Renormalization of Quantum Electrodynamics in a Classical Gravitational Field*

RYOYU UTIYAMA†
Institute of Field Physics, Department of Physics, University of North Carolina, Chapel Hill, North Carolina

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The divergences of the Green's functions of electrons and photons in a classical gravitational field are investigated and are found to be removable by the introduction of suitable counter terms into the Lagrangian. These counter terms are obtained by rewriting the conventional renormalization technique in a generally covariant way. It is shown that infinite renormalization constants identical to those appearing in conventional quantum electrodynamics are sufficient for the removal of all divergences also when a gravitational field is present. No other renormalization is necessary. The segregation of the divergences is accomplished by making use of the transformation properties of the Green's functions under (i) general coordinate transformations, (ii) Vierbein rotations, and (iii) gauge transformations.

INTRODUCTION

THE investigation of the Green's functions of electrons and photons interacting with a given classical gravitational field is of interest to the quantization program for general relativity for the following reasons.

Let us suppose that Feynman's method of path-integration is applicable to the quantization of the gravitational field interacting with a system of electrons and photons. The Green's function of an electron in this case is given by[1]

$$G(x,y)=N\int G(x,y|g_{\mu\nu})D[g_{\mu\nu}]$$

$$\times\exp\left[i\int(-g)^{\frac{1}{2}}g^{\mu\nu}R_{\mu\nu}d^4x\right]W[g]\prod_{\mu,\nu,x}dg_{\mu\nu}(x).$$

Here the function $G(x,y|g_{\mu\nu})$ is the Green's function for an electron interacting with the quantized electromagnetic field in a given classical gravitational field $g_{\mu\nu}(x)$. $D[g_{\mu\nu}]$ is a functional of the same $g_{\mu\nu}(x)$ which appears as a denominator in the definition of $G(x,y|g)$, namely,

$$D[g_{\mu\nu}]=\int\exp\left[i\int_{-\infty}^{\infty}L(\psi\bar{\psi}Ag_{\mu\nu})d^4x\right]$$

$$\times\prod_{\mu,x,y,z}[d\psi(x)d\bar{\psi}(y)dA_\mu(z)].$$

L is the Lagrangian density of the electron-photon system in the $g_{\mu\nu}$ field. Finally $W[g]$ is the weight function in the functional space which should be chosen so that $G(x,y)$ will have a correct transformation character under changes of the integration variables.

For the evaluation of $G(x,y)$ we must, first of all, know the behavior of $G(x,y|g_{\mu\nu})$ and remove the divergences from it.[2] Because of the derivative coupling of the gravitational field with the electron and photon fields, it will be expected that there occur an infinite number of different kinds of divergent diagrams in the course of the calculation of the S matrix. This unfavorable situation suggests that there may be serious barriers to the renormalization of quantum electrodynamics in a given classical gravitational field.

It is the purpose of the present paper to show that no such obstacles exist. The renormalization constants $Z_1=Z_2=Z$, Z_3 and δm in the conventional quantum electrodynamics are also sufficient for removing all the divergences when a classical gravitational field is present.

Schwinger's formalism[3] seems to be the most suitable approach to the present aim. Under some assumptions imposed on the gravitational field, it will be shown that the Green's functions of an electron and a photon in a classical gravitational field can be given together with the equations which should be satisfied by these Green's functions.

We shall see that these equations are the generally covariant substitutes of the familiar equations for Green's functions originally given by Schwinger.[4] The

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† On leave of absence from Osaka University, Osaka, Japan.

[1] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, English translation (Interscience Publishers, Inc., New York, 1959), Chap. 7.

[2] S. Deser [Revs. Modern Phys. **29**, 417 (1957)] has conjectured, on the basis of the Feynman functional integral, that the divergences of field theory should disappear, owing to a smearing of the Green's-function singularities, when account is taken of the quantized gravitational field itself. Although this seems to be a reasonable conjecture it should nevertheless be noted that when the functional integration is performed in the order indicated here, one cannot avoid dealing with the divergences arising from the electron-photon coupling, since they come first. The explanation of this paradoxical situation is not clear, although it would not be surprising if such a mathematically ill-defined object as a functional integral should depend, in value, on the precise manner (e.g., order) in which the integration is performed. In any case, since the renormalization described in the present paper is carried out in a completely covariant manner, the end result should be the same whether renormalizations are performed before or after the integration over the metric variables.

[3] J. Schwinger, Proc. Natl. Acad. Sci. (U. S.) **37**, 452 (1951).

[4] Although the metric field is not determined by dynamical laws but is here an externally given c-number function, one should not infer from this that general covariance is lost in the present discussion. The equations of the following sections hold for an arbitrary gravitational field in an arbitrary coordinate system pro-

singular behavior of the operators of the electron-self-energy type and the vacuum polarization type will be investigated in detail by expanding these operators in functional power series in the gravitational potential [strictly speaking, in power series in the deviations of $g_{\mu\nu}(x)$ from the Minkowskian metric $\eta_{\mu\nu}$]. According to Dyson's terminology, we shall have an infinite number of different kinds of primitive divergent diagrams corresponding to each term of the above functional series. The infinities corresponding to these primitive divergent diagrams, however, are fortunately not independent of each other, but should satisfy a set of recurrence formulae which are derived from the transformation characters of the operators concerned. Owing to this fortunate situation it will be seen that all the divergences appearing in the above series can be lumped as a compact expression containing an infinite constant identical to one of the well-known constants Z, Z_3, and δm.

It is evident that this result is consistent with both the principle of equivalence and the fact that the Green's functions of a bare electron and a bare photon in a classical gravitational field have the same strength of singularity as those in the case of no gravitational field.

1. LAGRANGIAN AND FIELD EQUATIONS

In order to describe the electron field by means of a spinor function $\psi(x)$ in a general Riemannian manifold having coordinates x^μ ($\mu=0, 1, 2, 3$), we introduce a local Lorentz system at each world point x. The four unit vectors at any world point x indicate the directions of the four coordinate axes of the local Lorentz frame defined at the point x and are, in general, functions of x. They are conventionally denoted by

$h_\mu{}^k(x)=$ the covariant μth component of the kth vector, ($k=0, 1, 2, 3$),

$h^0(x)=$ a time-like vector,

h^1, h^2, $h^3=$ a space-like vector.

These four vectors are called "Vierbeine" (four legs).[5] The spinor function $\psi(x)$ is defined with respect to the Vierbein at x and is subjected to a transformation only when the Vierbein is rotated, but behaves like a scalar function under general transformations of the coordinates x.

