficult to solve the energy gap equation (2.14) analytically for the case of mixed modes for all temperatures. Hence, we will obtain analytical expressions for a general mixture of modes near T_c.

Taking the energy gap equation (2.17) and expanding the factor $\{[\tanh(\beta E_k/2)]/E_k\}$ in powers of $(|\sum_m \epsilon_{0lm}Y_{lm}|^2)$, which goes to zero at $T=T_c$, we obtain for ϵ_{0lm}'s the following integral equation:

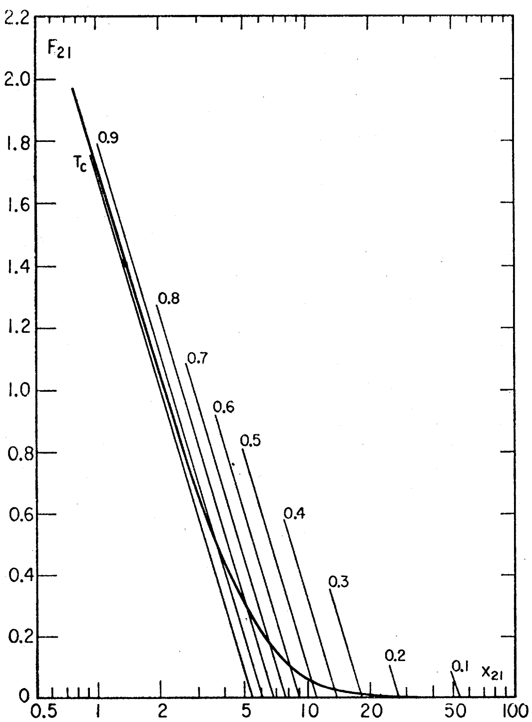
$$\frac{\epsilon_{02m}(T)}{N(0)V_2}=\int_0^\infty\frac{\tanh(\beta_c\epsilon/2)}{\epsilon}d\epsilon\,\epsilon_{02m}(T)+\int_0^\infty\frac{\operatorname{sech}^2(\beta_c\epsilon/2)}{2}d\epsilon\,(\beta-\beta_c)\epsilon_{02m}(T)-\frac{\beta^2}{8}\int_0^\infty\left(\frac{\tanh y}{y^3}-\frac{\operatorname{sech}^2 y}{y^2}\right)dy \times\int|\sum_{m'}\epsilon_{02m'}(T)Y_{2m'}|^2\sum_{m''}\epsilon_{02m''}(T)Y_{2m''}Y_{2m}^*d\Omega,$$



(a)



(c)



(b)

FIG. 2. Curves for $F_{2m}=2\sum_n(-1)^{n+1}\int|Y_{2m}|^2K_0(nx_{2m}|Y_{2m}|)d\Omega$ and lines for $y_{2m}=\ln\beta_c\epsilon_{02m}(0)-\ln x_{2m}-\ln(T/T_c)$ versus $x_{2m}=\beta\epsilon_{02m}(T)$ in the logarithmic scale. (a) $m=0$, (b) $m=1$, and (c) $m=2$.

or

$$\epsilon_{02m}K=\int Y_{2m}{}^*\sum_{n,\mu,\nu}\epsilon_{02n}Y_{2n}\epsilon_{02\mu}{}^*Y_{2\mu}{}^*\epsilon_{02\nu}Y_{2\nu}d\Omega, \quad (4.1)$$

where

$$K=[10.2/(\beta_c)^2](1-T/T_c).$$

By using the spherical harmonics for $l=2$, the above

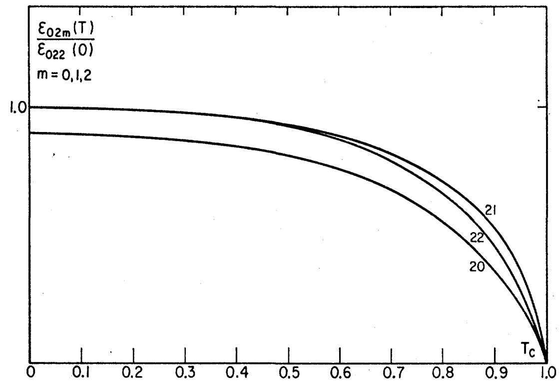


FIG. 3. Curves for energy gaps $\epsilon_{02m}(T)/\epsilon_{022}(0)$ versus temperature T/T_c.

