PHYSICAL REVIEW VOLUME 125, NUMBER 5 MARCH 1, 1962

Diffusion and Drift of Minority Carriers in Semiconductors for Comparable Capture and Scattering Mean Free Paths

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(Received October 13, 1961)

A method of treating transport of injected minority carriers is developed applicable to cases in which the physical dimension and the mean free path for capture may be less than the mean free path for scattering. The basic differential equations of scattering and capture are those of the conservation of flux method of McKelvey, Longini, and Brody, and the results agree with theirs, the new feature being a demonstration that the basic equations are equivalent to a continuity equation of the conventional form but with a diffusion constant reduced by including the effect of capture in shortening the mean free path. This method of treatment reduces the problems to a familiar form when suitable boundary conditions are introduced. The basic differential equations of scattering and capture are shown to correspond to certain simplifying and restricting assumptions about the carrier velocity distributions. The treatment is extended from the case of one dimension with zero electric field to three dimensions with electric fields.

I. INTRODUCTION

THE conventional method of dealing with minority carriers in semiconductors is based on various approximations. For the case of low electric fields, the flux of carriers is taken as the sum of the diffusion current and the drift current. The former is proportional to the concentration gradient and the latter to the concentration times the electric field. For materials of very short lifetime or small dimensions, in which the mean free path for scattering is no longer small compared to other lengths, the usual formulas for currents are not valid.

A recent attempt has been made by McKelvey, Longini, and Brody[1] to furnish a treatment which is more general and applicable to cases not having a restriction to the ratio of mean free paths. In this treatment (referred to as the MLB method) these authors do not use the diffusion equation or Fick's law in treating the diffusion currents. Their method is based upon considering fluxes of carriers. The influence of the material upon the flux is expressed through previously derived[2,3] transmission and absorption coefficients for regions of finite thickness bounded by parallel planes perpendicular to the flux direction. In this way, they avoid introducing the approximations of the diffusion equation, although their final results for solved problems coincide with the conventional ones when the mean free path for scattering is much less than the mean free path for recombination.

It is the purpose of this analysis to show that the MLB method is actually equivalent to a method involving the continuity equation with certain modifications in the diffusion constants and to generalize this treatment to include small electric fields and three dimensions.

[1] J. P. McKelvey, R. L. Longini, and T. P. Brody, Phys. Rev. 123, 51 (1961).
[2] J. W. Coltman, E. G. Ebbighausen, and W. Altar, J. Appl. Phys. 18, 530 (1947).
[3] R. L. Longini, J. Opt. Soc. Am. 39, 551 (1949).

II. DIFFERENTIAL EQUATIONS FOR THE FLUX FORMULATION

Figure 1 illustrates the definitions of the terms used in the MLB formulation. A thin region of thickness dx is considered. A flux to the right of carriers of strength r carriers per unit area per unit time is incident upon the left side of the slab and a corresponding flux to the left represented by l is incident upon the right side. The fluxes emerging from the slab are influenced by recombination, scattering, and generation in the region. (We shall consider electric fields in Sec. VI.) According to the assumptions of the model, a fraction kdx of the incident flux is reversed by the slab of thickness dx. A fraction wdx of the flux is combined within the slab. In addition, the rate of generation of carriers per unit volume is assumed to be g so that per unit area of the slab the total generation is gdx; half of this generation is supposed to emerge and add to the flux coming out of each side. These relationships are represented in Fig. 1.

Expressed in differential terms, the rate of change of the fluxes to the right and to the left are readily seen to be given by the following two equations:

$$dr/dx=k(l-r)-wr+g/2, \tag{1}$$

$$dl/dx=k(l-r)+wl-g/2. \tag{2}$$

The case of thermal equilibrium corresponds to random motion with half the particles moving to the right and

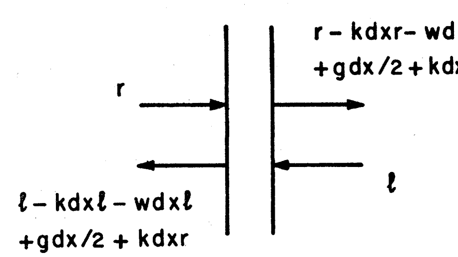

FIG. 1. The assumed effect of an infinitesimal layer upon the fluxes to right and to left.