The orthonormality properties of the Vierbein are taken as

$$h_\mu{}^k(x)h_\nu{}^l(x)g^{\mu\nu}(x)=\eta^{kl},$$

where $g^{\mu\nu}(x)$ is the contravariant metric tensor in the given curved space-time while η^{kl} is the Minkowskian metric in the local Lorentz frame, having the diagonal elements $(-1, 1, 1, 1)$.

For the sake of convenience we shall list some definitions:

$$g_{\mu\nu}(x)=\eta_{kl}h_\mu{}^k(x)h_\nu{}^l(x), \quad \text{(completeness)},$$
$$h_{k\mu}(x)=\eta_{kl}h_\mu{}^l(x),$$
$$h^{k\mu}(x)=g^{\mu\nu}(x)h_\nu{}^k(x).$$

In what follows, Vierbein components are denoted by Latin indices, while the Greek indices refer to the general coordinate system.

The familiar Dirac matrices γ^k are transformed to[6]

$$\boldsymbol{\gamma}^\mu(x)=\gamma^k h_k{}^\mu(x),$$

which satisfy the anticommutation relation

$$[\boldsymbol{\gamma}^\mu(x),\boldsymbol{\gamma}^\nu(x)]_+=2g^{\mu\nu}(x).$$

In terms of the quantities introduced above, the Lagrangian density of the system of electrons and photons interacting with the given classical gravitational field is written as

$$\begin{aligned}L(x)=&-[-g(x)]^{\frac{1}{2}}\bar{\psi}\{\boldsymbol{\gamma}^\mu(x)[\partial_\mu+B_\mu(x)+ieA_\mu(x)]+m\}\psi\\&-[-g(x)]^{\frac{1}{2}}\{f_{\mu\rho}f_{\nu\sigma}g^{\mu\nu}g^{\rho\sigma}/4+(\nabla_\mu A^\mu)^2/2\}\\&-iJ^\mu A_\mu-i\bar{\psi}\eta-i\bar{\eta}\psi,\end{aligned} \tag{1.1}$$

where the following abbreviations have been used:

$$\begin{aligned}f_{\mu\nu}&=\partial_\mu A_\nu-\partial_\nu A_\mu,\\g&=\det(g_{\mu\nu}),\\\bar{\psi}&=i\psi^*\gamma^0=\psi^*\beta,\\\nabla_\mu A^\mu&=\partial_\mu A^\mu+\Gamma_\mu{}^\mu{}_\nu A^\nu,\\\Gamma_\mu{}^\lambda{}_\nu&=\tfrac{1}{2}g^{\lambda\rho}(\partial_\nu g_{\rho\mu}+\partial_\mu g_{\nu\rho}-\partial_\rho g_{\mu\nu}).\end{aligned}$$

$B_\mu(x)$ is the affinity which is necessary for making the Lagrangian invariant under x-dependent rotations of Vierbein and is defined by

$$\begin{aligned}B_\mu(x)&=\tfrac{1}{8}h_l{}^\nu\nabla_\mu h_{k\nu}[\gamma^l,\gamma^k]=\tfrac{1}{8}[\boldsymbol{\gamma}^\nu(x),\nabla_\mu\boldsymbol{\gamma}_\nu(x)],\\\nabla_\mu h_{k\nu}&=\partial_\mu h_{k\nu}-\Gamma_\mu{}^\rho{}_\nu h_{k\rho},\end{aligned} \tag{1.2}$$

$iJ^\mu(x)$ is a given charge-current vector density (c number) satisfying the equation of continuity

$$\partial_\mu J^\mu(x)=0,$$

and η and $\bar{\eta}$ are anticommuting fictitious sources of the electron field which will be put equal to zero after all of the calculations have been finished.

The equations for the electron and photon fields are derived from (1.1):

$$i(-g)^{\frac{1}{2}}[\boldsymbol{\gamma}^\mu(x)\{\partial_\mu+B_\mu+ieA_\mu\}+m]\psi=\eta, \tag{1.3}$$

$$\begin{aligned}-i\partial_\mu[(-g)^{\frac{1}{2}}\bar{\psi}\boldsymbol{\gamma}^\mu]&+i(-g)^{\frac{1}{2}}\bar{\psi}\boldsymbol{\gamma}^\mu(B_\mu+ieA_\mu)\\&+i(-g)^{\frac{1}{2}}\bar{\psi}m=\bar{\eta},\end{aligned} \tag{1.4}$$

and

$$-i(-g)^{\frac{1}{2}}[\Box_c A^\mu+R^\mu{}_\rho A^\rho]=J^\mu+e(-g)^{\frac{1}{2}}\bar{\psi}\boldsymbol{\gamma}^\mu\psi, \tag{1.5}$$

vided an explicit transformation law (the customary one) is specified for the c-number variables. It is just the use of this transformation law, in fact, which is crucial for the derivation of the present results.

[5] T. Levi-Civita, Berlin. Ber. 137 (1929). H. Weyl, Z. Physik **56**, 330 (1929).

[6] The boldface letter is used to denote the x-dependent γ matrix.

where $\square_c$ stands for the D'Alembertian operator in the curved space-time and is defined by

$$\square_c A^\mu \equiv \nabla^\rho \nabla_\rho A^\mu,$$
$$\nabla_\rho \equiv \text{covariant derivative},$$

and

$$R_{\mu\nu} = \partial_\mu \Gamma_\nu{}^\rho{}_\rho - \partial_\rho \Gamma_\mu{}^\rho{}_\nu + \Gamma_\mu{}^\rho{}_\lambda \Gamma_\nu{}^\lambda{}_\rho - \Gamma_\mu{}^\rho{}_\nu \Gamma_\rho{}^\lambda{}_\lambda.$$

These equations are covariant or invariant under the following four groups of transformations[4]:

(i) The general coordinate transformation group

$$x^\mu \to x'^\mu = \text{function of } x,$$
$$h_{k\mu}(x) \to h'_{k\mu}(x') = (\partial x^\nu/\partial x^{\mu\prime}) h_{k\nu}(x),$$
$$\psi(x) \to \psi'(x') = \psi(x),$$
$$A_\mu(x) \to A'_\mu(x') = (\partial x^\nu/\partial x^{\mu\prime}) A_\nu(x),$$
$$\eta \to \eta'(x') = \eta(x) \partial(x^0 \cdots x^3)/\partial(x^{0\prime} \cdots x^{3\prime}),$$
$$J^\mu(x) \to J'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial(x^0 \cdots x^3)}{\partial(x'^0 \cdots x'^3)} J^\nu(x).$$