equation reduces to the following:

$$K\epsilon_{02m}=\epsilon_{02m}\sum_n|\epsilon_{02n}|^2\lambda_{|m||n|}+\epsilon_{02m}\sum_{n\neq m}|\epsilon_{02n}|^2\lambda_{|m||n|}$$

$$+(-1)^m\epsilon_{02-m}^*\sum_{n\neq m,\,n\neq -m}\epsilon_{02-n}\epsilon_{02n}(-1)^n\lambda_{|m||n|},\qquad(4.2)$$

where $\lambda_{|m||n|}=\int|Y_{2m}|^2|Y_{2n}|^2d\Omega$. This set of 10 coupled equations for ϵ_{02m}'s and ϵ_{02m}^*'s is solved in a generalized manner in Appendix A.

The free-energy expression represented by Eq. (2.18) can be obtained in powers of $(|\sum_m\epsilon_{0lm}Y_{lm}|^2=c^2)$ near T_c by using the same kind of Taylor expansion. Since $\partial F/\partial(c^2)|_{c^2=0,\,T=T_c}$ is identically zero, the free-energy expression is

$$F_s=F_n-\tfrac{1}{2}\int|c^2|^2d\Omega\frac{\beta^2}{8}N(0)\times\int_0^\infty\left(\frac{\tanh y}{y^3}-\frac{\operatorname{sech}^2 y}{y^2}\right)dy,\qquad(4.3)$$

where F_n is the normal free energy at that temperature. From Eq. (4.1) the second term on the right-hand side of (4.3) is $QK\sum_m|\epsilon_{0lm}|^2$, where

$$Q=\tfrac{1}{2}\int_0^\infty\frac{\beta^2}{8}N(0)\left(\frac{\tanh y}{y^3}-\frac{\operatorname{sech}^2 y}{y^2}\right)dy.\qquad(4.4)$$

From the set of equations (4.2), $\sum_m|\epsilon_{02m}|^2$ is solved in the Appendix, giving six solutions besides the pure mode solution. The most favorable solutions which give the highest value are obtained as

$$\textstyle\sum_m|\epsilon_{02m}|^2=K/\lambda,\qquad(4.5)$$

where

$$\lambda=\int|Y_{22}|^4d\Omega=\int|Y_{21}|^4d\Omega.$$

Hence, the lowest free energy obtained for the combined mode is $F_s=F_n-Q\times K^2/\lambda$. For the pure model $l=2$,

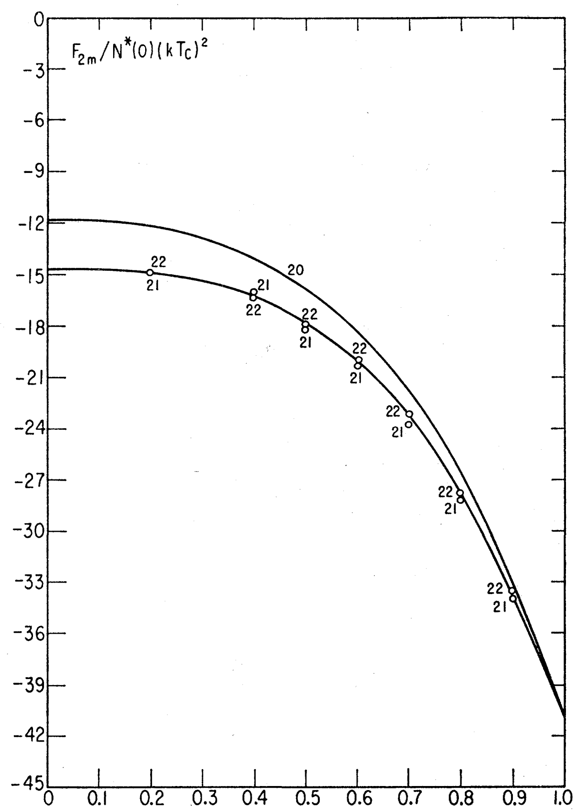


FIG. 4. Curves for free energies for pure state $l=2$; $m=0$, $m=1$, and $m=2$ versus T/T_c.