half to the left, and equal values of $g/2w$ for both r and l. The complete solution for Eqs. (1) and (2) may be obtained by adding exponential terms to this constant value so that the general solution may be represented as follows:

$$r=AR_\infty e^{qx}+Be^{-qx}+g/2w, \tag{3}$$

$$l=Ae^{qx}+BR_\infty e^{-qx}+g/2w. \tag{4}$$

The value of the coefficient q necessary to solve simultaneously Eqs. (1) and (2) is readily found to be

$$q=|(2kw+w^2)^{\frac{1}{2}}|, \quad k=(q^2-w^2)/2w, \tag{5}$$

and the ratio of values of the coefficients of the exponential terms is given by

$$R_\infty=k/(w+k+q)=(w+k-q)/k=(q-w)/(q+w). \tag{6}$$

A and B may be arbitrarily chosen to fit boundary conditions discussed later.

The fact that the coefficients k and w are taken as independent of the way in which the flux arises implies that certain assumptions and approximations have actually been made in the MLB treatment. For example, if a flux of carriers crosses a p-n junction and enters a region containing many recombination centers so that the distance of penetration is very small, then the flux will not decay exponentially as given by one of the terms in Eq. (3), but instead will initially decay more rapidly than at greater depths because initially the carriers which are moving nearly tangentially to the junction surface are recombined very near that surface. Further from the surface, only those carriers traveling approximately perpendicular to the surface will be found. These will have on the average greater depths of penetration, and an effective smaller value of w will occur for them. This problem is like the sorting for hard and soft components of radiation which occurs whenever an inhomogeneous flux is present. These variations are rejected by assuming a constant w in the MLB treatment. Similar comments apply to k.

The significance of the coefficient R_∞ may be understood as follows: For the case in which r and l are dominated by the e^{qx} term, it is evident that a flux of carriers is proceeding to the left. The strength of this flux to the left is proportional to A. Associated with it at any given plane there is a flux to the right smaller by the factor R_∞. It is evident that R_∞ represents the reflection coefficient for flux proceeding toward an infinite medium.

Essential to the MLB treatment are reflection coefficients for the situation represented in Fig. 2, in which a region of width a is subject to a flux $l(a)$ incident upon its right side at $x=a$. On the left side at $x=0$ it is assumed that no interference with the emerging flux occurs, and whatever emerges from that side does not return. This corresponds to a case in which $r(0)$ is zero:

$$r=0 \quad \text{at} \quad x=0. \tag{7}$$

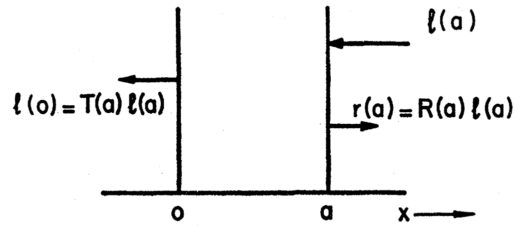

FIG. 2. The definition of the transmission and reflection coefficients.

Since we are dealing with the case in which the added flux dominates, we may neglect the term due to generation. The condition of Eq. (7) thus leads in Eq. (3) to

$$B=-AR_\infty. \tag{8}$$

According to this relationship between the coefficients A and B, the fluxes required for Fig. 2 are readily seen to be

$$l(0)=A(1-R_\infty^2), \tag{9}$$

$$l(a)=A(e^{qa}-R_\infty^2e^{-qa}), \tag{10}$$

$$r(a)=AR_\infty(e^{qa}-e^{-qa}). \tag{11}$$

From these it is readily seen that the reflection and transmission coefficients of Fig. 2 are

$$R(a)=r(a)/l(a)=R_\infty(e^{qa}-e^{-qa})/(e^{qa}-R_\infty^2e^{-qa}), \tag{12}$$

$$T(a)=l(0)/l(a)=(1-R_\infty^2)/(e^{qa}-R_\infty^2e^{-qa}). \tag{13}$$

These equations are identical with those used in the MLB treatment. In that treatment, however, differential equations for the fluxes are not given, and the expressions for reflection and transmission coefficients are obtained by deriving differential equations for them which are then integrated.

