Single Virtual Particle Exchange Model of High-Energy Inelastic Glancing Collisions*

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A general formulation, based upon the ideas of field theory, is given for the single virtual particle exchange model of high-energy inelastic glancing or "peripheral" collisions. The main assumption of the model is that there is a region of the final-state phase space in which the cross section is dominated by single virtual particle exchange graphs of a particular kind. A general discussion is given of the kinematics and of the region of applicability of the model. It is shown that, unlike the situation in elastic scattering, inelastic processes can occur with timelike momentum transfer, and an example in which this happens is discussed. The region in which it appears most reasonable to use the model is that characterized by final states which consist of two well-defined groups of particles. An argument based on unitarity is given which strongly suggests that the energy dependence of this part of the peripheral nucleon-nucleon cross section is less than logarithmic at very high energies.

1. INTRODUCTION

THE motivation for examining the single virtual particle exchange model of high-energy collisions comes from the fact that in pion-nucleon and nucleon-nucleon collisions, there is evidence, extending over a broad energy range, that: (1) important contributions to the elastic diffraction scattering come from an interaction range of the order of 1×10^{-13} cm, which is to be expected if the single virtual pion exchange interaction is important; (2) many inelastic events occur with small transverse momentum transfer; (3) in many events the final-state particles emerge in two well-collimated groups along the collision axis, which, in the over-all barycentric system, are the forward and backward cones.[1]

The recent development of the single virtual particle exchange or "peripheral" collision model[2–5] stems from its formulation in terms of the conjectured analyticity properties of the scattering matrix, as emphasized by Goebel[6] and by Chew and Low.[7] In particular, Chew and Low considered those Feynman graphs, for a general binary collision, in which a single final-state particle emerges from the target particle vertex and two or more final-state particles emerge from the projectile particle vertex, and focused attention on two important features expected of such graphs. (1) The propagator for the exchanged virtual particle has a first-order pole in the invariant square of its four-momentum, and for high enough scattering energies one can select a part of the final-state phase space (small target recoil) for which the production amplitude is close to the pole. (2) At the pole of the propagator the projectile particle vertex is equal to the scattering amplitude for a real collision of the projectile particle and the exchanged particle. These two considerations are equally applicable, because of relativistic invariance, to the case in which two or more final-state particles emerge from the target particle vertex and only one from the projectile particle vertex.

Extension of the model and of the field theoretical ideas to the case in which two or more final-state particles emerge from each of the two vertices was given by Dremin and Chernavskii[2] and by the authors.[4,5] In collisions of strongly interacting particles at energies above several Bev, such processes are expected to be more important than those of the kind considered by Goebel and by Chew and Low because of the strong vertex interactions and because of the greater available phase space if more than one final-state particle emerges from each of the two vertices.

A discussion in which the model is formulated with some detail has been given by the authors for a specific case in which two particles are emitted at each vertex, and for a definite energy.[4] In the present paper the discussion is extended to include the case of higher energies, in which many-particle final states become

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[1] O. Piccioni, *Proceedings of the 1958 Annual International Conference on High-Energy Physics at CERN* (CERN Scientific Information Service, Geneva, 1958), p. 65ff; reports by V. I. Veksler, I. E. Tamm, and E. L. Feinberg at the *Ninth Annual Conference on High-Energy Physics, Kiev, 1959* (Academy of Science, U.S.S.R., 1961); *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960); and International Conference on Theoretical Aspects of Very High-Energy Phenomena, 1961, CERN Report 61-22.

[2] I. M. Dremin and D. S. Chernavskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **38**, 229 (1960) [translation: Soviet Phys.—JETP **11**, 167 (1960)], Lebedev Institute of the Academy of Sciences, U.S.S.R., Rept. A-28, 1960 (unpublished). The total cross section given in these papers is too large by a factor 2, due to overcounting final states.

[3] F. Bonsignori and F. Selleri, Nuovo cimento **15**, 465 (1960); E. Ferrari, Phys. Rev. **120**, 988 (1960).

[4] F. Salzman and G. Salzman, Phys. Rev. **120**, 599 (1960). In this paper double pion production in $\pi^- - p$ collisions is considered at 5-Bev incident π^- energy.

[5] F. Salzman and G. Salzman, Phys. Rev. Letters **5**, 377 (1960).

[6] C. Goebel, Phys. Rev. Letters **1**, 337 (1958).

[7] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

important.[8,9] Particular emphasis is given to the general aspects of the kinematics (Sec. 2) and to the region of applicability of the model (Sec. 3). This region, which is obtained qualitatively, is the part of the phase space close to the pole of the propagator, in which it may be reasonable to assume that the cross section is dominated by single virtual particle exchange graphs of a particular kind. We are thus neglecting (1) the effects of "core" collisions, (2) the diffraction dissociation process discussed by Good and Walker, and (3) the effects of singularities which may exist for complex values of the invariant square of the momentum transfer near the pole of the propagator, discussed by Landshoff and Treiman.[10]

For the region of phase space of interest it is reasonable to neglect symmetrization between two like final-state particles that do not emerge from the same vertex, as is the case in very small angle, high-energy elastic collisions. There is thus an inelastic "classical limit," as in the elastic case, in which the particles that emerge from one vertex are "distinguishable" from those that come from the other vertex. It is this feature of the single virtual particle exchange graphs which leads naturally to the "two jet" final states observed in many high-energy collisions. The derivation of the cross section for such final states is given in Sec. 4.

In Sec. 5, an example is given of the application of the model to a collision process in which the momentum transfer may be timelike. Section 6 is concerned with certain very high energy aspects of the model in the case of nucleon-nucleon collisions.

2. FORMULATION OF THE KINEMATICS

The single virtual particle exchange model of inelastic collisions in which one or more final-state particles emerge from each of the two vertices is most simply formulated in terms of kinematical variables which are obtained as a direct generalization of the corresponding variables in the elastic scattering case. As shown in Figs. 1 and 2, the generalization is carried out by simply replacing each of the two final-state particles of the elastic collision case by a group of particles. The two incident particles A and A', with four-momenta p_i and p_i', collide by exchanging a single virtual particle, and two groups of final-state particles C and C' emerge, with total four-momenta P and P', respectively. It is convenient, in order to stress the kinematical symmetry of the two vertices Γ and Γ', to introduce, in addition

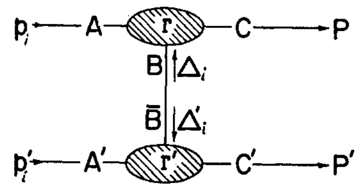

FIG. 1. Graph for a binary collision process in which the projectile particle A and the target particle A' collide by exchanging a single virtual quantum of the B field and two final-state particles emerge, C from the projectile particle vertex Γ, and C' from the target particle vertex Γ'. The four-momenta of A, A', B, C, and C' are denoted by p_i, p_i', Δ_i, P, and P', respectively. $\bar{B}$ is the antiparticle of B and has four-momentum $\Delta_i'=-\Delta_i$.

to the virtual particle B with four-momentum Δ_i directed at Γ, the virtual particle $\bar{B}$ with four-momentum Δ_i' directed at Γ'. Throughout the paper Δ_i and Δ_i' are the negatives of each other, and B and $\bar{B}$ are the antiparticles of each other. Of course the exchanged virtual particle may be different in the inelastic case than in the elastic case, but the conservation laws (charge, baryon number, etc.) must be satisfied at each vertex. The Chew-Low type graphs are a special case of Fig. 2 in which one of the groups C, C' consists of a single particle.

For convenience in the discussion we adopt the convention that A is the projectile particle and is incident from the left in the over-all barycentric system, in which the target particle A' is incident from the right. The rest masses w and w' of A and A' are given by

$$p_i^2=-w^2 \quad \text{and} \quad p_i'^2=-w'^2,$$

in units with $\hbar=c=1$, which are used throughout. The rest energies or "rest masses" W and W' of the groups C and C' are given by

$$P^2=-W^2 \quad \text{and} \quad P'^2=-W'^2.$$

In the elastic scattering case $W=w$ and $W'=w'$. The invariant square of the four-momentum Δ_i of the exchanged virtual particle B is denoted by Δ^2 and m_B is the rest mass of a real quantum of the B field.

Several coordinate systems are used, and are designated as follows: the over-all barycentric system by (U), the barycentric system of the group C by (W), the barycentric system of the group C' by (W'), the rest system of particle A by (w), and that of A' by (w'). The system (U) exists for any scattering problem, and each of the other rest systems exists unless the corresponding rest mass is zero. With the convention adopted here, (w') is normally the laboratory system (L). Components of a four-vector in a particular coordinate system are denoted by an additional subscript as follows: $(p_i)_L=(\mathbf{p}_{iL},w_L)$ are the three-momentum and energy of A in the laboratory system (L), $(P')_U=(\mathbf{P}_U',W_U')$ are those of the group C' in (U), and $(\Delta_i)_W=(\mathbf{\Delta}_{iW},\omega_{\Delta W})$ are those of B in (W). Magnitudes of three-vectors are denoted as follows: $p_{iL}=|\mathbf{p}_{iL}|$.

[8] Some of the results obtained in this paper have been previously reported by the authors (see references 5 and 9). The derivation given here includes that given in reference 4 as a special case, and is much simplified.

[9] F. Salzman and G. Salzman, Phys. Rev. **121**, 1541 (1961).

[10] M. L. Good and W. D. Walker, Phys. Rev. **120**, 1857 (1960). [An example in which this diffraction dissociation process may be as important for certain final states as the Chew-Low graph is given by P. Beckman, Nuovo cimento **20**, 812 (1961)]. P. V. Landshoff and S. B. Treiman, Nuovo cimento **19**, 1249 (1961).

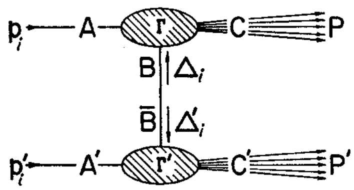

FIG. 2. Graph for a binary collision which differs from that of Fig. 1 in that in place of the two final-state particles, C and C' here represent two groups of final-state particles. In this case P and P' denote the total four-momenta of the groups C and C', respectively.

The total energy in (U) is designated by U and given by

$$(p_i+p_i')^2=-U^2, \tag{2.1}$$

from which it follows that the laboratory energy w_L of the projectile particle A is related to U by the equation

$$w^2+w'^2+2w_Lw'=U^2.$$

A number of relationships are useful in the discussion that follows. The energies, in (U), of particles A and A', and of groups C and C' are given by

$$w_U=(p_{iU}^2+w^2)^{\frac{1}{2}}, \tag{2.2a}$$

$$w_U'=(p_{iU}'^2+w'^2)^{\frac{1}{2}}, \tag{2.2b}$$

$$W_U=(P_U^2+W^2)^{\frac{1}{2}}, \tag{2.2c}$$

$$W_U'=(P_U'^2+W'^2)^{\frac{1}{2}}, \tag{2.2d}$$

respectively, which satisfy the equations

$$w_U+w_U'=U, \tag{2.3a}$$

$$W_U+W_U'=U. \tag{2.3b}$$

Expressions for these energies in (U), in terms of the four rest masses and U, may be obtained from the conservation of the total four-momentum,

$$p_i+p_i'=P+P'.$$

For example, from $p_i'=(P+P')-p_i$ one gets, by evaluating the square of each side in (U), the equation $-w'^2=-U^2-w^2+2Uw_U$, that is,

$$w_U=(U^2+w^2-w'^2)/(2U), \tag{2.4a}$$

and similarly,

$$w_U'=(U^2+w'^2-w^2)/(2U), \tag{2.4b}$$

$$W_U=(U^2+W^2-W'^2)/(2U), \tag{2.4c}$$

$$W_U'=(U^2+W'^2-W^2)/(2U). \tag{2.4d}$$

From conservation of four-momentum at each of the two vertices,

$$\Delta_i=P-p_i \quad \text{and} \quad \Delta_i'=P'-p_i',$$

it follows, by evaluating the squares in (U), that

$$\Delta^2=-W^2-w^2-2(\mathbf{P}_U\cdot\mathbf{p}_{iU}-W_Uw_U), \tag{2.5a}$$

$$\Delta^2=-W'^2-w'^2-2(\mathbf{P}_U'\cdot\mathbf{p}_{iU}'-W_U'w_U'). \tag{2.5b}$$

Of course, $p_{iU}=p_{iU}'$ and $P_U=P_U'$, and in an elastic collision all four are equal. From Eqs. (2.2a) and (2.4a) it then follows that

$$p_{iU}=p_{iU}'=\frac{U}{2}\left[1-2\frac{w^2+w'^2}{U^2}+\left(\frac{w^2-w'^2}{U^2}\right)^2\right]^{\frac{1}{2}}, \tag{2.6a}$$

and similarly

$$P_U=P_U'=\frac{U}{2}\left[1-2\frac{W^2+W'^2}{{}^2U}+\left(\frac{W^2-W'^2}{U^2}\right)^2\right]^{\frac{1}{2}}. \tag{2.6b}$$

In the case of elastic scattering the longest range part of the interaction due to the field B is $\sim m_B^{-1}$ and comes from the single virtual particle exchange graphs of Fig. 1 for small values of Δ^2. As is well known, in elastic scattering small Δ^2 means small-angle scattering in (U), since in this case $\Delta^2=(\mathbf{P}_U-\mathbf{p}_{iU})^2$ and is given by

$$\Delta^2=4p_{iU}P_U\sin^2(\theta/2), \tag{2.7}$$

where the equality $p_{iU}=P_U$ was used and the scattering angle θ in (U) is defined by

$$\mathbf{p}_{iU}\cdot\mathbf{P}_U=p_{iU}P_U\cos\theta. \tag{2.8}$$

Equation (2.7) implies that $\Delta^2\geq 0$ and is at least m_B^2 away from its value, $-m_B^2$, at the pole of the propagator. $\Delta^2=0$ implies $\Delta_i=0$ for elastic scattering. Thus Δ^2 is zero for elastic foward scattering and increases monotonically with θ.