$m=2$, the free energy can also be shown to be

$$F_s=F_n-QK^2/\lambda.\qquad(4.6)$$

Hence we see that the pure mode $l=2$, $m=2$, as well as $l=2$ and $m=1$, lead to the lowest free energy as also some other solutions corresponding to a mixing of different modes. Also it is found that for $l=2$, $m=0$, the free energy is higher than the pure $l=2$, $m=2$ and $m=1$ cases. It is also to be noted that some asymmetric mixing of modes; i.e., the case in which we mix $l=2$, $m=2$, $l=2$, $m=1$, and $l=2$, $m=0$, yields a lower energy equal to the pure $l=2$, $m=2$ case and all symmetric mixings, i.e., the cases in which m and $-m$ are mixed, lead to higher free energies for the system. All possible general solutions are obtained in Appendix A. In no case does the mixing lead to a free energy lower than the pure mode case $l=2$, $m=2$ or $m=1$.

Considering the case of the single mode by Eq. (4.3), we have

$$F_s=F_n-Q\int|\epsilon_{0lm}|^4|Y_{lm}|^4d\Omega.\qquad(4.7)$$

By solving the equation for the energy gap (4.1) for a pure mode, we obtain

$$|\epsilon_{02m}{}^2|=(10.2/\lambda)(1/\beta^2)(1-T/T_c).\qquad(4.8)$$

In view of the above (4.8), the absence of $(|\epsilon_{02m}|^2)$ term in (4.7) proves that the transition is one of second order.

A general discussion of the ground state is not attempted here.

V. CORRELATION LENGTH

Having introduced a directional correlation in the liquid to obtain the superfluid state, we are interested in calculating the length over which the correlation might persist.

According to the BCS theory, the correlation function for fermions of opposite spin is

$$\rho_A = n[\tfrac{1}{2}n + P_A(r)], \tag{5.1}$$

where r is the relative distance between two fermions, n the number of fermions of both spins for unit volume, and $P_A(r)$ is the following expression at $T=0$:

$$P_A(r) = \frac{1}{2m}\left(\frac{1}{2\pi}\right)^6 \int\int d\mathbf{k}d\mathbf{k}' \times \left\{ e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} \frac{|\epsilon_{02m}(0)|^2 Y_{2m}(\hat{k},\hat{k}_0) Y_{2m}(\hat{k}',\hat{k}_0)}{E\times E'} \right\}. \tag{5.2}$$

The range of the spatial correlation may be determined by investigating this $P_A(r)$. We evaluate the function

$$I(r) = \left(\frac{1}{2\pi}\right)^3 \int d\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{E} Y_{2m}(\hat{k},\hat{k}_0). \tag{5.3}$$

Expanding $e^{i\mathbf{k}\cdot\mathbf{r}}$ in terms of spherical harmonics, we get

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{l'm'} (i)^{l'} J_{l'}(kr) Y_{l'm'}^*(\hat{k},\hat{k}_0) Y_{l'm'}(\hat{k}_0,\hat{r}). \tag{5.4}$$

Since we are interested in a region of integration where $k_F r \gg 1$, we have the relation

$$J_{l'}(kr) \simeq (-1)^{l'+l'/2}(1/kr) \begin{Bmatrix} \sin(kr), & l'=\text{even} \\ \cos(kr), & l'=\text{odd} \end{Bmatrix} + \text{higher order terms in } 1/(kr), \tag{5.5}$$

and we set (as was done in BCS)

$$k \cong k_F + \left.\frac{\partial k}{\partial \epsilon_k}\right|_F \epsilon_k = k_F + \epsilon_k/hv_F, \tag{5.6}$$

where v_F is the Fermi velocity and

$$\begin{aligned} \sin[(k_F+\epsilon_k/hv_F)r] &\simeq \sin(k_F r)\cos(\epsilon_k r/hv_F), \\ \cos[(k_F+\epsilon_k/hv_F)r] &\simeq \cos(k_F r)\cos(\epsilon_k r/hv_F). \end{aligned} \tag{5.7}$$