The transmission and reflection properties of the slab of thickness a may thus be expressed by two equations with the reflection and transmission coefficients as coefficients which give the emergent fluxes in terms of the incident fluxes:

$$l(0)=T(a)l(a)+R(a)r(0), \tag{14}$$

$$r(a)=R(a)l(a)+T(a)r(0). \tag{15}$$

III. INTERPRETATION OF THE FLUX TREATMENT IN TERMS OF CONTINUITY EQUATIONS

As was discussed above, the constancy of the coefficients k and l imply that carriers of a single velocity distribution are involved in these fluxes. Since the treatment reduces correctly to the continuity equation method for small rates of recombination, the relationship between flux and concentration can be calculated in terms of the thermal equilibrium distribution. If we deal with the concentration of holes, and represent this by p, then the relationship of p to the velocity distribution of carriers $p(v_x,v_y,v_z)$ is given by integrating

the latter over all classes of velocities:

$$p=\int_{-\infty}^{+\infty}dv_x\int_{-\infty}^{+\infty}dv_y\int_{-\infty}^{+\infty}dv_z\,p(v_x,v_y,v_z). \tag{16}$$

It is evident that the density of carriers corresponding to a flux to the right is obtained by extending the integration over only positive values for v_x: Denoting the density of these carriers by p_r, we have

$$p_r=\int_{0}^{\infty}dv_x\int_{-\infty}^{+\infty}dv_y\int_{-\infty}^{+\infty}dv_z\,p(v_x,v_y,v_z). \tag{17}$$

The flux carried by these carriers is r and this is evidently given by

$$r=\int_{0}^{\infty}v_xdv_x\int_{-\infty}^{+\infty}dv_y\int_{-\infty}^{+\infty}dv_z\,p(v_x,v_y,v_z)=cp_r, \tag{18}$$

in which the coefficient c for a Maxwellian distribution of particles having a single effective mass m^* is evidently

$$c=(2kT/\pi m^*)^{\frac{1}{2}}. \tag{19}$$

In accordance with the relationship (16) which applies to carriers in thermal equilibrium, we may define in general a density-like quantity in terms of the fluxes r and l;

$$p=p_r+p_l=(r+l)/c, \tag{20}$$

where p_r and p_l are density-like quantities given by r and l individually.

The net flux to the right across any plane in the medium is clearly given by

$$F\equiv r-l. \tag{21}$$

The differential equations (1) and (2) may be expressed in terms of F and p. Adding (1) and (2) together and dividing by $(2k+w)$ readily yields

$$F=-(2k+w)^{-1}d(r+l)/dx=-D^*dp/dx, \tag{22}$$

where the diffusion-like coefficient D^* is evidently

$$D^*=c/(2k+w)=D2k/(2k+w), \tag{23}$$

where D is defined as

$$D\equiv c/2k. \tag{24}$$

If the flux treatment is to reduce to the continuity equation treatment, then it is evident that for small values of w/k, for which case D^* reduces to D, D must be the conventional diffusion constant.

From this it follows that a unique relationship between the coefficient k and the diffusion constant D, which applies in the case of the continuity equation, must be given by Eq. (24) in which c is uniquely defined by Eqs. (17)–(19).

A second equation in terms of F and p is obtained by subtracting Eq. (2) from Eq. (1). This is readily seen to yield

$$dF/dx=g-w(r+l)=g-p/\tau=-D^*d^2p/dx^2. \tag{25}$$

In this equation the last term is obtained by simply differentiating Eq. (22) to obtain dF/dx. The last equality in Eq. (25) is evidently equivalent to the ordinary continuity equation, and if the flux treatment is to reduce to this case for sufficiently small values of w, it is evident that the lifetime term τ must be given by

$$\tau=1/wc;\quad p_n=g\tau, \tag{26}$$

in which we have introduced p_n, the equilibrium density of holes in n-type material, as the value of p which can be constant throughout the material.