For the inelastic case we also define the "scattering angle" θ by Eq. (2.8). In this case small-angle scattering means that the center-of-mass motion of the group C is almost parallel to that of the projectile particle A, in (U). With this definition a natural generalization of Eq. (2.7) may be obtained. If half the sum of Eqs. (2.5a) and (2.5b) is formed, account taken of the fact that $\mathbf{P}_U\cdot\mathbf{p}_{iU}=\mathbf{P}_U'\cdot\mathbf{p}_{iU}'=p_{iU}P_U\cos\theta$, and that Eqs. (2.4a,b,c,d) imply that

$$2(w_UW_U+w_U'W_U')=U^2+[(w^2-w'^2)(W^2-W'^2)/(U^2)],$$

one obtains

$$\Delta^2=\frac{1}{2}\left[U^2-(w^2+w'^2+W^2+W'^2)+\frac{(w^2-w'^2)(W^2-W'^2)}{U^2}\right]-2p_{iU}P_U\cos\theta. \tag{2.9}$$

As in the elastic case, Δ^2 is smallest for inelastic forward scattering and increases monotonically with θ, the variables W, W' being held fixed. Thus, the maximum and minimum values of Δ^2 for given W and W' are given by

$$[\Delta^2(W,W')]_{\min}{}^{\max}=\frac{1}{2}\left[U^2-(w^2+w'^2+W^2+W'^2)+\frac{(w^2-w'^2)(W^2-W'^2)}{U^2}\right]\pm 2p_{iU}P_U. \tag{2.10}$$

Equation (2.9) can be written, with the use of Eq. (2.10) as

$$\Delta^2=[\Delta^2(W,W')]_{\min}+4p_{iU}P_U\sin^2(\theta/2). \quad (2.11)$$

It is easy to verify that $[\Delta^2(W,W')]_{\min}$ vanishes for elastic collisions, so that Eq. (2.11) is a natural generalization of Eq. (2.7).

In elastic scattering, since $[\Delta^2(W,W')]_{\min}$ vanishes, one has the strong condition, $\Delta^2\geq 0$. This is replaced by a weaker condition in the general case, where it is possible to have negative values of $[\Delta^2(W,W')]_{\min}$, and hence of Δ^2 under certain conditions. The generalization may be stated as the following theorem.

Theorem. If

$$(W-w)(W'-w')\geq 0, \quad (W+w)(W'+w')>0, \quad (2.12)$$

and $\Delta_i\neq 0$, then $\Delta^2>0$.

Proof. Suppose Δ_i is timelike or null. Then its energy component, ω_Δ, is either positive in all Lorentz coordinate systems or invariantly negative. Consider the case that it is positive. From the equations $\Delta_i+p_i=P$ and $P'-\Delta_i'=p_i'$ one obtains

$$W^2-w^2=-\Delta^2-2\Delta_i p_i \quad \text{and} \quad W'^2-w'^2=\Delta^2-2\Delta_i'P',$$

respectively, where $\Delta_i p_i$ is the invariant inner product of Δ_i and p_i, and $\Delta_i'P'$ is similarly defined. From the condition $(W+w)(W'+w')>0$ it follows that at least one of the coordinate systems (W), (w) exists, and also at least one of (W'), (w') exists. Evaluation of $\Delta_i p_i$ in the (W) or (w) system shows that it is negative. Therefore, since $\Delta^2\leq 0$, and $(W+w)>0$, it follows that $W-w>0$. Similarly it follows that $W'-w'<0$. Therefore, if $\Delta^2\leq 0$, $\omega_\Delta>0$, and $(W+w)(W'+w')>0$, then $(W-w)(W'-w')<0$. For the case that the energy component ω_Δ of Δ_i is invariantly negative, the same proof holds with primed and unprimed quantities interchanged. Thus in either case, if $\Delta^2\leq 0$ and $(W+w)(W'+w')>0$, then $(W-w)(W'-w')<0$, which is equivalent to the statement of the theorem.

The proof shows that, if $\Delta^2\leq 0$, the virtual particle behaves kinematically like a real particle in that it has an invariant sense of propagation, that is, one vertex loses rest mass and the other gains it. The condition $(W+w)(W'+w')>0$ is necessary because without it there is the possibility that $(W-w)(W'-w')=0$ and $\Delta^2=0$.

It may be noted that Eqs. (2.9)–(2.12) are invariant with respect to the interchange $w\leftrightarrow W$, $w'\leftrightarrow W'$. If we call a vertex excited, elastic, or de-excited, according as the change in rest mass is positive, zero, or negative, respectively, then from the theorem it follows that if both vertices are excited, both de-excited, or one elastic with nonzero rest mass and the other inelastic, then Δ^2 will always be more than $m_B{}^2$ away from its value at the pole. This will always be the case, for example, in inelastic nucleon-nucleon collisions in which one nucleon is contained in C and the other in C'.

The proof of the theorem and the reason that its converse is false, are easily visualized by examining the Minkowski energy-momentum diagram shown in Fig. 3.[11] The p_z axis is chosen along the collision axis, and the components of the four-momenta in (U) are shown. In accord with our convention, p_i represents the projectile particle incident from the left, and p_i' the target particle incident from the right, and $\mathbf{p}_{iU}=-\mathbf{p}_{iU}'$. The case shown is for $w>w'>0$, and correspondingly $w_U>w_U'$. The light cone drawn through the tip of $(p_i)_U$ touches the w-mass hyperboloid only at this point. Its future part is in the region of higher mass hyperboloids, and its past part is in that of lower mass hyperboloids. Similar statements hold for the light cone drawn through the tip of $(p_i')_U$. Addition of any timelike or null (but non-zero) four-momentum Δ_i to p_i thus gives $P=p_i+\Delta_i$ with W larger than or smaller

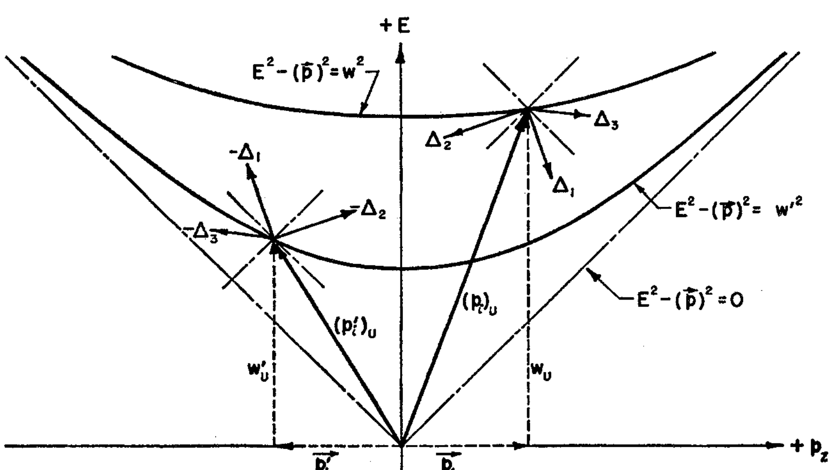

FIG. 3. A Minkowski energy-momentum diagram which illustrates the theorem (2.12) for the case that the rest masses w and w' of A and A' satisfy $w>w'>0$. The p_z axis in the over-all barycentric system (U) is taken to be the collision axis, and the energy and three-momentum components, in (U), of the four-momenta p_i and p_i' are shown. The effect of adding various kinds of four-momenta Δ_j $(j=1,2,3)$ to p_i and subtracting them from p_i' is illustrated. The vectors labeled Δ_j represent the components of these four-momenta in (U).

[11] This possibility was pointed out by K. Wilson (private communication)

than w according as Δ_i is in (or on) the future or past light cone, and correspondingly its subtraction from p_i' gives $P'=p_i'-\Delta_i$ with W' smaller than or larger than w', respectively. Thus, if $\Delta^2\leq 0$ and w, $w'>0$, then $(W-w)(W'-w')<0$, which is the main content of the theorem. The vector labeled Δ_1 is a typical timelike vector for the case $W-w<0$ and $W'-w'>0$. The spacelike vectors Δ_2 and Δ_3 illustrate why the converse of the theorem is false. For each of them the condition $\Delta^2>0$ holds, but in the case of Δ_2 one has $(W-w)(W'-w')<0$, while in the case of Δ_3, $(W-w)(W'-w')>0$.

To obtain more information about the allowed region in the Δ^2, W, W' "phase" space for a specified collision, that is, one in which U, w, w', and the mass spectra of C and C' are given, we consider Eq. (2.9). It is essentially quadratic in W^2 and W'^2 and is brought to simpler form if the variables W^2 and W'^2 are replaced by the dimensionless variables x and y defined by

$$W^2-w^2=U^2\left[x-\left(1+\frac{w^2-w'^2}{U^2}\right)y\right]-\Delta^2 = U^2\left[x-\left(\frac{2w_U}{U}\right)y\right]-\Delta^2, \quad (2.13a)$$

$$W'^2-w'^2=U^2\left[x+\left(1-\frac{w^2-w'^2}{U^2}\right)y\right]-\Delta^2 = U^2\left[x+\left(\frac{2w_U'}{U}\right)y\right]-\Delta^2. \quad (2.13b)$$

Roughly speaking, x is a measure of the total excitation of both vertices, and y is a measure of the "asymmetry" of the collision, that is, the amount by which the target particle vertex excitation exceeds that of the projectile particle vertex. For a given value of y, x increases monotonically with W and W'. For a given value of x, y increases monotonically with W' and decreases monotonically with W.