Then we have

$$I(r) = \frac{N(0)}{(2\pi)^3} \int d\Omega 2 \int_0^{\Delta E} \sum_{l'm'} (-1)^{2l'} \times \frac{Y_{l'm'}^*(\hat{k},\hat{k}_0) Y_{lm}(\hat{k},\hat{k}_0)}{2[\epsilon_k^2 + |\epsilon_{02m}(0)Y_{2m}|^2]^{\frac{1}{2}}} \cos\left(\frac{\epsilon_k r}{hv_F}\right) d\epsilon_k \times \frac{1}{k_F r} \begin{Bmatrix} \sin(k_F r) \\ \cos(k_F r) \end{Bmatrix} Y_{l'm'}(\hat{k}_0,\hat{r}). \tag{5.8}$$

Using the approximation for the limits as in Sec. III, we find

$$I(r) = \frac{N(0)}{(2\pi)^3} \frac{1}{k_F r} \begin{Bmatrix} \sin(k_F r) \\ \cos(k_F r) \end{Bmatrix} \sum_{l'm'} (4\pi)^{\frac{1}{2}} Y_{l'm'}(\hat{k}_0,\hat{r}) \times \int \frac{d\Omega}{(4\pi)^{\frac{1}{2}}} Y_{l'm'}^*(\hat{k},\hat{k}_0) Y_{2m}(\hat{k},\hat{k}_0) \times K_0\left(\frac{|\epsilon_{02m}(0)Y_{2m}(\hat{k},\hat{k}_0)|r}{hv_F}\right). \tag{5.9}$$

Since it is reasonable to expect a large contribution to the integral from $l'=2$, $m'=m$ in the summation, we approximate the value of the above as

$$I(r) = \left(\frac{1}{2\pi}\right)^3 \frac{N(0)}{k_F r} \sin(k_F r)(4\pi)^{\frac{1}{2}} Y_{2m}(\hat{k}_0,\hat{r}) \times \int \frac{d\Omega}{(4\pi)^{\frac{1}{2}}} |Y_{2m}(\hat{k}_0,\hat{k})|^2 \times K_0\left(\frac{|\epsilon_{02m}(0)|r|Y_{2m}(\hat{k},\hat{k}_0)|}{hv_F}\right). \tag{5.10}$$

The correlation length R can be taken as the difference between values of r corresponding to the maximum value of the function $I(r)$ and that corresponding to

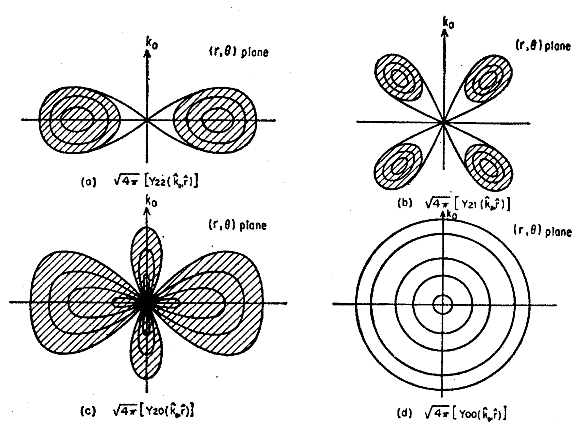


FIG. 5. Spatial patterns for the correlation function $I(r)$: $l=2$; (a) $m=2$, (b) $m=1$, (c) $m=0$, and (d) $l=0$; $m=0$ (BCS case).

$1/e$ times this maximum value. In particular, if we take the $l=2$, $m=0$ mode, the maximum value of $I(r)$ occurs at $r=0$ and assuming that $\sin(k_F r)/(k_F r)$ does not oscillate too much in this range of r, we find the the approximate value of R (the correlation length), using Fig. 1(c), to be about 400 A. The case of $l=2$, $m=1$ or 2 differs from the above case, since $I(r)$ vanishes at $r=0$, and its maximum occurs at a different point for these two modes.