A comparison of the case of high recombination rate with the ordinary continuity equation case can be obtained by considering the meaning of the attenuation constant q of Eq. (5). It is evident that Eq. (25) admits of a solution characterized by an attenuation constant equal to the reciprocal of the diffusion length. Algebraic manipulations show that this is identical with q:

$$q=1/(D^*\tau)^{\frac{1}{2}}=(2kw+w^2)^{\frac{1}{2}}=1/L^*. \tag{27}$$

In order to appreciate an important advantage of the MLB method, we consider the diffusion velocity for injected carriers decaying into uniform material:

$$\text{Diffusion velocity}\equiv v_D^*\equiv F/p=qD^*=(D^*/\tau)^{\frac{1}{2}}=wc/k$$
$$=v_D/[1+(v_D/c)^2]^{\frac{1}{2}}=c/[1+(c/v_D)^2]^{\frac{1}{2}}. \tag{28}$$

In this series of equations, the diffusion velocity is defined as the average velocity necessary for the carrier density p to carry the flux F. For the case in which the quantities involved vary as e^{qx}, this readily gives the expression qD^*. Straightforward algebraic manipulations yield the other expressions. In these, v_D is the diffusion velocity that will result from the conventional expression involving the ordinary diffusion constant D and the mean lifetime τ

$$v_D=(D/\tau)^{\frac{1}{2}}=(w/2k)^{\frac{1}{2}}c. \tag{29}$$

It is seen that the MLB method is equivalent to introducing expressions for the diffusion velocity which behave in the conventional manner when the diffusion velocity is much smaller than c but which converge to a limiting value of c when the conventional expression for diffusion velocity is larger than c.

It should be noted that the high capture condition of $w\gg k$ is equivalent to the $v_D\gg c$ as may be seen from Eq. (29).

The behavior for the diffusion velocity just discussed is represented in Fig. 3, in which it is seen that v_D^* is nearly equal to v_D for small values of v_D/c; but for large values of v_D/c, the effective diffusion velocity is less than v_D and approaches a limit of c as v_D/c approaches infinity or the lifetime approaches zero.

These results imply that the MLB treatment will probably have merit and yield expressions having an

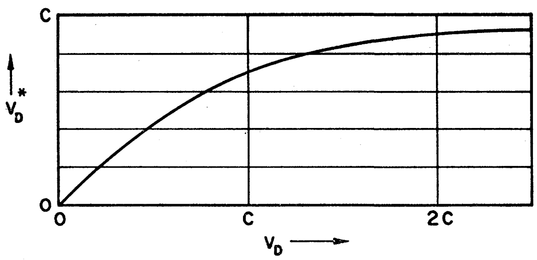

FIG. 3. Dependence of diffusion velocity upon lifetime expressed through continuity equation velocity v_D.

approximate validity when the mean free path for recombination becomes comparable to the mean free path between collisions so that $w \gg k$ and $v_D \gg c$; even though, as discussed in connection with Eqs. (1) and (2), the model does not realistically treat carriers subjected to scattering and recombination in this limiting case. The mathematics of the MLB treatment, when analyzed as indicated above, in effect handles the extreme cases by using the continuity equation but with an effective diffusion constant D^* smaller than the normal diffusion constant D by adding the effect of w in shortening the mean free path.

IV. BOUNDARY CONDITIONS IN TERMS OF DENSITY[4]

The formulation of the mathematical problem presented in Sec. III means that if a problem to be treated by the MLB method can be formulated in terms of the density p of minority carriers, then it can be completely solved in the usual way using the continuity equation. This follows from the fact that if the boundary conditions are known in respect to p then p and the flux F can be determined, and from these two quantities the individual components l and r of the flux needed in the MLB method can then also be determined by solving the simultaneous Eqs. (21) and (23) for r and l.

In this section we shall consider the formulation of boundary conditions for p in terms of definitions introduced in respect to r and l.