By the use of Eqs. (2.6a,b) and (2.13a,b), Eq. (2.9) may be written as

$$2p_{iU}^2-U^2x=2p_{iU}[p_{iU}^2+\Delta^2-U^2x+U^2y^2]^{\frac{1}{2}}\cos\theta. \quad (2.14)$$

By squaring Eq. (2.14) it then follows from the fact that $\cos^2\theta\leq 1$ for real angles θ, that the inequality

$$U^2x^2-4p_{iU}^2y^2\leq 4p_{iU}^2(\Delta^2/U^2) \quad (2.15)$$

is a necessary condition that points x, y must satisfy. In Figs. 4(a), (b), (c) the hyperbola

$$\frac{x^2}{(2p_{iU}/U)^2}-y^2=\frac{\Delta^2}{U^2} \quad (2.16)$$

is shown for the three cases of positive, zero, and negative Δ^2 as the curve labeled $H(\Delta^2)$. In each case the shaded area is the region where the inequality (2.15) is satisfied. The hyperbola $H(\Delta^2)$ is thus part of the phase-space boundary. In terms of the variables W, W',

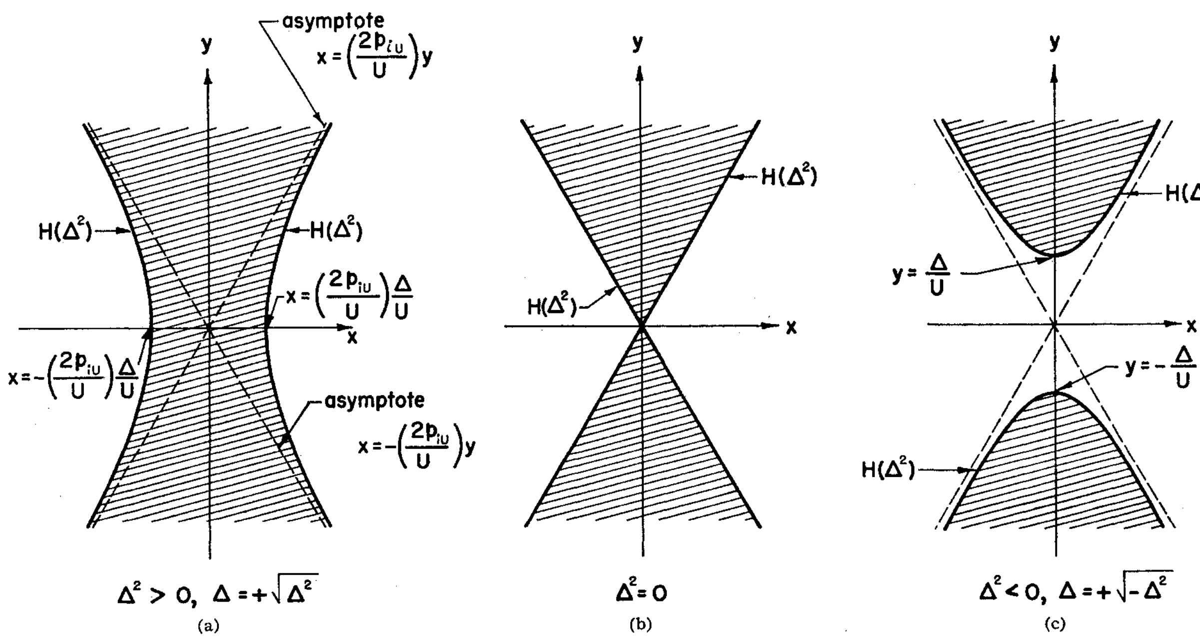

FIG. 4. The hyperbola of Eq. (2.16) is shown, as the curve labeled $H(\Delta^2)$, for the cases of positive, zero, and negative Δ^2. For each value of Δ^2 the allowed values of W and W' are represented by points x, y which must lie in the shaded part of the $x-y$ plane bounded by $H(\Delta^2)$.

Eq. (2.16) has the form

$$(W^2-w^2)(W'^2-w'^2)+\frac{1}{U^2}[W^2w'^2-W'^2w^2][(W^2-w^2)-(W'^2-w'^2)]$$
$$=\left[1-\frac{(W^2+W'^2+w^2+w'^2+\Delta^2)}{U^2}+\frac{(w^2-w'^2)(W^2-W'^2)}{U^4}\right]\Delta^2U^2. \quad (2.17)$$

The maximum and minimum values of Δ^2 given by Eq. (2.10) are the larger and smaller roots of Eq. (2.17), which is quadratic in Δ^2. If each of Δ^2, w^2, w'^2, W^2, W'^2 is much less than U^2, the right-hand member of Eq. (2.17) is well approximated by Δ^2U^2. If in addition each vertex is sufficiently inelastic, e.g., if $|W^2-w^2|\geq w^2$ and $|W'^2-w'^2|\geq w'^2$, then Eq. (2.17) is well approximated by

$$(W^2-w^2)(W'^2-w'^2)\approx\Delta^2U^2. \quad (2.18)$$

To obtain the other curves that complete the phase-space boundary for a given value of Δ^2, we note from Eq. (2.13a) that the points x, y for which W has a fixed constant value lie on the straight line

$$x=f(\Delta^2,W,y)\equiv\frac{2w_U}{U}y+\frac{W^2-w^2+\Delta^2}{U^2}, \quad (2.19a)$$

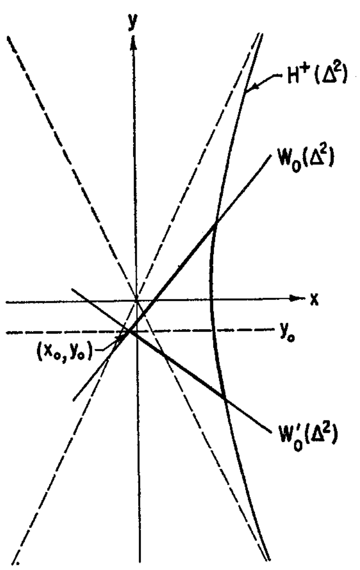

FIG. 5. The phase space for a positive value Δ^2, as it appears in the $x-y$ plane, is shown in a typical case. The asymptotes of the hyperbola $H(\Delta^2)$ are the unlabeled dashed lines. $H^+(\Delta^2)$ is the positive x branch of $H(\Delta^2)$. The line $W_0(\Delta^2)$ is the set of points x, y for which the group C has the smallest energy W_0 allowed by its energy spectrum. The allowed region must lie "to the right" of $W_0(\Delta^2)$. $W_0'(\Delta^2)$ likewise is the "lower bound" determined by the energy spectrum of the group C'. The phase space in this case is the "triangular" region bounded by the heavy lines.

and from Eq. (2.13b) that those points x, y for which W' is constant lie on the straight line

$$x=g(\Delta^2,W',y)\equiv-\frac{2w_U'}{U}y+\frac{W'^2-w'^2+\Delta^2}{U^2}. \quad (2.19b)$$

From Eq. (2.19a) it follows that the lines of constant W are a parallel set with slope $U/(2w_U)$, independent of Δ^2. The x intercept, $(W^2-w^2+\Delta^2)/U^2$, is a monotonically increasing function of W, and thus a line of larger W will be "to the right" of a line with smaller W. From Eq. (2.19b) it likewise follows that the lines of constant W' form a parallel set, and a line of larger W' lies "to the right" of a line with smaller W'.

Let W_0 and W_0' be the minimum values of the C and C' mass spectra. The line $W_0(\Delta^2)$ of minimum W, $x=f(\Delta^2,W_0,y)$, and $W_0'(\Delta^2)$, that of minimum W', $x=g(\Delta^2,W_0',y)$, are shown in Fig. 5 for the particular value of Δ^2. From the mass spectra, one thus obtains additional necessary conditions that points x, y must satisfy,

$$x\geq f(\Delta^2,W_0,y),$$
$$x\geq g(\Delta^2,W_0',y). \quad (2.20)$$

It may easily be shown, by considering the relevant slopes and intercepts, that for $\Delta^2>0$ each of the three curves $W_0(\Delta^2)$, $W_0'(\Delta^2)$, and $H^+(\Delta^2)$ [$H^+(\Delta^2)$ is the positive x branch of $H(\Delta^2)$] intersects the other two. If the intersection of the $W_0(\Delta^2)$ and $W_0'(\Delta^2)$ lines lies in the shaded region of Fig. 4(a), which is the case shown in Fig. 5, then the "triangular" region is the "phase space" for that value of Δ^2.

In order to trace the development of the phase space as Δ^2 varies, note that the separation of the x intercepts of the lines $W_0(\Delta^2)$ and $W_0'(\Delta^2)$ is independent of Δ^2. This, and their constant slopes, imply that as Δ^2 varies, their intersection point (x_0,y_0) moves along the line y_0, given by $y=[(W_0'^2-w'^2)-(W_0^2-w^2)]/(2U^2)$ and shown in Fig. 5. From Eqs. (2.19a,b) it follows that the x component of the intersection point, $x_0(\Delta^2)$, moves as Δ^2/U^2,

$$x_0(\Delta^2)=(\Delta^2/U^2)+\text{constant}.$$

For positive values of Δ^2, the x-axis intercept of $H^+(\Delta^2)$ moves proportionally to Δ/U. Consider the case in which $W_0>w$ and $W_0'>w'$, so that, by the theorem (2.12), only positive values of Δ^2 can occur. In this case Δ_0^2, the minimum value of $[\Delta^2(W,W')]_{\min}$ attainable in the collision, is given by

$$\Delta_0^2=[\Delta^2(W_0,W_0')]_{\min}.$$

For $\Delta^2=\Delta_0^2$ there is a single point of phase space, which consists of the common intersection of the lines $W_0(\Delta_0^2)$, $W_0'(\Delta_0^2)$, and of the curve $H^+(\Delta_0^2)$. The development of the phase space as Δ^2 increases is governed by the fact that the lines $W_0(\Delta^2)$ and $W_0'(\Delta^2)$ move "to the right" as Δ^2/U^2 and the curve $H^+(\Delta^2)$ moves "to the right" proportionally to Δ/U. For Δ^2

in the neighborhood of Δ_0^2, $H^+(\Delta^2)$ moves faster than $W_0(\Delta^2)$ and $W_0'(\Delta^2)$, and as Δ^2 increases the single point of the phase space for Δ_0^2 "grows" into a "triangular" region of the kind shown in Fig. 5. As Δ^2 increases further, $H^+(\Delta^2)$ moves more slowly than $W_0(\Delta^2)$ and $W_0'(\Delta^2)$, and the "triangular" region shrinks, finally becoming a single point when Δ^2 takes on the largest possible value attainable in the collision.

As can be seen from Fig. 4(c), the phase space in the case of negative values of Δ^2 may be more complicated because the allowed region may become disconnected. These questions are not further considered in the present paper. An example of a physical process in which negative values of Δ^2 do occur is given in Sec. 5.

In strong interactions the single virtual particle exchange graph of Fig. 2 is expected to be dominant only for small values of Δ^2. It follows from Eq. (2.17) that $[\Delta^2(W,W')]_{\min}$ approaches zero as U takes on increasingly large values. It is possible, even with small values of Δ^2, to have considerable excitation at each vertex if the excitation is small compared to U, that is, if the collision is one of small inelasticity. For excitations much smaller than U, P_U is close to $U/2$ and from Eq. (2.11) it then follows that small Δ^2 implies small θ.

For the case of excitation at each vertex, the exchanged virtual particle can behave kinematically as an incoming, "almost real" particle at each of the vertices Γ and Γ', as seen in the relevant coordinate systems (W) and (W'), respectively. The energies, $\omega_{\Delta W}$ of B in (W), and $\omega_{\Delta' W'}$ of $\bar{B}$ in (W'), are obtained from the conservation of four-momentum at each of the two vertices Γ and Γ'

$$p_i = P - \Delta_i \quad \text{and} \quad p_i' = P' - \Delta_i',$$

by evaluating the squares of these equations in (W) and (W'), respectively. One obtains

$$\omega_{\Delta W} = (W^2 - w^2 - \Delta^2)/(2W), \qquad (2.21a)$$

and

$$\omega_{\Delta' W'} = (W'^2 - w'^2 - \Delta^2)/(2W'). \qquad (2.21b)$$

For sufficiently large excitation at each vertex and small enough Δ^2, that is, if $W^2 - w^2 > \Delta^2$ and $W'^2 - w'^2 > \Delta^2$, the energy carried into each vertex by the exchanged virtual particle is positive as seen in (W) and (W'). This result is explained by the fact that in the case of double excitation Δ_i is spacelike, as shown by the theorem (2.12), and its energy component can change sign under the Lorentz transformation that connects (W) and (W').[12] In the limit $\Delta^2 \to -m_B^2$, Eq. (2.21a) gives the energy for a real particle B scattering with A at total energy W in the system (W). For Δ^2 in the physical region, if $\omega_{\Delta W}^2 \gg m_B^2, \Delta^2$, then in (W) the energy and momentum of the virtual particle B are close to those of the real particle B at the same total scattering energy W. In this sense the virtual B particle behaves kinematically as an "almost real" particle in (W). Likewise, if $\omega_{\Delta' W'}^2 \gg m_B^2, \Delta^2$, then the virtual $\bar{B}$ particle behaves kinematically as an "almost real" particle in (W'). All the kinematical considerations are of course valid whether the exchanged virtual particle is a boson or a fermion.

[12] A detailed explanation is given in reference 5.