From (5.10) we see that the expression for $I(r)$ has a factor $(4\pi)^{\frac{1}{2}}Y_{2m}(\hat{k}_0,\hat{r})$ which gives a peculiar pattern for the quantity $I(r)$ in the liquid. The space dependence of this pattern in a given plane represented by the shaded areas [inside of $(4\pi)^{\frac{1}{2}}Y_{2m}(\hat{k}_0,\hat{r})$ figures] is given in Fig. 5. The shaded areas represent those regions in which the function $I(r)$ has maximum strength; outside this region $I(r)$ tails off to zero. For the pure modes, $l=2$, $m=2$ or 1, the correlation vanishes near $r=0$, and for $m=0$ is a maximum at $r=0$. This is indicated by the amount of shading present in different parts of the patterns in the Fig. 5. From these considerations we note that the angle-dependent correlations vanish in some directions.

To find the correlation lengths at higher temperatures, we have to calculate the matrix element

$$\rho(r',r'';\sigma',\sigma'') =\langle\Psi_{\text{excited}}|\Psi_{\sigma''}^*(r'')\Psi_{\sigma'}^*(r')\Psi_{\sigma'}(r')\Psi_{\sigma''}(r'')|\Psi_{\text{excited}}\rangle, \tag{5.11}$$

where Ψ_{excited} are the excited-state wave functions. This matrix element can connect different possible initial and final states and, taking into account all such possible nonvanishing contributions, we obtain for the function $I(r)$ for higher temperatures the following expression

$$I(r)=\frac{1}{(2\pi)^3}\int d\mathbf{k}\frac{Y_{2m}(\hat{k},\hat{k}_0)}{[\epsilon_k^2+|\epsilon_{02m}(T)Y_{2m}|^2]^{\frac{1}{2}}} \times e^{i\mathbf{k}\cdot\mathbf{r}}\tanh[\tfrac{1}{2}\beta(\epsilon_k^2+|\epsilon_{02m}Y_{2m}|^2)^{\frac{1}{2}}]. \tag{5.12}$$

From the values of the quantity $\beta\epsilon_{02m}(T)$ determined from Figs. 2(a), 2(b), and 2(c) for different values of T/T_c, it is found that it exceeds unity considerably in ranges of temperatures from 0 to $0.8T_c$. Hence, as a very crude approximation, we can take $\tanh(\beta E_k/2)$ as 1 in this range. Hence, the function $I(r)$ is the same as that for $T=0$, except for the fact that $\epsilon_{02m}(0)$ is to be replaced by $\epsilon_{02m}(T)$ in Eq. (5.9). As before, for the $l=2$; $m=0$ mode at $r=0$, the function $I(r)$ reaches a constant value and hence the correlation length at any temperature T in this range will be of the form

$$R_{20}(T)=400[\epsilon_{020}(0)/\epsilon_{020}(T)]A. \tag{5.13}$$

For the other two modes $l=2$, $m=2$ and $l=2$, $m=1$ also, if, as a very crude approximation we assume the integrals represented by the curves 1(b), 1(c), etc. to be of the type e^{-Ax}, where A can be determined by finding the slope of the curve in the region between the points where it reaches the largest value and the point where this value is reduced by a factor of $1/e$, we can show that $R_{2m}(T)$ is of a form very similar to Eq. (5.13) and is given by

$$\begin{aligned} R_{22}&\cong 225[\epsilon_{022}(0)/\epsilon_{022}(T)]\text{A}, \\ R_{21}&\cong 245[\epsilon_{021}(0)/\epsilon_{021}(T)]\text{A}. \end{aligned} \tag{5.14}$$

At T_c the correlation strength function P_A defined in Eq. (5.1) goes to zero since P_A contains $|\epsilon_{02m}(T)|^2$ as a factor and liquid becomes a normal fluid.