(ii) The Vierbein rotation group

$$x^\mu = \text{no change},$$
$$h_{k\mu}(x) \to h'_{k\mu}(x) = a_k{}^l(x) h_{l\mu}(x),$$

with the restriction

$$a_k{}^l(x) a_m{}^n(x) \eta^{km} = \eta^{ln}.$$

In case of the infinitesimal transformation, we have

$$a_k{}^l(x) = \delta_k{}^l + \kappa \epsilon_k{}^l(x),$$
$$\epsilon_{kl}(x) = -\epsilon_{lk}(x),$$
$$\kappa = \text{infinitesimal parameter},$$

for which the field quantities are to be transformed as follows:

$$\psi_\alpha(x) \to \psi_\alpha'(x) = S_{\alpha\beta}(x) \psi_\beta(x),$$
$$\bar{\psi}_\alpha(x) \to \bar{\psi}_\alpha'(x) = \bar{\psi}_\beta(x) S_{\beta\alpha}{}^{-1}(x),$$
$$\eta \to \eta' = S\eta,$$
$$\bar{\eta} \to \bar{\eta}' = \bar{\eta} S^{-1},$$
$$\gamma^\mu \to \gamma'^\mu(x) = \gamma^\mu(x) + \kappa \epsilon_{kl} \gamma^k h^{l\mu}(x),$$
$$x,\ A_\mu,\ g_{\mu\nu},\ J^\mu = \text{not changed},$$

where S is

$$S_{\alpha\beta}(x) = \{1 + \tfrac{1}{8} \epsilon_{kl}(x) [\gamma^k, \gamma^l]\}_{\alpha\beta}.$$

The proof of the invariance of the Lagrangian density under these transformations is straightforward and familiar.

(iii) The gauge transformation group:

$$\psi \to \psi' = e^{-ie\lambda(x)} \psi,$$
$$\bar{\psi} \to \bar{\psi}' = e^{ie\lambda} \bar{\psi},$$
$$\eta \to \eta' = e^{-ie\lambda} \eta,$$
$$\bar{\eta} \to \bar{\eta}' = e^{ie\lambda} \bar{\eta},$$
$$A_\mu \to A_\mu' = A_\mu + \partial_\mu \lambda,$$

where $\lambda(x)$ is an arbitrary scalar c-number function satisfying

$$\square_c \lambda = \nabla^\rho \nabla_\rho \lambda = 0.$$

(iv) Charge conjugation:

$$\psi \to \psi^c = -C^{-1} \bar{\psi},$$
$$\bar{\psi} \to \bar{\psi}^c = \psi C,$$
$$\eta \to \eta^c = -C^{-1} \bar{\eta},$$
$$\bar{\eta} \to \bar{\eta}^c = \eta C,$$
$$A_\mu \to A^c{}_\mu = -A_\mu,$$
$$h_{k\mu} \to h^c{}_{k\mu} = h_{k\mu}.$$

Here the matrix C is independent of x and has the following properties:

$$C \gamma^k C^{-1} = -(\gamma^k)^T,$$
$$C^T = -C,$$
$$C^\dagger C = 1.$$

The invariance under charge conjugation leads to the fact that Furry's theorem continues to be valid even in the presence of a gravitational field. More precisely, the contribution from any Feynman diagram vanishes regardless of the number of external gravitational lines if the diagram has no external electron lines but has an odd number of external photon lines.

2. GREEN'S FUNCTIONS

The definition of the electron and photon Green's functions needs the following assumptions as its basis:

(i) The Riemann tensor $R_{\lambda\mu\nu\rho}$ vanishes rapidly for both

$$x^0 \to \pm\infty : \text{at any } \mathbf{x},$$

and

$$\mathbf{x} \to \pm\infty : \text{at any } x^0.$$

(ii) The given $g_{\mu\nu}(x)$ is a suitably well-behaved function all over the world and the hypersurfaces x^0=constant are space-like. These characters are to be retained under any general transformation of coordinates.

(iii) The electron charge $(-e)$ vanishes sufficiently gradually for $x^0 \to \pm\infty$.

By virtue of the postulate (i), it will be convenient to adopt a coordinate system in which

$$\lim_{x^0 \to \pm\infty} g_{\mu\nu}(x) \to \eta_{\mu\nu}. \tag{2.1}$$

This choice of coordinate system, together with postulate (iii) makes it possible to define the conventional creation and destruction operators for free electrons and photons at $x^0 = \pm\infty$. In addition, we can establish a complete set of eigenvectors for the free Hamiltonian at $x^0 = \pm\infty$.

The postulate (ii) enables us to establish the Heisenberg-Pauli scheme for the present system in a generally covariant way (even though this scheme is not manifestly covariant). Furthermore, the above three postu-

lates make it possible to define the T product and to represent formally any Heisenberg operator in terms of the incoming-field representation. Consequently they allow us to establish the formal S matrix.

Because of the existence of the external fields $g_{\mu\nu}(x)$ and $J^\mu(x)$, the vacuum state Ψ_0 at $x^0=-\infty$ is not generally the same as the vacuum state at $x^0=+\infty$. Accordingly in order to discuss the net effect of any operator F of the incoming-representation, the following quantity is of importance

$$\langle F\rangle \equiv (\Psi_0{}^*U(\infty,x^0)F(x)U(x^0,-\infty)\Psi_0)/(\Psi_0{}^*S\Psi_0),$$

where Ψ_0 means the vacuum state at $x^0=-\infty$, and U is the well-known U matrix giving the connection between any Heisenberg operator $F_H(x)$ and the corresponding incoming operator $F(x)$:

$$F_H(x)=U^{-1}(x^0,-\infty)F(x)U(x^0,-\infty),$$
$$\lim_{x_0\to-\infty} U(x^0,-\infty)=1,$$
$$\lim_{x_0\to+\infty} U(x^0,-\infty)=S.$$

In what follows, we shall make use of the following abbreviation:

$$\langle A(x),B(y)\cdots\rangle \equiv (\Psi_0{}^*T[S\,A(x)\,B(y)\cdots]\Psi_0)/(\Psi_0{}^*S\Psi_0), \quad (2.2)$$

which can be rewritten as

$$=(\Psi_0{}^*ST[A_H(x)B_H(y)\cdots]\Psi_0)/(\Psi_0{}^*S\Psi_0). \quad (2.2)'$$

Let us regard the field equations (1.3)–(1.5) as q-number equations of the Heisenberg picture and take the "expectation value" of these equations in the sense of (2.2)′. Keeping in mind the following relations[7]:

$$\begin{aligned}\frac{\delta}{\delta J^\mu(x)}(\Psi_0{}^*S\Psi_0)&=(\Psi_0{}^*SA_\mu(x)_H\Psi_0)\\&=(\Psi_0{}^*T[SA_\mu(x)]\Psi_0),\\ \delta_\eta(\Psi_0{}^*S\Psi_0)&=\int(\Psi_0{}^*S\bar\psi_H(x)\Psi_0)\delta\eta(x)dx,\\ \delta_{\bar\eta}(\Psi_0{}^*S\Psi_0)&=\int\delta\bar\eta(x)(\Psi_0{}^*S\psi_H(x)\Psi_0)dx,\end{aligned} \quad (2.3)$$

we can derive the equations for the one-electron and one-photon Green's functions, respectively, from (1.3) and (1.5). The relations (2.3) were first introduced by Schwinger[3] and are easily verified if we take into consideration the following expressions:

$$S=\prod_{j=-\infty}^{+\infty}T\left\{1-i\int_{-\infty}^{\infty}H_{\rm int}(x_j)d^3x\Delta x_j{}^0\right\},$$
$$H_{\rm int}(x)=\cdots+iJ^\mu A_\mu+i\bar\eta\psi+i\bar\psi\eta.$$