3. DISCUSSION OF THE REGION OF APPLICABILITY OF THE MODEL

The main assumption of the model is that there is a region of the final-state phase space in which the graph of Fig. 2 dominates the cross section, where B is the lowest mass field involved that couples the vertices Γ and Γ'. This is based upon the fact that the closest single-particle pole of the production amplitude is at $\Delta^2 = -m_B^2$, and the assumption that for small physical values of Δ^2 the B-particle propagator is still large enough to insure that this graph is dominant. In Sec. 2 it is shown that Δ^2 can have small values for the configuration of Fig. 2, in which the particles of the forward group C are produced at the projectile particle vertex Γ and those of the backward group C' are produced at the target particle vertex Γ'. On this basis it is assumed that graphs in which a single virtual particle of mass greater than m_B is exchanged and graphs in which more than one particle is exchanged may be neglected without further consideration.

There are in addition other single B-particle exchange graphs which differ from Fig. 2 in that one or more of the final-state particles is produced at the other vertex. The reasonableness of neglecting such graphs depends not only on the values of Δ^2, W, and W' of the original graph, but also on the four-momenta of the final-state particles that are "shifted" to give the alternate graphs. The magnitudes of the B-particle propagators in these graphs depend on the details of the kinematic structure of each of the original groups C and C', and in certain cases may be comparable to that of the original graph. We have not been able to make general quantitative estimates; however, qualitatively, it appears most reasonable to neglect these alternate B-particle exchange graphs for final states in which the two cones of particles are sufficiently "well defined." The groups C and C' are best defined in the case that, in (U), all the particles of C move together with a high velocity to the right and those of C' move together with a high velocity to the left. In this case there is no kinetic energy contribution of the particles to W or W', and $\theta = 0$. It is easy to verify that for this particular kind of final state, the invariant square of the four-momentum of the virtual B particle in an alternate graph, Δ_a^2, is much greater than Δ^2. For example, if each particle has mass m_B and U is sufficiently large, then for the alternate graph obtained from Fig. 2 by shifting a single particle one finds $\Delta_a^2 \sim m_B(\Delta U)^{\frac{1}{2}}$, and for the exchange graph obtained from Fig. 2 by shifting one particle of each group to the other vertex one finds $\Delta_a^2 \sim (m_B^2/\Delta)U$. In general, Δ_a^2 will be much greater than Δ^2 if the groups C and C' are "well defined," which is the case if W^2,

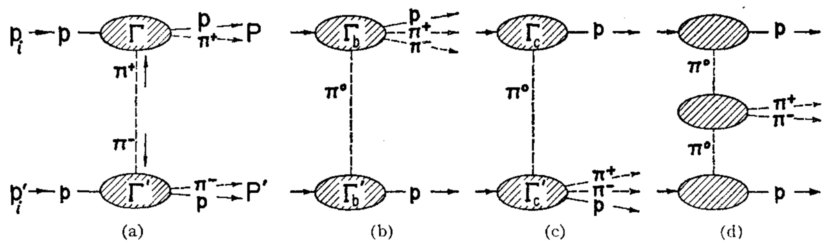

FIG. 6. The graph of (a) is the same kind as that of Fig. 2, for the particular case that two protons collide and produce two charged π mesons. The graphs of (b), (c), and (d) are other single virtual pion exchange graphs for the same process.

$W'^2 \ll U^2$, most of W and W' consist of the rest energies of the particles of C and C', respectively, and $\theta \approx 0$.

The condition $\Delta_a^2 \gg \Delta^2$ is not sufficient to guarantee that $[m_B^2+\Delta_a^2]^{-1} \ll [m_B^2+\Delta^2]^{-1}$ because Δ^2 can be $\ll m_B^2$ and Δ_a^2 can be $\approx m_B^2$. A simple example of this is one in which each particle has mass m_B, each of C and C' consists of two particles, $W=W'=2m_B \ll U$, and $\theta=0$. For this case the graph of Fig. 2 has $\Delta^2 \ll m_B^2$ and the alternate graph obtained by "shifting" a single final-state particle to the other vertex has $\Delta_a^2 \approx m_B^2$. The B-particle propagators for these two graphs are of comparable magnitude because of the finite rest mass of the B field. Of course, the finite mass gives an interaction range $\sim m_B^{-1}$, and the important part of the phase space is thus expected to occur for values of $\Delta^2 \gtrsim m_B^2$. If each of C and C' consists of many (n) particles, $W=W'=nm_B \ll U$, and $\theta=0$, then for an alternate graph, $\Delta_a^2 \gg m_B^2$ and its propagator is much smaller than that of Fig. 2. Implicit in the neglect of the alternate graphs, of which there are usually a large number for final states with many particles, is the assumption that the contribution of the sum of all such terms is also small.

If one tries to extend the model to the case in which W or W' is largely kinetic energy of the particles, this will include cases in which the two groups of final-state particles are not really defined kinematically, and a number of difficulties arise. In order to discuss these cases it is useful to define the "forward" direction in each of (W) and (W'); in (W) it is chosen to be that of the velocity vector $\mathbf{P}_U/W_U$ of the Lorentz transformation which transforms four-momenta in (W) to four-momenta in (U), and in (W') it is similarly chosen to be that of $\mathbf{P}_U'/W_U'$. Thus, forward particles in (W) are also forward in (U) and forward particles in (W') are backward in (U).

The lack of definition of the two groups arises from the possibility that an energetic backward particle in (W) may be backward in (U), and an energetic backward particle in (W') may be forward in (U). For small Δ^2 this part of the phase space is small and most events will lead to two well-defined groups, unless there is a strong preference for producing energetic backward particles in (W) or (W'). Such a strong preference would exist, for example, if the interaction at the Γ or at the Γ' vertex is itself as "peripheral" as the $A-A'$ collision.

An extreme example of this situation, in which there are not two well-defined groups, and in which it is not reasonable to consider only the graph of Fig. 2, is that of very high energy nucleon-nucleon collisions in which each of C and C' consists of a nucleon and a pion, as shown in Fig. 6(a) for particular charge states, and in which $W,W' \gg M+m_\pi$, where M and m_π are the nucleon and pion rest masses, respectively. Figure 6(a) shows the case in which A and A' are protons, C is a proton and a positive pion, and C' is a proton and a negative pion. Each vertex of Fig. 6(a) is close to that for elastic pion-nucleon scattering, which for sufficiently high energy is mainly diffraction scattering, and is itself characterized by small momentum transfer. It may be verified that for small θ, in each of (W) and (W') the pion emerges preferentially backwards and the nucleon forwards, and that for $\Delta^2 \approx m_\pi^2$, each pion has very low energy in (U). In such cases the final state consists of three groups, which are characterized in (U) as a very high energy forward proton, a very high energy backward proton, and two low-energy pions.

For such a final state the graph obtained from Fig. 6(a) by exchanging the two pions is of the same magnitude as that of Fig. 6(a) and cannot be properly neglected. In addition, there are two alternate single virtual pion exchange graphs with $\Delta_a^2 \ll \Delta^2$ which are expected to be important. These two alternate graphs, shown in Figs. 6(b) and 6(c), are obtained from Fig. 6(a) by shifting one or the other of the pions. Each vertex Γ_b and Γ_c' is itself "peripheral," because the proton which emerges from it has lost only a small fraction of its energy in the relevant coordinate system, and each should thus be represented by a single virtual pion exchange interaction as shown in Fig. 6(d), where the three groups of particles are connected by two virtual pions. If one tries to treat final states of this kind in terms of a "two-group" model, then, since the graph of Fig. 6(d) is included in both Fig. 6(b) and Fig. 6(c), there is the additional difficulty of overcounting such graphs, as has been pointed out by Goebel and by Amati and Fubini.[13]

This example also shows that a collision which produces a multi-group final state may be inadequately described by the corresponding "chain-of-pions" graph, in this case that of Fig. 6(d).

We restrict ourselves in this paper to those final

[13] C. Goebel, and D. Amati and S. Fubini, International Conference on Theoretical Aspects of Very High-Energy Phenomena, 1961, CERN (unpublished).

states in which there are two well-defined groups of particles and for which it is therefore reasonable to assume that the graph of Fig. 2 is dominant. It should however be emphasized that this is expected to include those multi-group final states in which the over-all groups C, C' are unambiguously defined. This is in fact the description proposed by the authors[4,9] to explain the "two-fireball" picture of ultra-high-energy nucleon-nucleon collisions, in which, in (U) each of C, C' consists of two groups, a very high energy nucleon and a trailing "fireball" of secondaries.

To the extent that it is reasonable to neglect other graphs, the dominance of Fig. 2 represents a "classical limit" for the production of the two groups of particles. There are no significant interference effects between these two groups and in this sense they are "distinguishable"; the forward group C comes from the projectile particle vertex and the backward group C' comes from the target particle vertex.

4. FIELD THEORETICAL DERIVATION

The cross section obtained from the graph of Fig. 2 will be derived for the case in which each of the groups C and C' produced in the reaction

$$A+A'\rightarrow C+C'$$

consists of two or more particles, each of $(W-w)$ and $(W'-w')$ is positive, A and A' are fermions, and the exchanged virtual particle B is a spinless boson. The resulting formula is the same for the case that either or both of A, A' are bosons, but depends upon the spin of the B particle.

Let C consist of m fermions (including antifermions) with rest masses m_l, four-momenta p_l, and spins σ_l, where $l=1,\cdots,m$, and n bosons with four-momenta k_j and spins s_j, where $j=1,\cdots,n$, and let the particles of C' be similarly labeled. Consider each Feynman graph that is improper because of a B-particle propagator which separates it into two parts, one of which contains the external lines for the particles of A and C, the other those for the particles of A' and C'. The sum of all such Feynman graphs is represented by the graph of Fig. 2, and gives the following contribution, in the (X) coordinate system, to the scattering matrix element for the process $A+A'\rightarrow C+C'$:

$$S_{fi}=-i(2\pi)^4\delta^4(p_i+p_i'-P-P')D_{F_c}(\Delta^2)\frac{1}{(2\pi)^3}\left(\frac{ww'}{w_Xw_X'}\right)^{\frac{1}{2}}\left[\frac{1}{(2\pi)^{\frac{3}{2}}}\right]^{m+m'+n+n'}\prod_{l=1}^{m}\left(\frac{m_l}{E_{lX}}\right)^{\frac{1}{2}}\prod_{l'=1}^{m'}\left(\frac{m_{l'}'}{E_{l'X}'}\right)^{\frac{1}{2}}$$
$$\times\prod_{j=1}^{n}\left(\frac{1}{2E_{jX}}\right)^{\frac{1}{2}}\prod_{j'=1}^{n'}\left(\frac{1}{2E_{j'X}'}\right)^{\frac{1}{2}}\langle P|\Gamma|\Delta_i,p_i\rangle\langle P'|\Gamma'|\Delta_i',p_i'\rangle,\quad(4.1)$$

where $\Delta_i'=-\Delta_i=p_i-P$, $D_{F_c}(\Delta^2)$ is the complete renormalized boson propagator, and in lowest order perturbation theory is equal to $(m_B^2+\Delta^2-i\epsilon)^{-1}$. The matrix elements of the two vertex operators Γ and Γ' are relativistically invariant. For simplicity, the charge and spin states of the particles are not indicated in the vertex matrix elements. Each of the two vertex matrix elements is automatically symmetrized with respect to the particles within its particular group C or C'. Justification for use of the weighting factor unity in obtaining the scattering matrix element S_{fi} of Eq. (4.1) from the graph of Fig. 2 is given in the Appendix, where it is also shown that the initial and final states involved in Eq. (4.1) are correctly normalized if the particles are treated as being nonidentical.

The invariant differential cross section obtained from the matrix element S_{fi} of Eq. (4.1) is

$$d\sigma=\frac{(2\pi)^4\delta^4(p_i+p_i'-P-P')}{[(p_ip_i')^2-w^2w'^2]^{\frac{1}{2}}}\frac{(G\Delta^2)}{(m_B^2+\Delta^2)^2}$$
$$\times w\sum_f\langle\sum_i\rangle_{\text{av}}|\langle P|\Gamma|\Delta_i,p_i\rangle|^2d\Omega$$
$$\times w'\sum_f\langle\sum_i\rangle_{\text{av}}|\langle P'|\Gamma'|\Delta_i',p_i'\rangle|^2d\Omega',\quad(4.2)$$

where $G(\Delta^2)$ is a form factor of the boson propagator normalized so that $G(-m_B^2)=1$, $\sum_f\langle\sum_i\rangle_{\text{av}}$ indicates the sum over final and average over initial spins, $d\Omega$ is the invariant phase-space volume element for the group C,

$$d\Omega=\left[\frac{1}{(2\pi)^3}\right]^{m+n}\prod_{l=1}^{m}m_l\frac{d^3p_l}{E_l}\prod_{j=1}^{n}\frac{1}{2}\frac{d^3k_j}{E_j},$$

$d\Omega'$ is similarly defined for the group C', the factor $[(p_ip_i')^2-w^2w'^2]^{-\frac{1}{2}}$ is the invariant part of the flux,[14] and, since each factor in Eq. (4.2) is separately relativistically invariant, no coordinate system label is indicated.