VI. SPECIFIC HEAT

In the previous sections we have discussed the various thermodynamic aspects of the system, though we are only sure of the lowest excitation mode of the system near T_c, where the energy corresponding to the $l=2$, $m=2$ mode is one of the lowest possible energy state of the system. Any linear combination of the different modes does not lead to a lower energy near T_c. Near $T=0$, and in the intermediate range of temperatures, the $l=2$, $m=2$ mode, as well as the $l=2$, $m=1$ mode, yield the same energy (for the system) which is lower than that of the $l=2$, $m=0$ mode. Though we are not certain whether any linear combination of the modes will lead to lower free energy of the system in these lower temperature ranges, we shall draw the entropy and specific heat curves for the $l=2$, $m=2$ mode throughout the whole range. Near T_c at least it will represent the true state of affairs.

The expression for entropy is

$$S=4N^*(0)k\beta\int d\Omega\int_0^{\Delta E}\frac{\epsilon_k^2+E_\mathbf{k}^2}{E_\mathbf{k}}f(\beta E_\mathbf{k})d\epsilon_k. \tag{6.1}$$

Effecting the same change in the integration variable as was done before in Eq. (3.5) and approximating the upper limit of integration to infinity, we obtain

$$S=4N^*(0)k^2Tx_{2m}^2\sum_{n=1}^{\infty}(-1)^{n+1} \times\int d\Omega|Y_{2m}|^2K_2(n|x_{2m}Y_{2m}|),$$

where

$$x_{2m}=\epsilon_{02m}(T)\beta. \tag{6.2}$$

For different temperatures the value of x_{2m} can be found out as was done in Sec. III and the corresponding value of the entropy calculated. The entropy curve for $l=2$, $m=2$ mode is shown in Fig. 6.

The specific heat $C_s=T(dS/dT)$ can be calculated at each point of the entropy curve and a graph for the specific heat of the system is drawn in Fig. 7. The curve shows a discontinuity at T_c corresponding to the transition from the normal to the superfluid phase. The

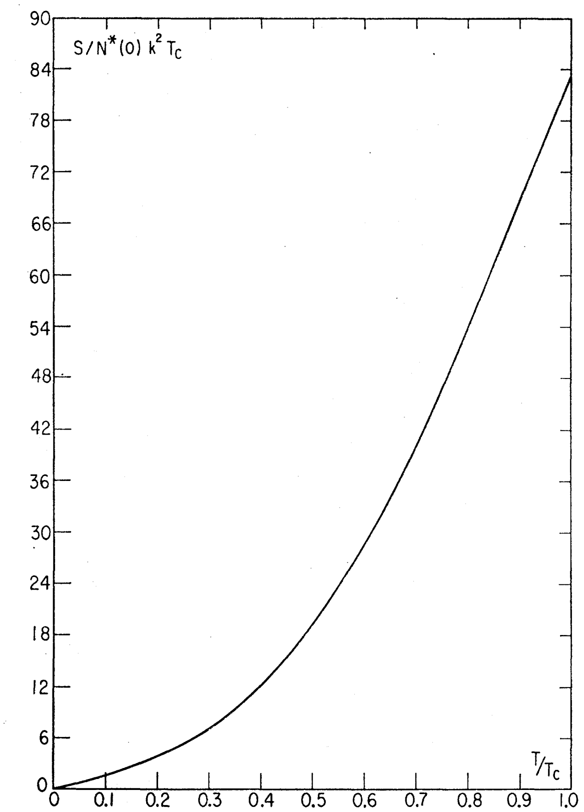


FIG. 6. Entropy curve for $l=2$, $m=2$ versus T/T_c.

ratio of the jump in the specific heat to the normal specific heat at T_c can be calculated as

$$(C_s(l=2, m=2)-C_n)/C_n=1.08. \tag{6.3}$$

In conclusion, we may point out that the thermodynamic behavior of liquid He[3] discussed in the previous chapters can be analyzed very well near T_c; and the extension to lower temperatures can be accomplished only by more detailed numerical computations.