A. Surface Recombination Velocity Expressed in Terms of the Reflection Coefficient

If we suppose that a body of homogeneous material terminates in the positive x direction on the surface having a reflection coefficient R_0 for flux to the right, then we can re-express this condition in terms of a surface recombination velocity s. In this treatment we use the principle of detailed balance and assume that under conditions of thermal equilibrium, absorption of carriers at the surface is just made up by generation from the surface in amount g_s per unit area. We assume that this generation is independent of the carrier density or flux incident upon the surface. Accordingly, we may represent the flux leaving the surface to the left according to

$$l = R_0 r + g_s. \tag{30}$$

The carrier density immediately in front of the surface is denoted as p, where evidently

$$cp = r + l = (1+R_0)r + g_s. \tag{31}$$

The net flux into the surface is denoted by F and this is evidently equal to the loss of the incident flux minus the generated flux leaving the surface:

$$F = (1-R_0)r - g_s = (p-p_n)s. \tag{32}$$

The last equation in (32) expresses F in terms of the usual definition of a surface recombination velocity, which gives a flux into the surface proportional to the excess of the carrier density over the thermal equilibrium carrier density denoted by p_n. In order for Eq. (32) to become an identity when the relationship between p and r is given by (31), we must have

$$s = (1-R_0)c/(1+R_0), \tag{33}$$

$$g_s = (1-R_0)cp_n/2. \tag{34}$$

Using these definitions, we may treat the surface on the basis of the usual diffusion continuity equation, introducing the diffusion constant and the lifetime as discussed in Sec. III.

B. Boundary Condition at a Slab Bounded on the Far Side by a Surface Constant R_0

We shall here derive a result needed for discussing the boundary condition at an abrupt p-n junction. The case considered consists of a slab of thickness a and coefficients $R(a)$ and $T(a)$ of Eqs. (12) and (13) bounded on the far side by a surface of reflection coefficient R_0.

We shall assume the same coordinates as Fig. 2 and Eqs. (7) to (15) of Sec. II. The incident flux $r(0)$ is regarded as known with the other three fluxes $l(0)$, $r(a)$, and $l(a)$ as unknown. In order to determine the reflection coefficient $R \equiv l(0)/r(0)$ of the composite structure, it is necessary to have three independent linear equations relating $l(0)$, $r(a)$, and $l(a)$ to $r(0)$. Two of these equations are given by (14) and (15) describing the transmission properties of the slab. The third equation is

$$l(a) = R_0 r(a), \tag{35}$$

which is Eq. (28) for the condition in which the incident flux $r(0)$ dominates the situation, so that the g_s term may be neglected, a condition similar to that discussed with Eq. (8).

The solution of the three equations leads to

$$R \equiv l(0)/r(0) = R(a) + \{T(a)^2 R_0/[1-R(a)R_0]\}. \tag{36}$$

[4] The author was stimulated to consider these problems after seeing a manuscript of J. P. McKelvey, [J. Appl. Phys. (to be published)]. In this manus ript several of these problems were treated by the MLB method, yielding some of the same results.

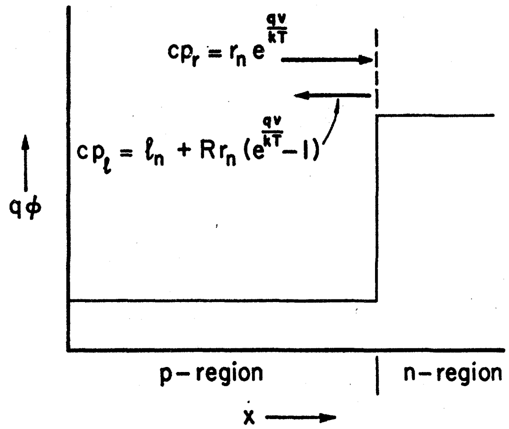

FIG. 4. Boundary condition at a p-n junction showing the potential energy $q\phi$ of a hole versus distance. (The fluxes shown represent values just inside the n region.)