For sufficiently large excitation at each vertex and small enough Δ^2, it follows from Eqs. (2.21a,b) that the vertex Γ is kinematically close to that for the physical process

$$A+B\rightarrow C,\quad(4.3\text{a})$$

and the vertex Γ' is close to that for the process

$$A'+\bar{B}\rightarrow C'.\quad(4.3\text{b})$$

It is thus reasonable to express Eq. (4.2) in terms of the off-the-mass-shell cross sections that correspond to these physical processes; that for the process of Eq. (4.3a) is

[14] C. Møller, Kgl. Danske Videnskab. Selskab, Mat-fys. Medd. **23**, 1 (1945).

defined as follows:

$$\sigma_{A+B\to C}(\Delta^2;W)=\frac{(2\pi)^4}{p_W W}w\int d\Omega\,\delta^4(p_i+\Delta_i-P)$$
$$\times\textstyle\sum_f\langle\sum_i\rangle_{\text{av}}|\langle P|\Gamma|\Delta_i,p_i\rangle|^2,\quad(4.4)$$

where p_W is the magnitude of the three-momentum in (W) of a real B-particle scattering with A at energy W, given by

$$W=(p_W^2+w^2)^{\frac{1}{2}}+(p_W^2+m_B^2)^{\frac{1}{2}},\qquad(4.5)$$

and the off-the-mass-shell cross section that corresponds to the process of Eq. (4.3b) $\sigma_{A'+\bar{B}\to C'}(\Delta^2;W')$, is similarly defined. It may be shown, in the case that the B particle is a spinless boson, that each of the off-the-mass-shell cross sections depends only on Δ^2 and the energy. In order to emphasize that Δ^2 represents the off-the-mass-shell nature of the cross section and is not a phase-space variable of the group C, it is separated by a semicolon from the energy variable in Eq. (4.4). Equation (4.4) is the simplest definition consistent with the requirement that

$$\lim_{\Delta^2\to -m_B^2}\sigma_{A+B\to C}(\Delta^2;W)=\sigma_{A+B\to C}(W),$$

the physical cross section.

The vertex matrix elements that occur in Eq. (4.1) are the "inelastic" analogs of the familiar vertex functions that arise, for example, in elastic electron-nucleon scattering, and reflect, as do those, the fact that the exchanged virtual particle is not on its mass shell. The ability to extract these off-the-mass-shell scattering amplitudes, or more conveniently, the corresponding off-the-mass-shell scattering cross sections, will depend upon the dominance of single virtual particle exchange graphs, and upon the closeness of $G(\Delta^2)$ to 1, as does the ability to extract the nucleon electromagnetic form factors in elastic electron-nucleon scattering. There is also the possibility for making a Chew-Low type of extrapolation from the physical region to the pole of the propagator, at which each of the vertices is the scattering cross section for a real collision.

In order to express Eq. (4.2) in terms of the off-the-mass-shell cross sections, an energy-momentum delta function is introduced for each vertex by writing

$$\delta^4(p_i+p_i'-P-P')$$
$$=\int d^4\Delta_i\delta^4(p_i+\Delta_i-P)\delta^4(p_i'+\Delta_i'-P'),$$

where, as usual $\Delta_i'=-\Delta_i$. By integrating Eq. (4.2) over $d\Omega$ and $d\Omega'$, and making the indicated substitutions, one obtains the differential cross section with respect to Δ_i,

$$d\sigma=\frac{4}{(2\pi)^4 p_{iU}U}d^4\Delta_i\frac{G(\Delta^2)}{(m_B^2+\Delta^2)^2}p_W W$$
$$\times\sigma_{A+B\to C}(\Delta^2;W)p_{W'}'W'\sigma_{A'+\bar{B}\to C'}(\Delta^2;W'),\quad(4.6)$$

where the invariant flux factor has been evaluated in (U). Since each of the off-the-mass-shell cross sections contains an energy-momentum conserving δ function for its vertex, this is another restriction on the phase-space variables, which is compensated for by the re-introduction of the four-momentum Δ_i as additional phase-space variables. A more convenient form of this result is obtained by replacing the variables Δ_i by Δ^2, W, W', and an azimuthal angle φ. From the equation $\Delta_i=P-p_i$ it follows that $d^4\Delta_i=d^4P$. In (U),

$$d^4P=d^3P_U dW_U=P_U^2dP_U\sin\theta d\theta d\varphi dW_U.$$

One obtains from the relation $P_U^2=W_U^2-W^2$ that

$$P_U^2dP_U=-(W_U^2-W^2)^{\frac{1}{2}}WdW,$$

from Eq. (2.9) that

$$\sin\theta d\theta=d(\Delta^2)/[2p_{iU}(W_U^2-W^2)^{\frac{1}{2}}],$$

and from Eq. (2.4c) that

$$dW_U=-W'dW'/U.$$

Therefore,

$$d^4\Delta_i=(1/2p_{iU}U)WdWd(\Delta^2)d\varphi\,W'dW'.$$

If this is substituted into Eq. (4.6) and the integrations with respect to φ, W, and W' are performed, one obtains

$$\frac{d\sigma_{A+A'\to C+C'}}{d(\Delta^2)}$$
$$=\frac{2}{(2\pi)^3p_{iU}^2U^2}\frac{G(\Delta^2)}{(m_B^2+\Delta^2)^2}$$
$$\times\int_{\text{allowed }W,\,W'}dWdW'\;p_WW^2p_{W'}'W'^2$$
$$\times\sigma_{A+B\to C}(\Delta^2;W)\sigma_{A'+\bar{B}\to C'}(\Delta^2;W'),\quad(4.7)$$

where the integration is restricted to the portion of the $W-W'$ plane which is allowed for the particular value of Δ^2. This is a useful form in which to write the cross section because the model is expected to be valid only for small values of Δ^2.

A similar result is obtained if instead of the total off-the-mass-shell cross sections, the differential cross sections with respect to $d\Omega$ and $d\Omega'$ (which depend also on Δ_i) are used.

The only reference to the number of particles in the final state in Eq. (4.7) is through the labels C and C',

which denote the constituents of each group. If one sums the various possibilities for the group C, this leads to the replacement of $\sigma_{A+B\to C}(\Delta^2;W)$ in Eq. (4.7) by $\sigma_{AB}{}^{\text{tot}}(\Delta^2;W)$. If one also sums the various possibilities for the group C', then likewise $\sigma_{A'+\bar{B}\to C'}(\Delta^2;W')$ is replaced by $\sigma_{A'\bar{B}}{}^{\text{tot}}(\Delta^2;W')$. If in addition the various charge states of the exchanged B particle are summed, one obtains

$$\frac{d\sigma_{AA'}(\Delta^2,U)}{d(\Delta^2)} = \frac{2}{(2\pi)^3 p_{iU}{}^2 U^2}\frac{1}{(m_B{}^2+\Delta^2)^2} \times \int_{\text{allowed } W,\, W'} dW dW'\, p_W W^2 p_{W'}{}' W'^2 \times \sum_B G(\Delta^2)\sigma_{AB}{}^{\text{tot}}(\Delta^2;W)\sigma_{A'\bar{B}}{}^{\text{tot}}(\Delta^2;W'). \quad (4.8)$$

To the extent that it is reasonable to sum over all the states C and C', this expression gives the total differential cross section with respect to Δ^2 for the single B-particle exchange interactions of the particles A and A'.

5. THE REACTION $\pi+N\to\gamma+N^*$

The single virtual pion exchange graph for the pion-photon production process, $\pi+N\to\gamma+N^*$, shown in Fig. 7, is an example in which Δ^2 can be negative, that is, it can be less than $m_\pi{}^2$ away from its value at the pole of the propagator. Figure 7 is obtained from the graph of Fig. 2 by taking A, A', B, C, and C' to be π meson, nucleon, virtual π meson, photon, and "excited nucleon" group, respectively. Therefore, $w=m_\pi$, $w'=M$, $m_B=m_\pi$, $W=0$, and $W'>M$. In this reaction, since the Γ vertex is de-excited, it follows from theorem (2.12) that negative Δ^2 can occur provided the Γ' vertex is excited, that is, if $W'>M$. Although the possibility of obtaining timelike momentum transfers provided the original motivation for examining this process, it in fact turns out that the part of the phase space for which Δ^2 is negative does not make an important contribution to the cross section. Nevertheless, a sizeable differential cross section is predicted for the production of high-energy photons at very small angles. For this reason, and also because of the insight gained to reactions of this kind, some details of this example are presented here.

As indicated in Fig. 7, the initial-state pion and nucleon have four-momenta p_i and p_i', respectively. The laboratory components are

$$(p_i)_L=(\mathbf{p}_{iL},\omega_{iL}) \quad\text{and}\quad (p_i')_L=(\mathbf{0},M).$$

The final-state photon γ and "excited nucleon" group

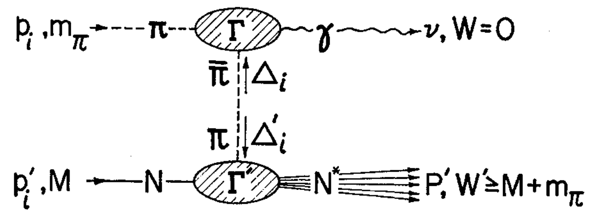

FIG. 7. Single virtual pion exchange graph for the pion photoproduction process in which a single high-energy photon is produced at a very small angle by a very high energy π meson.

N^* have four-momenta P and P', respectively. The laboratory components are

$$(P)_L=(\boldsymbol{\nu}_L,\nu_L) \quad\text{and}\quad (P')_L=(\mathbf{P}_L',W_L').$$

We restrict ourselves to values of $W'\geq M+m_\pi$ so that N^* includes at least one nucleon and one pion. From Eq. (2.4c), the photon energy in (U), ν_U is given by

$$\nu_U=(U^2-W'^2)/(2U), \quad (5.1)$$

and from Eq. (2.11), Δ^2 is given by

$$\Delta^2=[\Delta^2(0,W')]_{\min}+4p_{iU}\nu_U\sin^2(\theta/2). \quad (5.2)$$

With $W'\geq M+m_\pi$ one finds that

$$-m_\pi{}^2<[\Delta^2(0,W')]_{\min}<0.$$

If W' is close to U, that is $\nu_U\approx 0$, then $[\Delta^2(0,W')]_{\min}\approx -m_\pi{}^2$. As W' decreases towards $M+m_\pi$, ν_U increases towards its maximum value, $\sim U/2$, and $[\Delta^2(0,W')]_{\min}$ increases towards zero. For values of $W'^2\ll U^2$ one may use Eq. (2.18), which gives the minimum value of Δ^2 for given values of W and W'. One obtains

$$[\Delta^2(0,W')]_{\min}\approx -m_\pi{}^2(W'^2-M^2)/U^2. \quad (5.3)$$

This shows that $[\Delta^2(0,W')]_{\min}$ is least negative and of very small magnitude for the minimum value' $W'=M+m_\pi$, and becomes more negative as W, increases.

Thus, negative values of Δ^2 close to $-m_\pi{}^2$ permit only soft photon emission. This process is better visualized as one in which the incident pion radiates a soft photon, continues almost completely unperturbed, and then interacts with the nucleon. For soft-photon emission other graphs than that of Fig. 7 are also expected to be important.

For the particular case that a very high energy photon is produced at a very small angle, the graph of Fig. 7 is expected to be dominant. The formula for the cross section is similar to that given in Eq. (4.7) except that because of the unique value of the mass W, there is no integration over this variable and in place of a scattering cross section for the Γ vertex one has the modulus squared of an "emission" vertex function.