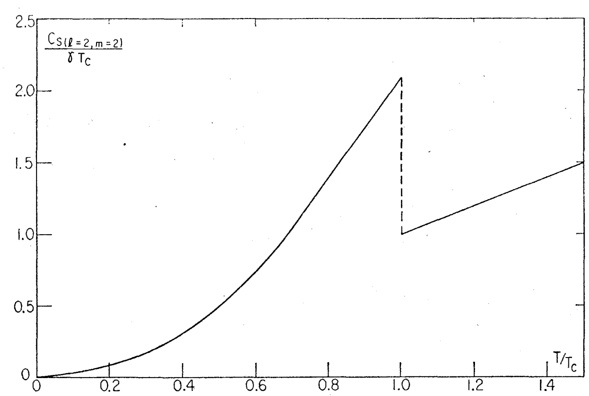


FIG. 7. Specific heat curve for $l=2$, $m=2$ case.

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APPENDIX A

We will here solve Eq. (4.2) in the most general case and we are interested in the expression for the quantity $\sum_n|\epsilon_{02n}|^2$ which enters into our free-energy calculations [Eq. (4.3)] in the case of general mixing of modes. Let us rewrite Eq. (4.2) as

$$[K-2\sum_n|\epsilon_{02n}|^2\lambda_{|m||n|}+|\epsilon_{02m}|^2\lambda_{|m||m|}]\epsilon_{02m} = (-)^m\epsilon_{02-m}{}^* \sum_{n\neq m,\, n\neq -m}\epsilon_{20n}\epsilon_{02-n}(-)^n\lambda_{|m||n|}. \tag{A1}$$

Let us assume the most general form for the ϵ_{02m}'s as

$$\epsilon_{02m}=\eta_m e^{i\alpha_m};\quad \epsilon_{02-m}=\eta_{-m}e^{i\alpha_{-m}}, \tag{A2}$$

where η_m is the modulus of the quantity ϵ_{02m}. Equation (A1) can be written

$$\frac{\eta_m}{\eta_{-m}}[K+|\epsilon_{02m}|^2\lambda_{|m||n|}-2\sum_n|\epsilon_{02n}|^2\lambda_{|m||n|}] = (-)^m e^{-i(\alpha_m+\alpha_{-m})}\sum_{n\neq m,\, n\neq -m}\eta_n\eta_{-n}e^{i(\alpha_n+\alpha_{-n})}. \tag{A3}$$

Replacing the m by $(-m)$ in the above, we obtain an equation which, when multiplied on both sides by Eq. (A3), yields

$$\eta_m{}^2(K-2\sum_n|\epsilon_{02n}|^2\lambda_{|m|\,|n|}+\eta_m{}^2\lambda_{|m|\,|n|}) = \eta_{-m}{}^2(K-2\sum_n|\epsilon_{02n}|^2\lambda_{|m|\,|n|}+\eta_{-m}{}^2\lambda_{|m|\,|m|}). \tag{A4}$$

Two possibilities arise, namely, either $\eta_m{}^2=\eta_{-m}{}^2$ or $\eta_m{}^2\neq\eta_{-m}{}^2$. In the former case, Eq. (A4) is only an identity. Hence if we take the latter case $\eta_m{}^2\neq\eta_{-m}{}^2$, we arrive at the result that

$$K/\lambda=\eta_2{}^2+\eta_1{}^2+\eta_{-2}{}^2+\eta_{-1}{}^2+\eta_0{}^2=\sum_n|\epsilon_{02n}|^2, \tag{A5}$$

by virtue of the values of $\lambda_{|m|\,|n|}$ obtained earlier, $\lambda_{12}=\lambda_{10}=\lambda_{20}=\lambda/2$ and $\lambda_{00}=\frac{3}{2}\lambda$.