[7] We must be careful about the position of $\delta\eta$ and $\delta\bar\eta$ in the definition (2.3), because of the anticommuting character of η and $\bar\eta$.

The one-electron and one-photon Green's functions are defined as follows:

$$G(x,y)\equiv[\delta\langle\psi(x)\rangle/\delta\eta(y)]_{\eta\to0}=\langle\psi(x),\bar\psi(y)\rangle,$$
$$\begin{aligned}\mathcal{G}_{\mu\nu}(x,y)&\equiv\delta\langle A_\mu(x)\rangle/\delta J^\nu(y)=\delta\langle A_\nu(y)\rangle/\delta J^\mu(x)\\&=\langle A_\mu(x),A_\nu(y)\rangle-\langle A_\mu(x)\rangle\langle A_\nu(y)\rangle=\mathcal{G}_{\nu\mu}(y,x).\end{aligned}$$

The equations which are satisfied by G and $\mathcal{G}$ are

$$i(-g)^{\frac12}[\gamma^\mu(\partial_\mu+B_\mu+ie\langle A_\mu(x)\rangle)+m]G(x,y)+ie^2(-g)^{\frac12}\int\Sigma(x,u)[-g(u)]^{\frac12}G(u,y)du=\delta^4(x-y), \quad (2.4)$$

$$-i(-g)^{\frac12}[\Box_c g^{\mu\rho}(x)+R^{\mu\rho}(x)]\mathcal{G}_{\rho\nu}(x,y)+ie^2(-g)^{\frac12}\int P^{\mu\rho}(x,u)[-g(u)]^{\frac12}\mathcal{G}_{\rho\nu}(u,y)du=\delta^\mu{}_\nu\delta^4(x-y), \quad (2.5)$$

where $\langle A_\mu\rangle$ is a function of J satisfying the equation

$$-i(-g)^{\frac12}[\Box_c g^{\mu\rho}+R^{\mu\rho}]\langle A_\rho(x)\rangle+e(-g)^{\frac12}\,{\rm Tr}[\gamma^\mu(x)G(x,x)]=J^\mu(x). \quad (2.6)$$

The mass operator Σ in (2.4) and the polarization operator P in (2.5) are defined by

$$\Sigma(x,y)=i\gamma^\mu(x)\int G(x,u)[-g(u)]^{\frac12}\times\Gamma^\nu(u,y;v)\mathcal{G}_{\mu\nu}(x,v)[-g(v)]^{\frac12}dudv, \quad (2.7)$$

$$P^{\mu\nu}(x,y)=-i\,{\rm Tr}\int\{\gamma^\mu(x)G(x,u)[-g(u)]^{\frac12}\times\Gamma^\nu(u,v;y)[-g(v)]^{\frac12}G(v,x)\}dudv, \quad (2.8)$$

where the vertex operator Γ is

$$\Gamma^\nu(x,y;z)=\frac{-1}{e[-g(z)]^{\frac12}}\frac{\delta G^{-1}(x,y)}{\delta\langle A_\nu(z)\rangle}, \quad (2.9)$$

$G^{-1}(x,y)$ is defined by

$$[-g(x)]^{\frac12}\int G^{-1}(x,u)[-g(u)]^{\frac12}G(u,y)du=\int G(x,u)[-g(u)]^{\frac12}G^{-1}(u,y)[-g(y)]^{\frac12}du=\delta^4(x-y).$$

In the present case G^{-1} and Γ turn out to be

$$G^{-1}(x,y)=i\{\gamma^\mu(x)(\partial_\mu+B_\mu+ie\langle A_\mu(x)\rangle)+m\}\times\delta^4(x-y)/[-g(y)]^{\frac12}+ie^2\Sigma(x,y), \quad (2.10)$$

and

$$\Gamma^\nu(x,y;z)=\frac{\gamma^\nu(x)\delta(x-y)\delta(x-z)}{[-g(y)]^{\frac{1}{4}}[-g(z)]^{\frac{1}{4}}}-\frac{ie}{[-g(z)]^{\frac{1}{2}}}\frac{\delta\Sigma(x,y)}{\delta\langle A_\nu(z)\rangle}$$
$$\equiv\gamma^\nu(x)\delta(x-y)\delta(x-z)/[-g(y)]^{\frac{1}{4}}[-g(z)]^{\frac{1}{4}}+e^2\Lambda^\nu(x,y;z),\quad(2.11)$$

respectively. The derivation of these relations is essentially given in Schwinger's paper[3] and will not be repeated here.

3. TRANSFORMATION PROPERTIES OF THE GREEN'S FUNCTIONS

(i) Vierbein Rotations

Equations (2.4)–(2.6) are form-invariant under the transformation

$$\begin{aligned}h_{k\mu}(x)\rightarrow h_{k\mu}(x)&+\kappa\epsilon_k{}^l(x)h_{l\mu}(x)\\&=h'_{k\mu}(x)\quad(\kappa=\text{infinitesimal parameter}),\\G(x,y)&\rightarrow G'(x,y)=S(x)G(x,y)S^{-1}(y),\\G^{-1}(x,y)&\rightarrow G'^{-1}(x,y)=S(x)G^{-1}(x,y)S^{-1}(y),\\\Gamma^\mu(x,y;z)&\rightarrow\Gamma'^\mu(x,y;z)=S(x)\Gamma^\mu(x,y;z)S^{-1}(y),\\\Sigma(x,y)&\rightarrow\Sigma'(x,y)=S(x)\Sigma(x,y)S^{-1}(y),\\&\mathcal{G}_{\mu\nu}P^{\mu\nu}\langle A_\rho\rangle=\text{not changed},\end{aligned}\quad(3.1)$$