The differential cross section is given by[14a]

$$d^2\sigma=\frac{1}{8(2\pi)^2 p_{iL}^2M^2}\sum_{\lambda=1}^{2}|\Gamma^{(\lambda)}|^2\frac{G(\Delta^2)}{(\Delta^2+m_\pi^2)^2}d(\Delta^2)\times p_W W d(W^2)\sigma_{\pi N}{}^{\rm tot}(\Delta^2;W),\quad(5.4)$$

where the invariant flux factor has been evaluated in (L), the primes have been dropped from the variables related to the target particle vertex, $\sum_\lambda|\Gamma^{(\lambda)}|^2$ is the $\pi-\pi-\gamma$ vertex function squared and summed over the photon polarization λ and is given in (L) by

$$\sum_{\lambda=1}^{2}|\Gamma^{(\lambda)}|^2=4e^2\sum_{\lambda=1}^{2}(\mathbf{p}_{iL}\cdot\boldsymbol{\varepsilon}_L{}^{(\lambda)})^2F(\Delta^2)=4e^2p_{iL}^2\sin^2\theta_L F(\Delta^2),\quad(5.5)$$

where $\boldsymbol{\varepsilon}_L{}^{(\lambda)}$ is the photon polarization vector for the state λ, θ_L is defined by $\mathbf{p}_{iL}\cdot\mathbf{v}_L=p_{iL}\nu_L\cos\theta_L$, and $F(\Delta^2)$ is a pion-electromagnetic form factor present because the exchanged pion is virtual. This is not the usual pion-electromagnetic form factor that occurs when the photon is virtual. The function $\Delta^2+m_\pi^2$ may be expressed in terms of the laboratory angle, θ_L,

$$\Delta^2+m_\pi^2=2\nu_L\omega_{iL}(1-\beta_{iL}\cos\theta_L),\quad(5.6)$$

where $\beta_{iL}=p_{iL}/\omega_{iL}$. For $\beta_{iL}\approx1$ and $\theta_L\approx0$ Eq. (5.6) becomes

$$\Delta^2+m_\pi^2=m_\pi^2\frac{\nu_L}{\omega_{iL}}\left[1+\left(\frac{\omega_{iL}}{m_\pi}\theta_L\right)^2\right].\quad(5.7)$$

From Eqs. (5.4), (5.5), and (5.7) it follows that if $F(\Delta^2)$, $G(\Delta^2)$, and $\sigma_{\pi N}{}^{\rm tot}(\Delta^2;W)$ are not rapidly varying functions of Δ^2 for small values of Δ^2, then the differential cross section will be maximum in the region where $\theta_L\approx m_\pi/\omega_{iL}$. By use of this result in Eq. (5.7), it follows that if $\nu_L\approx\omega_{iL}$, which is the case of interest here, then the maximum differential cross section occurs for values of $\Delta^2\approx m_\pi^2$.

In order to obtain the differential cross section with respect to the laboratory angles and energy of the photon, we note that, from Eq. (5.6), one has

$$|d(\Delta^2)|=2\nu_L p_{iL}d(\cos\theta_L)=(\nu_L p_{iL}/\pi)d\Omega_L.\quad(5.8)$$

The variable W can be expressed in terms of ν_L and θ_L, as follows:

$$W^2=W_L^2-P_L^2=(\omega_{iL}+M-\nu_L)^2-(\mathbf{p}_{iL}-\mathbf{v}_L)^2=-2[M+(\omega_{iL}-p_{iL})+2p_{iL}\sin^2(\theta_L/2)]\nu_L+2\omega_{iL}M+M^2+m_\pi^2,\quad(5.9)$$

from which it follows that for a highly relativistic incident pion and for angles $\theta_L\approx m_\pi/\omega_{iL}$,

$$|d(W^2)|=2Md\nu_L.\quad(5.10)$$

The factor p_WW in Eq. (5.4) may also be expressed in terms of ν_L and θ_L. Since p_W is the magnitude of the three-momentum in the barycentric system for a real pion scattering with a nucleon at barycentric energy W, the relation

$$p_WW=p_LM\quad(5.11)$$

holds, where p_L is the magnitude of the same pion's three-momentum in the laboratory system. Its laboratory energy ω_L is related to W by the equation

$$\omega_L=(W^2-M^2-m_\pi^2)/(2M).\quad(5.12)$$

Substitution of Eq. (5.9) into Eq. (5.12) gives

$$\omega_L=(\omega_{iL}-\nu_L)-(\nu_L/M)[(\omega_{iL}-p_{iL})+2p_{iL}\sin^2(\theta_L/2)],\quad(5.13)$$

from which it follows that for a highly relativistic incident pion and for angles $\theta_L\approx m_\pi/\omega_{iL}$,

$$\omega_L\approx(\omega_{iL}-\nu_L)-(m_\pi/M)(\nu_L/\omega_{iL})m_\pi\approx(\omega_{iL}-\nu_L).\quad(5.14)$$

Substitution of these results into Eq. (5.4) gives the differential cross section, for highly relativistic incident pions and angles $\theta_L\approx m_\pi/\omega_{iL}$,

$$\frac{\partial^2\sigma}{\partial\Omega_L\partial\nu_L}\approx\frac{\alpha}{\pi^2}\frac{(\omega_{iL}\theta_L/m_\pi)^2F(\Delta^2)G(\Delta^2)}{[1+(\omega_{iL}\theta_L/m_\pi)^2]^2}\times\frac{\omega_{iL}[(\omega_{iL}-\nu_L)^2-m_\pi^2]^{\frac12}}{m_\pi^2\nu_L}\sigma_{\pi N}{}^{\rm tot}(\Delta^2;\omega_L),\quad(5.15)$$

where α is the fine-structure constant and ω_L is given approximately by Eq. (5.14). As an example of the magnitude of the differential cross section that one obtains, if $\omega_{iL}=25$ Bev, $\nu_L=20$ Bev, $\theta_L=m_\pi/\omega_{iL}$, $F(\Delta^2)=1$, $G(\Delta^2)=1$, and $\sigma_{\pi N}{}^{\rm tot}(\Delta^2;\omega_L=5\text{ Bev})=30$ mb, then

$$\partial^2\sigma/\partial\Omega_L\partial\nu_L\approx1.8\text{ mb/(sr-Bev)}.$$

One thus has the possibility of producing a well-collimated high-energy photon beam from a beam of highly relativistic π mesons. Study of this reaction would of course be of theoretical interest in connection with the single virtual particle exchange model.

6. HIGH-ENERGY ASPECTS

The single virtual particle exchange model is applied here to extremely high energy nucleon-nucleon collisions, the strong interaction process for which the most high-energy experimental information is available. In this case, the following assumptions lead to a prediction for the asymptotic behavior of the nucleon-nucleon cross section:

[14a] *Note added in proof*. It has been shown by F. E. Low [Phys. Rev. **110**, 974 (1958)], that for a process with an electromagnetic vertex of a spinless boson at which the photon and one boson are real and the other boson is virtual, $F(\Delta^2)G(\Delta^2)=1$ is an exact equation for all Δ^2. This should be taken account of when Eq. (5.5) is substituted into Eq. (5.4), and in Eq. (5.15). Thus, the cross section for the process shown in Fig. 7 contains only one unknown function, the off-the-mass-shell pion-nucleon cross section. We thank Professor Low for calling this result to our attention, and for an informative discussion.

I. The model is valid for sufficiently small values of Δ^2.

II. For sufficiently large values of W and small enough values of Δ^2, that is, $\Delta^2 \sim m_\pi^2$, the off-the-mass-shell pion-nucleon cross section equals the physical cross section,

$$\sigma_{\pi+N\to C}(\Delta^2;W)=\sigma_{\pi+N\to C}(W).$$

III. For sufficiently large values of W the physical pion-nucleon total cross section is constant,

$$\sigma_{\pi N}{}^{\mathrm{tot}}(W)=\sigma_{\pi N}.$$

By using the fact that for large values of U and for values of Δ^2 of interest the important part of the phase space occurs for large values of W and W', one then obtains from Eq. (4.8) and the above assumptions

$$\frac{d\sigma_{NN}(\Delta^2,U)}{d(\Delta^2)}\approx\frac{3}{16\pi^3}(\sigma_{\pi N})^2\left(\frac{\Delta^2}{\Delta^2+m_\pi^2}\right)^2\ln\left(\frac{\Delta U}{M^2}\right), \quad (6.1)$$

where the region of phase space with either W or $W'<\sqrt{2}M$ has been neglected, the approximations $p_{iU}=U/2$, $p_W=\frac{1}{2}W[1-(M^2/W^2)]$, $p_{W'}{}'=\frac{1}{2}W'[1-(M^2/W'^2)]$ have been made, the approximate boundary given by Eq. (2.18) has been used, U was taken to be large enough so that $4\ln[\Delta U/M^2]\gg 1$, and $G(\Delta^2)$ was set equal to one. The factor 3 in Eq. (6.1) arises from the sum over the three possible charge states of the exchanged virtual pion, and $\sigma_{\pi N}$ is therefore the average total pion-nucleon cross section.

The logarithmic energy dependence given by Eq. (6.1) for the asymptotic nucleon-nucleon differential cross section is thus different than the asymptotic energy dependence assumed for the pion-nucleon cross section, which does not seem reasonable since they are both due to the same strong finite-range interaction.

Gribov,[15] and also Berestetsky and Pomeranchuk[16] have shown that if the usual assumption of constant pion-nucleon cross section at very high energy[17] is given up, and that if instead one requires asymptotically the same energy dependence of the nucleon-nucleon and pion-nucleon cross sections, then it follows from the model that the physical cross sections would have to go to zero faster than the inverse logarithm of the energy, $(\ln E)^{-1}$.

Chernavskii and Dremin[2] take a more phenomenological approach based on the approximately constant value of $\sigma_{\pi N}{}^{\mathrm{tot}}(W)$ observed experimentally for energies above the well-known "resonances." They suggest that $\sigma_{\pi N}{}^{\mathrm{tot}}(W)$ is constant but that $\sigma_{\pi N}{}^{\mathrm{tot}}(\Delta^2;W)$ approaches zero as W becomes very large.

[15] V. N. Gribov, Nuclear Phys. **22**, 249 (1961).

[16] V. B. Berestetsky and I. Ya. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 1078 (1960) [translation: Soviet Phys.—JETP **12**, 752 (1961)], Nuclear Phys. **22**, 629 (1961). The total cross section given in this paper appears to be too large by a factor 2, due to overcounting final states.

[17] I. Ya. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 725 (1958) [translation: Soviet Phys.—JETP **7**, 499 (1958)].

Equation (6.1) was obtained with the use of the constant pion-nucleon total cross section. As pointed out in Sec. 3, the model is not expected to be valid for that part of the pion-nucleon cross section which is itself as peripheral as the over-all nucleon-nucleon collision. In particular, it may be necessary to exclude the pion-nucleon elastic diffraction scattering. However, the same logarithmic behavior would be obtained if asymptotically there is a constant fraction of the pion-nucleon cross section, $f\sigma_{\pi N}>0$, due to a hard-core part of the interaction. It is of course not known whether such a hard core exists.

It is generally assumed that the nucleon-nucleon cross section becomes constant asymptotically.[17] It has been shown by Froissart[18] that the cross section obtained from a two scalar particle scattering amplitude which satisfies the Mandelstam representation cannot increase faster than $\ln^2 U$. A stronger statement than this has not yet been obtained on general grounds. Thus, although the logarithmic behavior of Eq. (6.1) is questionable, it is not necessarily incorrect. It is shown here that the same assumptions which lead to the asymptotic $\ln U$ dependence, together with an additional assumption concerning the total angular momentum in (U), yield a cross section which violates unitarity.

The additional assumption, which seems to be a very physical one, is that for W^2, $W'^2\ll U^2$, the total angular momentum in (U) is due almost entirely to the translational motion of the groups C and C' in (U) and that the internal angular momenta of each of these groups is negligible by comparison. This assumption is justified to the extent that the groups C and C' are well defined, that is, have little internal kinetic energy and large translational kinetic energy in (U).

To demonstrate the violation of unitarity, consider the part δ_f of the differential cross section obtained from Eq. (4.7) by summing over those combinations of final groups C, C' which come from the core part of the pion-nucleon interaction

$$\frac{\partial^3\delta_f\sigma_{NN}(\Delta^2,W,W',U)}{\partial(\Delta^2)\partial W\partial W'}=\frac{2\times 3}{(2\pi)^3 p_{iU}{}^2U^2}\frac{1}{(\Delta^2+m_\pi^2)^2}p_W W^2 p_{W'}{}'W'^2 \times\sum\nolimits_f \sigma_{\pi+N\to C}(\Delta^2;W)\sigma_{\pi+N\to C'}(\Delta^2;W'), \quad (6.2)$$

where $G(\Delta^2)$ was set equal to one, $\sum_f$ denotes the restricted sum over the groups C, C', and the sum over the three charge states of the exchanged virtual pion is replaced by three times the product of the charge-averaged pion-nucleon cross section. For large values of

[18] M. Froissart, Phys. Rev. **123**, 1053 (1961).