By reasoning similar to that of Sec. IV(A), it follows that if the density at $x=0$ is p_n under conditions of thermal equilibrium, the composite structure of slab plus surface R_0 must be characterized by a generation due to g and g_s given by

$$g_s(\text{at } x=0) = (1-R)cp_n/2. \qquad (37)$$

This will give the emergent flux across $x=0$ when the incident flux $r(0)$ is reduced to zero.

C. Boundary Condition at an Abrupt p-n Junction

We shall now assume that the structure discussed in Sec. IV(B) bounds an abrupt p-n junction. By abrupt we mean that the potential rise may be thought of as occurring over so small a distance that the carrier density incident upon the structure from the left side and moving toward the right side is in equilibrium with the majority carriers to the left of the junction. This means that as illustrated in Fig. 4 this density increases exponentially with the applied forward bias, so that we may write for the density of incident carriers denoted by p_r as given by Eq. (20)

$$p_r = p_{rn} \exp(qV/kT), \qquad (38)$$

in which V is the applied forward voltage and $p_{rn} = p_n/2$' where p_n is the thermal equilibrium minority carrier density within the n-type material.

(It may be helpful to note that the hole fluxes l and r just outside the n region of Fig. 4 are only very slightly disturbed from their thermal equilibrium values of $cp_p/2$. This value is larger than cp_{rn} in the familiar ratio $n_n p_p/n_i^2$ so that disturbances which affect the hole flux in the n region by large factors produce only relatively small fractional disturbances in the p region.)

The minority carrier density for carriers moving to the left just in the n-type region may be written as

$$p_l = p_{ln} + Rp_{rn}[\exp(qV/kT) - 1], \qquad (39)$$

in which the second term represents the increment in carriers due to the reflection of the increment in incident carriers. Equation (39) reduces correctly to the normal value for no applied voltage and also correctly represents the reflection of the increment.

Combining (38) and (39) leads to a total carrier density just inside the n layer given by

$$p = p_r + p_l = p_n + (p_n/2)(1+R)[\exp(qV/kT) - 1]. \qquad (40)$$

It is seen that this equation reduces to the usual continuity equation form when R approaches unity. For cases in which the recombination rate is large so that $R(a)$ is less than unity, or the layer is very thin, and R_0 is substantially less than unity, then R is less than unity and $(1+R)/2$ in Eq. (40) may be significantly less than unity so that (40) differs from the usual boundary condition.

In any event if (40) can be evaluated, it suffices to give a boundary condition at the p-n junction for p. The diffusion equation (25) then suffices to determine p throughout the region involved and the currents can thereafter be calculated in the usual way using, however, the modified value D^* instead of D.

V. CURRENT VOLTAGE CHARACTERISTIC OF A p-n JUNCTION BY BOTH METHODS[4]

In this section we shall illustrate how the current voltage characteristic of a p-n junction may be derived on the basis of the MLB method and also on the basis of the continuity equation method of Sec. III, together with the boundary conditions of Sec. IV. We shall assume that the junction has an abrupt potential rise as discussed in connection with Fig. 4. Just inside the body of uniform n-type material, the deviation in total hole density due to the applied voltage may be represented as

$$\Delta p = \Delta p_r + \Delta p_l. \qquad (41)$$

As discussed in Sec. IV, the deviation of carriers corresponding to flux towards the right is given by

$$\Delta p_r \equiv p_r - p_{rn} = (p_n/2)[\exp(qV/kT) - 1], \qquad (42)$$

where Eq. (38) gives p_r.

Since both methods of treatment are linear in disturbances in the carrier density, it follows that the flux itself must be proportional to the deviation in density and thus of the form

$$F = F_s[\exp(qV/kT) - 1]. \qquad (43)$$

The constant F_s, which corresponds to the saturation current in conventional junction theory, is calculated differently on the basis of the type of treatment given to the problem.

Our object in this section is to show that the results of the two different treatments lead to one and the same current-voltage relationship.

If we assume that the structure dealt with is like that discussed in Sec. IV(B) and is characterized by a

reflection coefficient R, then the MLB method leads readily to the following value for F_s:

$$F_s=(1-R)(cp_n/2). \tag{44}$$

In the preceding expression the second factor is the thermal equilibrium flux to the right, and the first factor represents the fraction of this that finally constitutes carrier current across the p-n junction.