where $S(x)$ is given by

$$S(x)=1+\tfrac{1}{8}\kappa\epsilon_{kl}(x)[\gamma^k,\gamma^l].\quad(3.2)$$

It will be easily seen that these transformation laws are consistent with the definitions of Σ, Γ, and P. Since the new Green's functions G', $\mathcal{G}'$, and $\langle A'\rangle$ satisfy equations of the same form as the original ones [*except for the explicit change of* $h_{k\mu}(x)$], and since the boundary conditions in both cases are the same, we have the following relations between the new and the original quantities:

$$\begin{aligned}G'(x,y)&=G(x,y,h'(x))=SG(x,y,h)S^{-1},\\\mathcal{G}'_{\mu\nu}(x,y)&=\mathcal{G}_{\mu\nu}(x,y,h')=\mathcal{G}_{\mu\nu}(x,y,h),\\P'^{\mu\rho}(x,y)&=P^{\mu\rho}(x,y,h')=P^{\mu\rho}(x,y,h),\\\Sigma'(x,y)&=\Sigma(x,y,h')=S\Sigma(x,y,h)S^{-1}.\end{aligned}\quad(3.3)$$

Inserting (3.1) and (3.2) into (3.3), we have

$$\tfrac{1}{8}\{\epsilon_{kl}(x)[\gamma^k,\gamma^l]G(x,y)-\epsilon_{kl}(y)G(x,y)[\gamma^k,\gamma^l]\}=\int\frac{\delta G(x,y,h)}{\delta h_{k\mu}(z)}\epsilon_{kl}(z)h_\mu{}^l(z)dz\quad(3.4)$$

(as well as similar relations for Σ and Γ) and

$$\frac{1}{2}\int\left\{\frac{\delta\mathcal{G}}{\delta h_{k\mu}(z)}h_\mu{}^l(z)-\frac{\delta\mathcal{G}}{\delta h_{l\mu}(z)}h_\mu{}^k(z)\right\}\epsilon_{kl}(z)dz=0\quad(3.5)$$

(and similar relations for $\langle A\rangle$ and $P^{\mu\rho}$).

Since $\epsilon_{kl}=-\epsilon_{lk}$ is completely arbitrary, (3.5) leads to

$$\frac{\delta\mathcal{G}}{\delta h_{k\mu}(z)}h_\mu{}^l(z)=\frac{\delta\mathcal{G}}{\delta h_{l\mu}(z)}h_\mu{}^k(z),$$

which shows that $\mathcal{G}$ is a functional of $g_{\mu\nu}=h_{k\mu}h_{l\nu}\eta^{kl}$ but does not explicitly depend on $h_{k\mu}$. The same is also true for $P^{\mu\rho}$ and $\langle A_\mu\rangle$.

(ii) Gauge Transformations

The gauge transformation which leaves Eqs. (2.4)–(2.6) invariant is defined by

$$\begin{aligned}\langle A_\mu(x)\rangle&\rightarrow\langle A'_\mu(x)\rangle=\langle A_\mu(x)\rangle+\partial_\mu\lambda(x),\\G(x,y)&\rightarrow G'(x,y)=e^{-ie\lambda(x)}G(x,y)e^{ie\lambda(y)},\\\mathcal{G}_{\mu\nu}&\rightarrow\mathcal{G}'_{\mu\nu}=\mathcal{G}_{\mu\nu},\end{aligned}\quad(3.6)$$

where the arbitrary scalar function λ is restricted by

$$\Box_c\lambda=0.$$

It is seen that G^{-1}, Γ, and Σ are transformed in the same manner as that of G, while $P^{\mu\nu}$ is gauge invariant. We can easily check the consistency of the above law of transformation with the definitions of $\mathcal{G}$, Γ, Σ, and P.

The argument which led to the relations (3.4) and (3.5) leads us, in the case of infinitesimal gauge transformations, to the relation

$$-ie\{\lambda(x)-\lambda(y)\}G(x,y)=\int\frac{\delta G(x,y,\langle A\rangle)}{\delta\langle A_\mu(z)\rangle}\partial_\mu\lambda(z)dz,$$

where G has been regarded as a functional of $\langle A_\mu(z)\rangle$ instead of J^μ. In a similar way we have the important relation

$$\begin{aligned}-ie[\lambda(x)-\lambda(y)]\Sigma(x,y)&=\int\frac{\delta\Sigma(x,y,\langle A\rangle)}{\delta\langle A_\mu(z)\rangle}\partial_\mu\lambda(z)dz\\&=ie\int\Lambda^\mu(x,y;z)[-g(z)]^{\frac{1}{2}}\partial_\mu\lambda(z)dz,\end{aligned}\quad(3.7)$$

where the definition (2.11) of Λ^μ has been used.

The relation (3.7) is the basis of Ward's identity. Applying similar arguments to $P^{\mu\nu}$, we have

$$\frac{\partial}{\partial z^\rho}\left(\frac{\delta P^{\mu\nu}(x,y,\langle A\rangle)}{\delta\langle A_\rho(z)\rangle}\right)=0.\quad(3.8)$$

(iii) General Coordinate Transformations

Consider the infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \kappa\xi^\mu(x),$$

where $\xi^\mu(x)$ = an arbitrary and infinitesimal function of x and κ = small parameter. Under this transformation the δ function and $[-g(x)]^{\frac{1}{2}}$ are changed in the same way:

$$\delta^4(x-y) \to \delta^4(x'-y') = \frac{\partial(x^0\cdots x^3)}{\partial(x'^0\cdots x'^3)}\delta^4(x-y),$$

$$[-g(x)]^{\frac{1}{2}} \to [-g'(x')]^{\frac{1}{2}} = \frac{\partial(x^0\cdots x^3)}{\partial(x'^0\cdots x'^3)}[-g(x)]^{\frac{1}{2}}.$$

Keeping in mind the above-mentioned fact, we can prove that the covariance of the basic equations for the Green's functions is guaranteed by the following transformation laws:

$$\begin{aligned} G(x,y) &\to G'(x',y') = G(x,y), \\ \Sigma(x,y) &\to \Sigma'(x',y') = \Sigma(x,y), \\ \Gamma^\mu(x;y;z) &\to \Gamma'^\mu(x',y';z') = \frac{\partial z'^\mu}{\partial z^\nu}\Gamma^\nu(x,y;z), \\ \mathcal{G}_{\mu\nu}(x,y) &\to \mathcal{G}'_{\mu\nu}(x',y') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial y^\sigma}{\partial y'^\nu}\mathcal{G}_{\rho\sigma}(x,y), \\ P^{\mu\nu}(x,y) &\to P'^{\mu\nu}(x',y') = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial y'^\nu}{\partial y^\sigma}P^{\rho\sigma}(x,y). \end{aligned} \tag{3.9}$$

It is easily shown that these laws are compatible with the definitions of Γ, Σ, P, etc.