W and W' and small values of Δ^2, assumption II enables one to make the replacement

$$\sum_f \sigma_{\pi+N\to C}(\Delta^2;W)\sigma_{\pi+N\to C'}(\Delta^2;W') = \sum_f \sigma_{\pi+N\to C}(W)\sigma_{\pi+N\to C'}(W'). \quad (6.3)$$

The specific consequence of the additional assumption is that in the equation resulting from substitution of (6.3) into (6.2), one ignores the contribution to the total angular momentum in (U) coming from the various angular momentum states of the pion-nucleon cross sections. With this assumption, the only angular dependence remaining in the right-hand member is that of the propagator, and is due solely to the translational motion of the two groups in (U).

From Eq. (2.11) one has

$$\Delta^2+m_\pi^2=2p_{iU}P_U(a-\cos\theta),$$

where

$$a=1+\tfrac{1}{2}\{([\Delta^2(W,W')]_{\min}+m_\pi^2)/p_{iU}P_U\}. \quad (6.4)$$

The propagator $(\Delta^2+m_\pi^2)^{-1}$ can be expanded in partial waves as follows:

$$\frac{1}{\Delta^2+m_\pi^2}=\frac{1}{2p_{iU}P_U}\sum_l(2l+1)Q_l(a)P_l(\cos\theta), \quad (6.5)$$

where $Q_l(a)$ are the Legendre functions of the second kind,

$$Q_l(a)=\tfrac{1}{2}\int_{-1}^{+1}\frac{P_l(x)}{a-x}dx \quad \text{for} \quad a>1.$$

Substituting Eqs. (6.3) and (6.5) into (6.2), and using the fact, obtained from Eq. (6.4), that $|d(\Delta^2)| = 2p_{iU}P_U|d(\cos\theta)|$, one obtains

$$\frac{\partial^3\delta_f\sigma_{NN}(\Delta^2,W,W',U)}{\partial(\cos\theta)\partial W\partial W'} = \frac{6}{(2\pi)^3 p_{iU}^2U^2}p_W W^2 p_{W'}'W'^2 \sum_f \sigma_{\pi+N\to C}(W) \times\sigma_{\pi+N\to C'}(W')\frac{1}{2p_{iU}P_U}\sum_{ll'}(2l+1)(2l'+1)Q_l(a) \times Q_{l'}(a)P_l(\cos\theta)P_{l'}(\cos\theta). \quad (6.6)$$

In order to obtain the contribution of particular partial waves, we wish to integrate over the angle θ and to use the orthogonality of the Legendre functions, $P_l(\cos\theta)$. Although we are interested only in small values of Δ^2, which correspond to small values of θ, the integration over θ may be extended because the expression (6.6) is very sharply peaked about $\theta=0$ provided that $a\approx 1$. Integration with respect to θ gives

$$\frac{\partial^2\delta_f\sigma_{NN}(W,W',U)}{\partial W\partial W'} = \frac{\pi}{p_{iU}^2}\sum_l(2l+1)\left\{\frac{3p_WW^2p_{W'}'W'^2}{4\pi^4p_{iU}P_UU^2} \times\sum_f \sigma_{\pi+N\to C}(W)\sigma_{\pi+N\to C'}(W')(Q_l(a))^2\right\}. \quad (6.7)$$

As may be seen from Eq. (6.4) the condition $a\approx 1$ will be satisfied for values of W and W' for which $[\Delta^2(W,W')]_{\min}\ll p_{iU}P_U$. For convenience in the following, we further restrict W and W' to values for which

$$[\Delta^2(W,W')]_{\min}\leq \tau^2 \ll m_\pi^2,$$

where τ^2 is a constant. This part, δ_τ, of the cross section in the lth partial wave is given by

$$\delta_\tau\delta_f\sigma_{NN}^{(l)}(U) = \frac{\pi}{p_{iU}^2}(2l+1)\left\{\frac{3}{4\pi^4p_{iU}U^2}\int_\tau dWdW'\frac{p_WW^2p_{W'}'W'^2}{P_U} \times\sum_f \sigma_{\pi+N\to C}(W)\sigma_{\pi+N\to C'}(W')(Q_l(a))^2\right\}. \quad (6.8)$$

The explicit dependence of $Q_l(a)$ and of P_U on W and W' are needed to perform the integrations. If a is written as

$$a=1+\tfrac{1}{2}\eta^2,$$

where

$$\eta^2=\frac{[\Delta^2(W,W')]_{\min}+m_\pi^2}{p_{iU}P_U},$$

then, for $\eta\ll 1$ and l large enough so that $l\eta\gg 1$,[19]

$$Q_l(1+\tfrac{1}{2}\eta^2)\approx K_0(l\eta)\approx(\pi/2)^{\frac{1}{2}}e^{-l\eta}/(l\eta)^{\frac{1}{2}}, \quad \text{for} \quad \eta\ll 1 \quad \text{and} \quad l\eta\gg 1.$$

Substitution of this into Eq. (6.8) gives

$$\delta_\tau\delta_f\sigma_{NN}^{(l)}(U) = \frac{\pi}{p_{iU}^2}(2l+1)\left\{\frac{3}{8\pi^3p_{iU}U^2}\int_\tau dWdW'\,p_WW^2p_{W'}'W'^2 \times\sum_f\sigma_{\pi+N\to C}(W)\sigma_{\pi+N\to C'}(W') \times\left[\frac{\exp\{-2l[(\Delta_m^2+m_\pi^2)/p_{iU}P_U]^{\frac{1}{2}}\}}{P_Ul[(\Delta_m^2+m_\pi^2)/p_{iU}P_U]^{\frac{1}{2}}}\right]\right\},$$

$$\text{for} \left(\frac{\Delta_m^2+m_\pi^2}{p_{iU}P_U}\right)^{\frac{1}{2}}\ll 1 \text{ and } l\left(\frac{\Delta_m^2+m_\pi^2}{p_{iU}P_U}\right)^{\frac{1}{2}}\gg 1, \quad (6.9)$$

[19] G. N. Watson, *Theory of Bessel Functions* (The Macmillan Company, New York, 1944), p. 156.

where Δ_m^2 means $[\Delta^2(W,W')]_{\min}$. For sufficiently large U, Δ_m^2 is well approximated by Eq. (2.18),

$$\Delta_m^2=(W^2-M^2)(W'^2-M^2)/U^2, \qquad (6.10)$$

provided that there is sufficient excitation at each vertex, e.g., if W, $W'\geq\sqrt{2}M$.

The square bracket containing the exponential factor in Eq. (6.9) depends upon the variables W and W' through Δ_m^2 and P_U. By use of Eq. (2.6b) for P_U, and Eq. (6.10) for Δ_m^2, the bracket is expanded in powers of W^2/U^2 and W'^2/U^2. For $\Delta_m^2\ll m_\pi^2$ the leading term is

$$\frac{\exp[-2(l/p_{iU})m_\pi]}{lm_\pi}\exp\left[-\frac{l}{p_{iU}}m_\pi\frac{\Delta_m^2}{m_\pi^2}\right], \qquad (6.11)$$

where the approximation $p_{iU}=U/2$ given by Eq. (2.6a) has been used. Since $\Delta_m^2\leq\tau^2$ the second exponential factor in expression (6.11) satisfies the inequality

$$\exp\left[-\frac{l}{p_{iU}}m_\pi\frac{\Delta_m^2}{m_\pi^2}\right]\geq\exp\left[-\frac{l}{p_{iU}}m_\pi\frac{\tau^2}{m_\pi^2}\right]. \qquad (6.12)$$

Let b_l be the impact parameter for the lth partial wave,

$$b_l=l/p_{iU}. \qquad (6.13)$$

With the use of Eqs. (6.11)–(6.13) one obtains from (6.9) the following inequality:

$$\delta_\tau\delta_f\sigma_{NN}^{(l)}(U)\geq\frac{\pi}{p_{iU}^2}(2l+1)\left\{\frac{3}{8\pi^3p_{iU}^2U^2}\frac{\exp[-2b_lm_\pi]}{b_lm_\pi}\times\exp\left[-b_lm_\pi\frac{\tau^2}{m_\pi^2}\right]\int_\tau dWdW'\,p_WW^2p_{W'}'W'^2\times{\sum}_f\,\sigma_{\pi+N\to C}(W)\sigma_{\pi+N\to C'}(W')\right\},$$
$$\text{for}\quad b_l\gg m_\pi^{-1}\quad\text{and}\quad\tau^2\ll m_\pi^2. \qquad (6.14)$$

For sufficiently large values of U, even though Δ_m^2 is very small, the important part of the region τ occurs for values of W and W' in the asymptotic region. If one assumes that the restricted sum in the right-hand member of Eq. (6.14) remains larger than a finite constant, $(f\sigma_{\pi N})^2>0$, then

$$\int_\tau dWdW'\,p_WW^2p_{W'}'W'^2\,{\sum}_f\,\sigma_{\pi+N\to C}(W)\sigma_{\pi+N\to C'}(W')\geq(f\sigma_{\pi N})^2\int_\tau dWdW'\,p_WW^2p_{W'}'W'^2. \qquad (6.15)$$

The equation for the "upper" boundary of the region τ is obtained by replacing Δ_m^2 in Eq. (6.10) by τ^2. One then obtains

$$\int_\tau dWdW'\,p_WW^2p_{W'}'W'^2=\frac{1}{16}\tau^4U^4\ln\left(\frac{\tau U}{M^2}\right), \qquad (6.16)$$

where the contribution from the region with either W or $W'<\sqrt{2}M$ is neglected, the approximations $p_W=\frac{1}{2}W[1-(M^2/W^2)]$ and $p_{W'}'=\frac{1}{2}W'[1-(M^2/W'^2)]$ are used and U is taken to be large enough so that $4\ln(\tau U/M^2)\gg1$. Using the inequality (6.15) and Eq. (6.16), one then obtains from the inequality (6.14),

$$\delta_\tau\delta_f\sigma_{NN}^{(l)}(U)\geq\frac{\pi}{p_{iU}^2}(2l+1)\left\{\frac{3(f\sigma_{\pi N})^2\tau}{32\pi^3}\frac{\exp[-2b_lm_\pi]}{b_lm_\pi}\times\exp\left[-b_lm_\pi\frac{\tau^2}{m_\pi^2}\right]\ln\left(\frac{\tau U}{M^2}\right)\right\},$$
$$\text{for}\quad b_l\gg m_\pi^{-1},\quad\tau^2\ll m_\pi^2,\quad\text{and}\quad\tau U\gg M^2, \qquad (6.17)$$

where p_{iU} was replaced by $U/2$ in the curly bracket. Unitarity is violated if the curly bracket exceeds 1. b_l may be kept constant as U is increased by increasing l. Therefore, for given values of τ and b_l, unitarity is violated for sufficiently large U because of the logarithmic dependence on U.

The $W-W'$ phase-space integrand in Eq. (6.16) is the same one involved in obtaining Eq. (6.1). Furthermore, the region τ of the $W-W'$ plane defined by the condition $[\Delta^2(W,W')]_{\min}\leq\tau^2$ is the same as the allowed region of the $W-W'$ plane for $\Delta^2=\tau^2$. The source of the logarithmic dependence on energy of Eq. (6.1) is thus essentially the same as that in Eq. (6.16), and it is this which suggests that the result expressed by Eq. (6.1) also violates unitarity.