On the basis of the continuity equation and boundary condition treatment, an equal value for F_s can be derived. The reasoning is as follows: In accordance with the treatment of Sec. IV(C), the carrier density just inside the uniform n-layer is given by Eq. (40). This excess carrier density diffuses deeper into the n-layer at an effective diffusion velocity denoted by v_D^{**}. This leads to a saturation flux coefficient given by

$$F_s=(p_n/2)(1+R)v_D^{**}. \tag{45}$$

The effective diffusion velocity v_D^{**} in this case can be calculated in a straightforward way using the boundary conditions for the surface recombination velocity given by Eq. (33) and the corresponding solution for the diffusion equation throughout the region of width a. This result can be shown to be equivalent to Eq. (44). In this treatment, however, we shall simplify a comparison of Eqs. (44) and (45) by restricting the considerations to the case in which the width a becomes infinite so that R reduces to R_∞ and v_D^{**} becomes simply v_D^*.

For this particular assumption the F_s values given by the MLB method and the continuity equation method can readily be seen to be equal. The necessary manipulations are as shown below:

$$\begin{aligned}2F_s/p_n&=(1+R_\infty)v_D^*=[2q/(q+w)](cw/q)\\&=2cw/(q+w)=(1-R_\infty)c=2R_\infty^{\frac{1}{2}}v_D.\end{aligned} \tag{46}$$

In the preceding equation, the term following the second equality is obtained by expressing the two factors of the preceding term in terms of c, w, and q. It can be verified that the final term, when expressed in terms of the same variables, reduces to the same expression. The final term permits expressing the saturation current in a form similar to that used in conventional p-n junction theory.

The saturation flux coefficient F_s can be interpreted in terms of two of the preceding expressions:

$$F_s=p_nv_D^*(1+R_\infty)/2=R_\infty^{\frac{1}{2}}p_nv_D. \tag{47}$$

In the above equation the term following the first equals sign is seen to be similar to the classical continuity equation value, which would be p_nv_D except that it has the actual effective diffusion velocity times a factor involving R_∞. When R_∞ approaches unity the fractional term approaches unity and v_D^* reduces to v_D: Thus this term reduces to the classical formula for small ratios of w/k. For large values of w/k, for which R_∞ approaches zero, this expression reduces to the limiting flux value $p_nc/2$ regardless of how large the recombination term w becomes. The last expression in Eq. (47) shows that the saturation flux can also be written simply as $R_\infty^{\frac{1}{2}}$ times the classical expression. This shows that as the lifetime approaches infinity and v_d also approaches infinity, the correction term approaches zero. It is not so easy to see quickly from this form that the limiting value for F_s is the thermal equilibrium flux in one direction.

Similar expressions can, of course, be derived for electron current injected into p-type material, but the generalizations for this case are obvious and will not be discussed here.

VI. EXTENSION OF THE METHOD TO THREE DIMENSIONS AND TO INCLUDE SMALL ELECTRIC FIELDS

The presence of an electric field $E=-\mathrm{grad}\phi$ may be expected to influence the transmission of carriers through a thin slab of thickness dx. This effect should be most pronounced for flux of carriers which are decelerated upon entering the slab so that some of them are reversed, adding to the effect of scattering described by k. The flux which is accelerated rather than retarded by the field should be little affected. The same basic problems are involved in linearizing the effect of the electric field as are involved in taking w and k as constant as discussed following Eq. (6). However, a mathematically consistent generalization of Eqs. (1) and (2) can be made simply by adding linear terms in the electric field and the fluxes as follows:

$$dr/dx=-k(r-l)-wr+(g/2)+hE(r+l), \tag{48}$$

$$dl/dx=-k(r-l)+wl-(g/2)+hE(r+l). \tag{49}$$

Manipulation of these equations in the same way as carried out in Sec. III leads readily to

$$\begin{aligned}F&=-[c/(2k+w)]dp/dx+[2hc/(2k+w)]Ep\\&=-D^*dp/dx+\mu^*Ep.\end{aligned} \tag{50}$$