The relationships (3.9) between original and transformed quantities give us important information. Let us, for example, consider the mass operator $\Sigma(x,y)$ which, as is easily seen from the equations (2.4), (2.7), and (2.9), is a functional of $h_{k\mu}(x)$ and $\langle A_\mu(x)\rangle$. Σ can be represented as a functional Taylor series in $h_{k\mu}$ and $\langle A_\mu\rangle$ in the following way[8]:

$$\begin{aligned} \Sigma(x,y;h,\langle A\rangle) &= \sum_{n,m=0}^{\infty}\frac{e^m}{n!m!}\int\prod_{j=1}^{n}h_{k_j\mu_j}(y_j)\prod_{j=1}^{m}\langle A_{\nu_j}(z_j)\rangle \\ &\quad\times f_{n,m}\{(k_1\cdots k_n),(\mu_1\cdots\mu_n);(\nu_1\cdots\nu_m); \\ &\quad x,y,y_1,\cdots,y_n;z_1,\cdots,z_m\} \\ &\quad\times\prod_{j=1}^{n}d^4y_j\prod_{j=1}^{m}d^4z_j. \end{aligned} \tag{3.10}$$

[8] The coefficient $f_{n,m}$ is a function of x, y, $\cdots z_n$ and has a set of contravariant indices $(\mu_1\cdots\mu_n)$, $(\nu_1\cdots\nu_m)$ together with a set of Vierbein indices $(k_1\cdots k_n)$. For the sake of convenience in printing, this unpreferable notation was adopted.

After the coordinate transformation has been made, the new equations for the transformed Green's functions still have the same features as the original equations apart from the difference in the explicit forms of $h_{k\mu}$ and $\langle A_\mu\rangle$. Therefore, the transformed Σ' should have the form

$$\begin{aligned} \Sigma'(x',y') &= \sum_{n,m=0}^{\infty}\frac{e^m}{n!m!}\int\prod_{j=1}^{n}h'_{k_j\mu_j}(y_j')\prod_{j=1}^{m}\langle A'_{\nu_j}(z_j')\rangle \\ &\quad\times f_{n,m}\{(k_1,\cdots,k_n),(\mu_1,\cdots,\mu_n); \\ &\quad(\nu_1,\cdots,\nu_m),x',y',y_1',\cdots,y_n',z_1',\cdots,z_m'\} \\ &\quad\times\prod_{j=1}^{n}d^4y_j'\prod_{j=1}^{m}d^4z_j'. \end{aligned}$$

Noticing the fact that the primes on $y_1'\cdots y_n'$, $z_1'\cdots z_m'$ can be removed (since these are dummy integration variables), we have the following relation:

$$\begin{aligned} \Sigma'(x,y)-\Sigma(x,y) &= \kappa\xi^\mu(x)\partial\Sigma(x,y,h,\langle A\rangle)/\partial x^\mu+\kappa\xi^\mu(y)\partial\Sigma/\partial y^\mu \\ &\quad+\int\left\{\frac{\delta\Sigma}{\delta h_{k\mu}(z)}\bar{\delta}h_{k\mu}(z)+\frac{\delta\Sigma}{\delta A_\mu(z)}\bar{\delta}\langle A_\mu(z)\rangle\right\}dz=0, \end{aligned} \tag{3.11}$$

where

$$\bar{\delta}h_{k\mu}(z)\equiv h'_{k\mu}(z)-h_{k\mu}(z)=-\kappa\frac{\partial\xi^\nu(z)}{\partial z^\mu}h_{k\nu}(z)-\kappa\xi^\nu(z)\frac{\partial h_{k\mu}}{\partial z^\nu},$$

and

$$\bar{\delta}\langle A_\mu(z)\rangle=\langle A_\mu'(z)\rangle-\langle A_\mu(z)\rangle = -\kappa\frac{\partial\xi^\rho(z)}{\partial z^\mu}\langle A_\rho(z)\rangle-\kappa\xi^\rho(z)\partial_\rho\langle A_\mu(z)\rangle.$$

Similarly,

$$\begin{aligned} &\kappa\frac{\partial\xi^\mu(x)}{\partial x^\rho}P^{\rho\nu}(x,y)+\kappa\frac{\partial\xi^\nu(y)}{\partial y^\sigma}P^{\mu\sigma}(x,y) \\ &= \kappa\xi^\rho(x)\frac{\partial}{\partial x^\rho}P^{\mu\nu}+\kappa\xi^\rho(y)\frac{\partial}{\partial y^\rho}P^{\mu\nu} \\ &\quad+\int\left\{\frac{\delta P^{\mu\nu}}{\delta g_{\rho\sigma}(z)}\bar{\delta}g_{\rho\sigma}(z)+\frac{\delta P^{\mu\nu}}{\delta\langle A_\mu(z)\rangle}\bar{\delta}\langle A_\mu(z)\rangle\right\}dz, \end{aligned} \tag{3.12}$$

with the definition

$$\bar{\delta}g_{\mu\nu}(x)=-\kappa\frac{\partial\xi^\rho}{\partial x^\mu}g_{\rho\nu}-\kappa\frac{\partial\xi^\rho}{\partial x^\nu}g_{\mu\rho}-\kappa\xi^\rho\partial_\rho g_{\mu\nu}.$$

These relations will play a basic role in the segregation of the singular parts of Σ and $P^{\mu\nu}$.

4. THE SINGULAR PARTS OF Σ, Γ, AND P

The differential equation (2.4) can be changed into an integral equation by introducing the Green's function G^0 defined by

$$i(-g)^{\frac{1}{2}}[\gamma^\mu(\partial_\mu+B_\mu)+m]G^0(x,y)=\delta^4(x-y), \quad (4.1)$$

with the same boundary condition as that for the ordinary $S^F(x-y)$.