It may be shown from Eq. (6.9) that for a given value U and for l sufficiently large, $\delta_\tau\delta_f\sigma_{NN}^{(l)}(U)$ decreases as $\exp[-2b_lm_\pi]$. $\delta_\tau\delta_f\sigma_{NN}^{(l)}(U)$ comes from the longest range part of the interaction because it is due to the smallest values of $[\Delta^2(W,W')]_{\min}$. This exponential decrease of the part of the cross section due to the long-range part of the force is expected for a finite-range interaction, as has been pointed out by Gribov for diffraction scattering.[15]

The above violation of unitarity follows from assumptions I and II, that of a hard core in pion-nucleon scattering, and the neglect of the internal angular momentum of each of the two groups, in (U). The applicability of the single pion exchange model and the assumption that the internal angular momentum of each of the groups is a negligible part of the total angular momentum in (U) seem to be physically reasonable for collisions resulting in two well-defined groups of particles. From this one may conclude that $\sum_f\sigma_{\pi+N\to C}(\Delta^2;W)\sigma_{\pi+N\to C'}(\Delta^2;W')$ goes to zero as W or W' becomes infinite, either because of the Δ^2 dependence or because there is no hard core in pion-nucleon scattering. In either of these cases the energy dependence of this part of the peripheral $N-N$ cross section would be less than logarithmic. It could of course still be an important part of the total cross section.

7. DISCUSSION

The single virtual particle exchange model has already led to increased understanding of high-energy inelastic collisions. In particular, it explains naturally certain qualitative features of nucleon-nucleon collisions over an extended energy range.[9] It also predicts the importance of other small momentum transfer processes such as high-energy photon production at small angles by very high energy π mesons in the reaction $\pi+N\rightarrow\gamma+N^*$, as discussed in Sec. 5.

Extensive detailed confirmation of the model is still lacking. In general, an effective method for testing the model is to relate the cross sections for small momentum transfer events in various interactions. For example, from nucleon-nucleon inelastic collisions one may obtain the pion-nucleon off-the-mass-shell cross sections, provided that $G(\Delta^2)$ is close to unity. These may be used to predict the composition of the backward cone in pion-nucleon collisions. To the extent that this confirms the model, one has obtained the off-the-mass-shell pion-nucleon cross sections and has begun to learn about the range of validity of the model.

It should be emphasized that the single virtual particle exchange ideas, as currently formulated, constitute a model rather than a theory. There is thus no theoretically precise and at the same time experimentally useful statement that one can make, in the case of strong interactions, about the range of validity of the model. One does not know, for example, how important is the Δ^2 dependence of the off-the-mass-shell cross sections that occur in the model, nor up to what values of Δ^2 the single virtual particle exchange graphs dominate the cross section, nor how well defined the two groups must be in order that the "classical" limit be valid, in which the interference effects between the two groups of final-state particles can be neglected. Much additional experimental information will be needed before these questions can be answered.

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APPENDIX

Symmetry and Normalization

Justification is given here for use of the weighting factor unity in obtaining the matrix element of Eq. (4.1) from the graph of Fig. 2 for the case that the exchanged virtual particle is a spinless boson. The discussion deals first with the scattering operator, then with the normalization of the state vectors. It is shown that there may be final states for which the weighting factor is not unity, but that these states are physically insignificant in a scattering problem.

Consider the well-known invariant perturbation expansion for the scattering operator in the interaction picture,

$$S=1+\sum_{n=1}^{\infty}\frac{1}{n!}\int d^4x_1\cdots d^4x_n\,T(i\mathcal{L}(x_1)\cdots i\mathcal{L}(x_n)),\quad\text{(A1)}$$

where $\mathcal{L}(x)$ is the interaction Lagrange density. After expansion of the chronological products into unpaired and chronologically paired operators by Wick's theorem, identification of the various terms with Feynman graphs in configuration space is made.

Consider all the Feynman graphs of Fig. 2, in configuration space. Each of them is improper because of a B-particle propagator which separates the graph into two parts, the Γ-vertex part which has $m+n+1$ external lines for the particles of A and C, and the Γ'-vertex part with $m'+n'+1$ external lines for the particles of A' and C'. Suppose that the least power of $\mathcal{L}(x)$ that contributes to the Γ vertex is s and the least power that contributes to the Γ' vertex is s'. Then the set of all lowest order Feynman graphs of Fig. 2 comes from the $s+s'$ term of the scattering operator,

$$S_{s+s'}=\frac{1}{(s+s')!}\int d^4x_1\cdots d^4x_{s+s'}\times T(i\mathcal{L}(x_1)\cdots i\mathcal{L}(x_{s+s'})).\quad\text{(A2)}$$

Two nth order Feynman graphs in configuration space are topologically equal to each other if they differ only by a permutation of the indices $x_1,\cdots,x_n$ labeling their vertices. Topologically equal graphs give equal contributions to the scattering operator. The set of graphs of order $s+s'$ of Fig. 2 may be divided into a number of subsets of topologically equal graphs. Let $\mathcal{S}_i$ denote the ith such subset, and let $\mathcal{G}_{ij}$ denote the jth element of $\mathcal{S}_i$.

Consider a particular graph $\mathcal{G}_{ij}$ in which the s elementary vertices of the Γ-vertex part are labeled $x_1,\cdots,x_s$, the s' elementary vertices of the Γ'-vertex part are labeled $x_{s+1},\cdots,x_{s+s'}$, and the virtual B particle propagates from x_{s+1} to x_s. The term of $\mathcal{L}(x_s)$ which annihilates the B particle is written as $\mathfrak{M}(x_s)\phi_B(x_s)$, and that of $\mathcal{L}(x_{s+1})$ which creates it as $\mathfrak{N}(x_{s+1})\phi_B^*(x_{s+1})$. Since there are no other propagators joining any of the vertices $x_1,\cdots,x_s$ to any of the vertices $x_{s+1},\cdots,x_{s+s'}$, the part of $S_{s+s'}$ which generates the graph $\mathcal{G}_{ij}$ may be factored as follows:

$$\frac{1}{(s+s')!}\int d^4x_1\cdots d^4x_{s+s'}T(i\mathcal{L}(x_1)\cdots i\mathfrak{M}(x_s))\times C(\phi_B(x_s)\phi_B^*(x_{s+1}))T(i\mathfrak{N}(x_{s+1})\cdots i\mathcal{L}(x_{s+s'})),\quad\text{(A3)}$$

where $C(\phi_B(x_s)\phi_B^*(x_{s+1}))$ is the chronological pairing of the relevant B-field operators. Let $\Gamma_{ij}(x_1,\cdots,x_s)$ be the term of the Wick expansion of $T(i\mathcal{L}(x_1)\cdots i\mathfrak{M}(x_s))$ which gives the Γ-vertex part of $\mathcal{G}_{ij}$, and let $\Gamma_{ij}'(x_{s+1},\cdots,x_{s+s'})$ be the term of the Wick expansion of $T(i\mathfrak{N}(x_{s+1})\cdots i\mathcal{L}(x_{s+s'}))$ which gives the Γ'-vertex

part of $\mathcal{G}_{ij}$. The part of S that represents the particular graph $\mathcal{G}_{ij}$ may then be written

$$\frac{1}{(s+s')!}\int d^4x_1\cdots d^4x_{s+s'}\Gamma_{ij}(x_1,\cdots,x_s) \times C(\phi_B(x_s)\phi_B{}^*(x_{s+1}))\Gamma_{ij}{}'(x_{s+1},\cdots,x_{s+s'}). \quad \text{(A4)}$$

Suppose that the Γ-vertex part of $\mathcal{G}_{ij}$ is invariant under a group of r_i permutations of the labels $x_1, \cdots, x_s$ of its s elementary vertices, and that the Γ'-vertex part is invariant under a group of r_i' permutations of the labels $x_{s+1}, \cdots, x_{s+s'}$ of its s' elementary vertices. If $\mathcal{G}_{ij}$ is invariant under the exchange $A\leftrightarrow A'$, $C\leftrightarrow C'$, it has bilateral symmetry with respect to the B-particle propagator. Let r_{Bi} equal 2 or 1 according as $\mathcal{G}_{ij}$ has or has not this bilateral symmetry, respectively. The number of topologically equal graphs in $\mathcal{S}_i$ is $(s+s')!/r_{Bi}r_ir_i'$ and the total contribution to the scattering operator of the set $\mathcal{S}_i$ is therefore

$$\frac{1}{r_{Bi}}\int d^4x_1\cdots d^4x_{s+s'}\frac{1}{r_i}\Gamma_{ij}(x_1,\cdots,x_s) \times C(\phi_B(x_s)\phi_B{}^*(x_{s+1}))\frac{1}{r_i'}\Gamma_{ij}{}'(x_{s+1},\cdots,x_{s+s'}). \quad \text{(A5)}$$

The contribution to the scattering operator for the real scattering process $A+B\rightarrow C$ from the Feynman graphs obtained from those of the set $\mathcal{S}_i$ by opening the B-particle lines and retaining the Γ-vertex parts is

$$\int d^4x_1\cdots d^4x_s\,\frac{1}{r_i}\Gamma_{ij}(x_1,\cdots,x_s)\phi_B(x_s). \quad \text{(A6)}$$

The factor $1/r_i$ is therefore needed in order to give the correct contribution to the off-the-mass-shell scattering amplitude for the process $A+B\rightarrow C$, and $1/r_i'$ is likewise needed for the process $A'+\bar{B}\rightarrow C'$.

When the transition from the configuration operator (A5) to the momentum representation is carried out, each Feynman graph in momentum space will be produced r_{Bi} times. The factor $1/r_{Bi}$ can thus be ignored in expression (A5) provided that, in the case of bilaterally symmetric graphs, one also ignores the duplication of graphs in momentum space due to this symmetry. With this understanding, the over-all weighting factor in expression (A5) is unity, and this is true for the contribution of each of the subsets of topologically equal graphs of order $s+s'$ of Fig. 2.

Addition of the contribution from all higher order graphs of Fig. 2 serves to "dress" the bare Γ- and Γ'-vertex operators and the B-particle propagator of the $s+s'$ order contributions, but does not change the over-all weighting factor. Consideration of the symmetries of the scattering operator therefore leads to an over-all weighting factor of unity.

In addition, one has to consider normalization of initial and final states.

If V is the (infinite) volume of all space, then a state of one particle with definite four-momentum $[\mathbf{p},E=+(\mathbf{p}^2+m^2)^{\frac{1}{2}}]$ and spin index i is represented by

$$\Phi_1=\frac{(2\pi)^{\frac{3}{2}}}{V^{\frac{1}{2}}}a_i{}^{(+)}(\mathbf{p})\Phi_0,$$

where Φ_0, the vacuum state, is normalized to 1, Φ_1 is the one-particle state, and $a_1{}^{(+)}(\mathbf{p})$ the creation operator for the particular particle. A state of two particles that are similar but not completely identical, e.g., two electrons with different momenta, is represented by

$$\Phi_2=\frac{(2\pi)^3}{V}a_{i_1}{}^{(+)}(\mathbf{p}_1)a_{i_2}{}^{(+)}(\mathbf{p}_2)\Phi_0. \quad \text{(A7)}$$

Only if $i_1=i_2$, $\mathbf{p}_1=\mathbf{p}_2$, and the particles are bosons does the normalization condition $\Phi_2{}^*\Phi_2=1$, require a factor $1/\sqrt{2}$ on the right-hand side.[20] Equation (A7) is thus the correctly normalized expression for the initial state of a binary collision process.

For the case of bosons the sudden change in the explicit form of the right-hand side of Eq. (A7) as the two particles become completely identical is of course due to the fact that the commutator brackets involve the factor $\delta^3(\mathbf{p}_1-\mathbf{p}_2)$, and that suddenly an additional term appears in $\Phi_2{}^*\Phi_2$ when $\mathbf{p}_1=\mathbf{p}_2$ (provided $i_1=i_2$). Since $\Phi_2{}^*\Phi_2$ is the probability for having the two specified particles in the state Φ_2, it must be continuous at the point $\mathbf{p}_1=\mathbf{p}_2$, and the factor $1/\sqrt{2}$ is therefore present in Φ_2 at $\mathbf{p}_1=\mathbf{p}_2$ but nowhere else. This leads to the curious result that the differential cross section for a production process in which two similar bosons are present in the final state has a discontinuity at points in the phase space at which the two similar bosons have equal momenta. In fact, the differential cross section at $\mathbf{p}_1-\mathbf{p}_2=\mathbf{0}$ is half its value in the immediate neighborhood, where $\mathbf{p}_1-\mathbf{p}_2$ is almost, but not quite, zero. Such isolated surfaces in the phase space cannot, it seems, be experimentally detected, and for this reason the result is experimentally unimportant.

The initial and final states therefore do not introduce any additional weighting factors. Thus the graph of Fig. 2 leads to the matrix element of Eq. (4.1).

[20] The expression in the book by N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), p. 259, for an N-particle state is thus generally incorrect.