In this equation an effective mobility μ^*, which varies with $w/2k$ in the same ratio as does D^* of Eq. (23), has been defined as follows:

$$\mu^*\equiv 2hc/(2k+w). \tag{51}$$

The other equation obtained by subtracting (48) from (49) is the same as before:

$$dF/dx=g-p/\tau. \tag{52}$$

It should be noted that the Einstein relationship applies between μ^* and D^*, since it must apply between μ and D if the model is to reduce to the correct continuity equation. Thus we have

$$\mu^*=\mu 2k/(2k+w)=(q/kT)D^*, \tag{53}$$

where μ is the value for μ^* when w is set equal to zero and is thus the ordinary conductivity mobility.

The results of Eqs. (50) and (53) indicate another way of looking at the effect of the recombination terms upon the continuity equation. In case the carrier density is uniform, then it is evident that the effect of the electric field in producing carrier flux or current is reduced due to the recombination and emission of carriers from the recombination centers. This recombination and emission in effect adds an additional scattering mechanism whose importance is in the ratio of $w/2k$ compared to normal scattering; the factor 2 arises from the fact that recombination and subsequent emission simply randomizes the motion of the carrier and thus reduces the velocity to zero on each collision; scattering in accordance with the definition k reverses carrier velocity and thus is twice as effective.

Thus it appears that the flux method is equivalent to an effective reduction of mobility and diffusion constant, together with modifications of the boundary conditions like those discussed in Sec. V.

The formulation presented above allows a ready generalization to three dimensions. The complete set of differential equations for the density p is as follows:

$$\mathbf{F} = -D^* \operatorname{grad} p - \mu^* p \operatorname{grad} \phi, \tag{54}$$

$$\operatorname{div} \mathbf{F} = g - p/\tau, \tag{55}$$

$$D^* = c/(2k+w) = D/[1+(w/2k)], \tag{56}$$

$$\mu^* = qD^*/kT. \tag{57}$$

The effect of these equations is to modify the mobility and diffusion constant by introducing an effective influence of the recombination constant upon them. It is evident that this modification does not affect the Boltzmann distribution for carrier density under the thermal equilibrium conditions, in which case Eq. (54) vanishes. As discussed above, the modification of the diffusion constant leads to reasonable results for limiting currents for high recombination rates.

The combination of Eqs. (54) to (57), with suitable boundary conditions like those discussed in Sec. IV, formulates the flux treatment of McKelvey *et al.* into a generalized and modified form of the continuity theory.

ACKNOWLEDGMENT

The writer would like to express his appreciation for the cooperative attitude of the staff of Sequoia Hospital in Redwood City, California, which made the preparation of this manuscript possible.

APPENDIX

Significance of the Constants k and w with Respect to Familiar Quantities

In their article[1] McKelvey, Longini, and Brody derive a relationship between the mean free path for scattering and the quantity k. This relationship is derived on an arbitrary basis which brings out the fact that the constancy of the quantity k is physically inconsistent with the model assumed. The treatment given in their Appendix B calculates the average distance which a carrier fluxing across a plane travels without scattering. This distance is then identified with k. However, k is uniquely defined in their mathematical theory in terms of scattering in an infinitesimal distance. If their Appendix B formulation is used with the differential definition of k, a k value is obtained which is twice as large as theirs; the difference arises from the fact that the carriers crossing the plane are initially in the first infinitesimal distance scattered to a greater degree than are ones that have crossed a greater distance.

These considerations raise the question of what physical model might actually correspond accurately to the mathematics of scattering used in the MLB method. There does appear to be one applicable to a one-dimensional case, but this is a very artificial one. It is a model in which a large number of additional scattering centers of unusual property must be assumed. If one assumes that there are present centers producing a mean free path much smaller than that associated with either k or w and having, in addition, the property that these centers never reverse the direction of carriers, they will insure that the right-hand flux and the left-hand flux, r and l, have each always the same composition of carrier velocities no matter how the carriers have originated. It is hard to see how a generalization of such centers could lead to the intuitively reasonable three-dimensional formulation presented in Sec. VI.