Namely, (2.4) becomes

$$G(x,y)=G^0(x,y)+e\int G^0(x,u)[-g(u)]^{\frac{1}{2}}\gamma^\mu\langle A_\mu(u)\rangle$$
$$\times G(u,y)du-ie^2\int G^0(x,u)$$
$$\times[-g(u)]^{\frac{1}{2}}\Sigma(u,v)[-g(v)]^{\frac{1}{2}}G(v,y)dudv. \quad (4.2)$$

In a similar way the Green's function $\mathcal{G}^0{}_{\mu\nu}$ defined by

$$-i[-g(x)]^{\frac{1}{2}}[\Box_c g^{\mu\rho}(x)+R^{\mu\rho}(x)]\mathcal{G}^0{}_{\rho\nu}(x,y)$$
$$=\delta^4(x-y)\delta_\nu{}^\mu, \quad (4.3)$$

allows us to rewrite (2.5) and (2.6) as

$$\mathcal{G}_{\mu\nu}(x,y)=\mathcal{G}^0{}_{\mu\nu}(x,y)-ie^2\int\mathcal{G}^0{}_{\mu\rho}(x,u)[-g(u)]^{\frac{1}{2}}$$
$$\times P^{\rho\sigma}(u,v)[-g(v)]^{\frac{1}{2}}\mathcal{G}_{\sigma\nu}(v,y)dudv, \quad (4.4)$$

and

$$\langle A_\mu(x)\rangle=\int\mathcal{G}^0{}_{\mu\nu}(x,y)J^\nu(y)dy-e\int\mathcal{G}^0{}_{\mu\nu}(x,y)$$
$$\times[-g(y)]^{\frac{1}{2}}\,\mathrm{Tr}[\gamma^\nu(y)G(y,y)]dy. \quad (4.5)$$

The method of iteration gives G, $\mathcal{G}$, and $\langle A\rangle$ as expansions in series with respect to the charge, each term of which corresponds to a Feynman-diagram in which each internal electron and photon line stand for G^0 and $\mathcal{G}^0$, respectively in place of the conventional S^F and D^F.

As is well known,[9,10] Green's functions in a curved manifold defined by (4.1) and (4.3) have singularities on the light cone[11] of exactly the same type as those in a flat manifold. The postulate (*ii*) mentioned in Sec. 2 is therefore necessary for the definition of the T product in a generally covariant manner.

In general, the analytic form of G^0 and $\mathcal{G}^0$ is very hard to obtain. However, if we put

$$h_{k\mu}(x)=\eta_{k\mu}+\kappa a_{k\mu}(x),$$
$$h^{k\mu}(x)=\eta^{k\mu}-\kappa a^{\mu k}(x)+\cdots$$

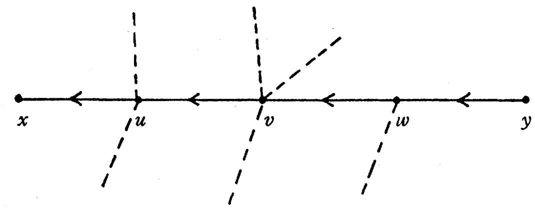

FIG. 1. Diagram representing some typical term of G^0.

(κ=small parameter) and expand, for example, (4.1) with respect to the parameter κ, we are able to obtain G^0 as a power series in κ. The first few terms of G^0 obtained in this way are, for example,

$$G^0(x,y)=S^F(x-y)-\kappa S^F(x-y)\eta^{k\mu}a_{k\mu}(y)$$
$$+i\kappa\int S^F(x-u)a_\rho{}^\mu(u)\gamma^\rho\frac{\partial}{\partial u^\mu}S^F(u-y)du$$
$$+\frac{i}{8}\kappa\int S^F(x-u)\gamma^\mu[\gamma^\sigma,\gamma^\lambda]$$
$$\times\{\partial_\mu a_{\sigma\lambda}(u)-\partial_\lambda\phi_{\sigma\mu}(u)\}$$
$$\times S^F(u-y)du+\cdots, \quad (4.6)$$

where

$$\phi_{\sigma\mu}=a_{\sigma\mu}+a_{\mu\sigma},$$

and S^F is defined by

$$i(\gamma^\mu\partial_\mu+m)S^F(x)=\delta^4(x).$$

It is seen that G^0 may be expressed as a sum of terms each of which can be represented by a diagram of the type shown in Fig. 1. In Fig. 1, the lines $x\leftarrow u$, $u\leftarrow v$, etc., correspond to $S^F(x-u)$, $S^F(u-v)$, etc. At each vertex there appears an arbitrary number of dashed lines corresponding to products of $a_{k\mu}$. These dashed lines may be called external gravitational lines or, more simply, "g lines."

The important fact is that at each vertex there appears a product of γ-matrices and *at most* one derivative operating either on an adjacent S^F or on one of $a_{k\mu}$'s at this vertex. The fact that we have only one derivative at each vertex of the open polygon is due to the fact that B_μ is linear in the first derivatives of $a_{k\mu}$ and the fact that Eq. (4.1) for G^0 is of the first differential order.

Accordingly in the momentum representation $G^0(p,p')$, defined by

$$G^0(p,p')=\frac{1}{(2\pi)^4}\int G^0(x,x')e^{-ipx+ip'x'}dxdx',$$

turns out to have the asymptotic form

$$G^0(p,p')\to O(p^{-1}) \quad\text{or}\quad O(p'^{-1})$$

for p (and p')$\gg m$. This favorable feature can be easily seen, for example, in the third term of (4.6), where the

[9] B. S. DeWitt and R. W. Brehme, Ann. Phys. **9**, 220 (1960).

[10] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, Connecticut, 1923).

[11] Consider a fixed point A and any other point B, for which the geodesic distance $\sigma(B,A)$ between B and A vanishes. The light cone at the point A is defined by the set of all the points $\{B\}$ for which $\sigma(B,A)=0$.

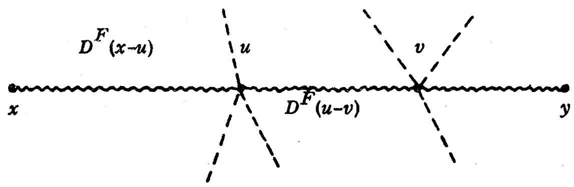

FIG. 2. A typical diagram of $\mathcal{G}^0$.

asymptotic contribution of the derivative is cancelled out by that of the last S^F.

In a similar way it is easy to see that $\mathcal{G}^0{}_{\mu\nu}$ is represented by a series each term of which corresponds to a diagram similar to the one shown in Fig. 2. Each vertex has an arbitrary number of g *lines* and at most one second-order derivative. The Fourier-transform $\mathcal{G}^0(k,k')$ has the asymptotic behavior $\mathcal{G}^0(k,k')\rightarrow O(k^{-2})$ or $O(k'^{-2})$ for k, $k'\gg m$.

It should be noted that since the gravitational field is nonquantized, G^0 and $\mathcal{G}^0$ have no closed diagrams of the self-energy type, consequently, both Green's functions have no singularities other than those existing on the light-cone.