Taking the other possibility $\epsilon_{02m}{}^2=\epsilon_{02-m}{}^2$, we can assume

$$\epsilon_{02m}=\eta_{|m|}e^{i\alpha_m}\quad \epsilon_{02-m}=\eta_{|m|}e^{i\alpha_{-m}}. \tag{A6}$$

Putting these values of ϵ_{02m} in Eq. (A1) we obtain

$$(K-\sum_n \eta_n{}^2\lambda_{|m||n|}-\sum_{n\neq m}\eta_n{}^2\lambda_{|m||n|})=(-)^m e^{-i(\alpha_m+\alpha_{-m})}$$
$$\times[\sum_{n\neq m,\, n\neq -m}\eta_{|n|}{}^2\lambda_{|m||n|}e^{i(\alpha_n+\alpha_{-n})}]. \quad \text{(A7)}$$

Since the left-hand side of Eq. (A7) is real, the right-hand side also should be real, and hence we obtain the condition

$$\sum_{n\neq m,\, n\neq -m}\eta_{|n|}{}^2\lambda_{|m||n|}(-)^n \times\sin(\alpha_n+\alpha_{-n}-\alpha_m-\alpha_{-m})=0, \quad \text{(A8)}$$

for all values of m.

If we identify groups of angles $(2\alpha_0-\alpha_1-\alpha_{-1})$, $(2\alpha_0-\alpha_2-\alpha_{-2})$, and $(\alpha_1+\alpha_{-1}-\alpha_2-\alpha_{-2})$ as a, b, and c, respectively, we obtain the following conditions from (A8):

$$(\eta_{|2|}{}^2)/(\eta_{|1|}{}^2)=\sin a/\sin b; \quad \text{(i)}$$
$$-2(\eta_{|2|}{}^2/\eta_0{}^2)=-\sin a/\sin c; \quad \text{(ii)} \quad \text{(A9)}$$

and

$$2(\eta_{|1|}{}^2/\eta_0{}^2)=\sin b/\sin c. \quad \text{(iii)}$$

Writing the real part of Eq. (A7) for the value of $m=2$, we get

$$K=3\eta_2{}^2\lambda_{22}+4\eta_1{}^2\lambda_{21}+2\eta_0{}^2\lambda_{20}+[\eta_0{}^2\lambda_{20}\cos(2\alpha_0-\alpha_2-\alpha_{-2}) -2\eta_1{}^2\lambda_{21}\cos(\alpha_1+\alpha_{-1}-\alpha_2-\alpha_{-2})]. \quad \text{(A10)}$$

If the angles a, b, c are not zero or multiples of π, we can make use of Eqs. (A9) and obtain

$$\textstyle\sum_n \eta_{|n|}{}^2=K/\lambda. \quad \text{(A11)}$$

If, however, a, b, and c are odd or even multiples of π, we can go to the real part of Eq. (A7) and write them as follows for the three values of $m=1$, 2, and 0.

$$\begin{aligned}K/\lambda&=3\eta_{|2|}{}^2+2\eta_{|1|}{}^2(1-\tfrac{1}{2}R)+\eta_0{}^2(1+\tfrac{1}{2}M),\\ K/\lambda&=2\eta_{|2|}{}^2(1-\tfrac{1}{2}R)+3\eta_{|1|}{}^2+\eta_0{}^2(1-\tfrac{1}{2}L),\\ K/\lambda&=2\eta_{|2|}{}^2(1+\tfrac{1}{2}M)+2\eta_{|1|}{}^2(1-\tfrac{1}{2}L)+\tfrac{3}{2}\eta_0{}^2,\end{aligned} \quad \text{(A12)}$$

where $L=\cos a$, $M=\cos b$, and $R=\cos c$. In the above, L and M can individually take values of ±1 according as a, b assume even or odd multiples of π. The value of R is determined by those of L and M, since $c=(b-a)$. Thus there are four possible cases and if we solve these four sets of equations, we find that the sum $\sum_n \eta_{|n|}{}^2$ takes the value either $2K/3\lambda$ or K/λ. We can similarly consider asymmetric mixing of either (1) ϵ_{02-2} and ϵ_{02-1} are both missing, or (2) one of them is missing. The results are such that there are no greater values for $\sum_m|\epsilon_{02m}|^2$ than K/λ. The case of a special symmetric mixing of the $+m$ and $-m$ mode is shown to lead to $(\frac{2}{3})(K/\lambda)$ for the value of $\sum_m|\epsilon_{02m}|^2$. Thus the general mixing of all modes in view of the Eqs. (4.3) and (4.6) does not lead to a free energy lower than the case of the pure mode $l=2$, $m=2$.