Now let us insert these expressions for G^0 and $\mathcal{G}^0$ into the power-series expansions (in e) of G and $\mathcal{G}$. The fact that $\mathcal{G}^0$ and G^0 have the same asymptotic behavior as D^F and S^F for large momenta enables one to apply Dyson's criterion[12] for divergent diagrams to the present problem. The primitive divergent diagrams in the present case are:

(i) diagrams of the electron self-energy type, with or without any number of g lines (linearly divergent);

(ii) diagrams of the vacuum polarization type, with or without any number of g lines (quadratically divergent);

(iii) diagrams of the electromagnetic vertex type, with or without any number of g lines (logarithmically divergent).

5. SEGREGATION OF DIVERGENCES

Equation (4.3) shows that the Green's function G can be made free of divergences if the mass operator Σ is regularized by a suitable subtraction method.

In order to separate the divergences involved in Σ, let us put

$$h_{k\mu}(x)=\eta_{k\mu}+\kappa a_{k\mu}(x), \tag{5.1}$$

and express all quantities depending on $h_{k\mu}$ [for example $h^{k\mu}$, $(-g)^{\frac{1}{2}}$, etc.] as power series in the parameter[13] κ. The use of such expansions does not conflict with the fundamental postulate (i) and (ii) in Sec. 2, as long as we restrict ourselves to classical gravitational fields.

In the case of the quantized gravitational field, however, the dynamical operator $a_{k\mu}$ at any fixed point x can presumably take arbitrarily large values (i.e., the range of its spectrum), thus destroying the convergence of such expansions. For the classical field, on the other hand, the maximum value of $a_{k\mu}(x)$ can be restricted by choosing suitable initial values for $h_{k\mu}$ even in the case where the gravitational field is produced by the electron-photon system (for example, by the expectation value of the energy-momentum tensor of these fields) according to the Einstein equation.

Since the mass operator $\Sigma(x,y)$ or $\Sigma^*(x,y)=\Sigma(x,y)\times[-g(y)]^{\frac{1}{2}}$ is a functional of J^μ and $h_{k\mu}$ or $A_\mu(x)$ and $h_{k\mu}$, it is reasonable to expand it in a double functional series[14]

$$\begin{aligned}\Sigma^*(x,y)=\sum_{n,m=0}^{\infty}\frac{\kappa^n e^m}{n!m!}\int\prod_{j=1}^{n}a_{k_j\mu_j}(z_j),\prod_{j=1}^{m}A_{\nu_j}(y_j)\\ \times b_{n,m}\{(k_1,\cdots k_n),(\mu_1,\cdots\mu_n);\\ (\nu_1,\cdots\nu_m);x,y,z_1,\cdots,z_n,y_1,\cdots,y_m\}\\ \times\prod_{j=1}^{n}d^4z_j\prod_{j=1}^{m}d^4y_j.\end{aligned} \tag{5.2}$$

It will be seen later that for our present discussion Σ^* is more convenient than Σ.

In (5.2), the coefficient $b_{n,m}$ is a sum of complicated products of S^F, D^F and the charge e and the constant Dirac matrices γ, and all the gravitational potentials $a(x)$'s have been factored out. Namely,

$$\begin{aligned}b_{n,m}\{(k_1,\cdots,k_n),(\mu_1,\cdots,\mu_n);\\ (\nu_1,\cdots,\nu_m),x,y,z_1,\cdots,z_n,y_1,\cdots,y_m\}\\ =\sum_{l=0}^{\infty}e^l b_{n,m,l}\{(k_1,\cdots,k_n),(\mu_1,\cdots,\mu_n);\\ (\nu_1,\cdots,\nu_m),x,y,z_1,\cdots,y_1,\cdots\},\end{aligned}$$

where $b_{n,m,l}$ is a complicated product of D^F, S^F, and γ and their derivatives; accordingly it is a function of relative coordinates;

$$\begin{aligned}b_{n,m,l}\{(k_1,\cdots,k_n),\ (\mu_1\cdots);\ (\nu_1\cdots),\\ x-y,\ z_1-y,\ \cdots,\ z_n-y,\ y_1-y,\ \cdots,\ y_m-y\}.\end{aligned}$$

Following the conventional line of arguments, let us introduce the momentum representation:

$$\begin{aligned}\Sigma^*(p,p')=\frac{1}{(2\pi)^4}\int e^{-ipx}\Sigma^*(x,x')e^{ip'x'}dxdx'\\ =(2\pi)^2\sum_{n,m,l=0}^{\infty}\frac{\kappa^n e^{m+l}}{n!m!}\int\prod_{j=1}^{n}a_{k_j\mu_j}(p_j)\prod_{j=1}^{m}A_{\nu_j}(q_j)\\ \times b_{n,m,l}\{(k_1,\cdots),(\mu_1,\cdots);\\ (\nu_1,\cdots\nu_m),p_0,p_1,\cdots,p_n,q_1,\cdots,q_m\}\\ \times\delta^4[p'-\sum_0^n p_j-\sum_1^m q_j]\\ \times dp_1\cdots dp_n dq_1\cdots dq_m,\end{aligned} \tag{5.3}$$

[12] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[13] The small parameter κ is chosen to be the same one as the parameter appearing in the general coordinate transformation in Sec. 3 (iii).

[14] The coefficient $b_{n,m}$ is a function of $(x,y,z_1,\cdots,y_m)$ and is a contravariant tensor under Lorentz transformations with contravariant indices $(k_1,\cdots,k_n)$, $(\mu_1,\cdots,\mu_n)$, $(\nu_1,\cdots,\nu_m)$.

where the following notation has been introduced

$$b_{n,m,l}\{(k_1\cdots),(\mu_1\cdots);(\nu_1,\cdots\nu_m),p_0,p_1\cdots p_n,q_1\cdots q_m\}$$

$$=(2\pi)^{-(2n+2m+2)}\int d^4x,\,d^4z_1,\,\cdots d^4z_n,\,d^4y_1,\,\cdots d^4y_m$$

$$\times b_{n,m,l}\{(k_1\cdots),\,(\mu_1\cdots);\,(\nu_1,\cdots\nu_m),$$

$$x-y,\,z_1-y,\,\cdots,\,z_n-y,\,y_1-y,\,\cdots y_m-y\}$$

$$\times\exp[-ip_0(x-y)-i\sum_1^n p_j(z_j-y)-i\sum_1^m q_j(y_j-y)],$$

$$A_\nu(q)=(2\pi)^{-2}\int\langle A_\nu(y)\rangle e^{iqy}dy,$$

$$a_{k\mu}(p)=(2\pi)^{-2}\int a_{k\mu}(z)e^{ipz}dz,$$

and

$$p_0=p.$$

The coefficient

$$b_{n,m,l}\{(k,\cdots),(\mu,\cdots);(\nu_1,\cdots),p_0p_1\cdots q_m\}$$

is represented by the Feynman diagram shown in Fig. 3. In general, this diagram involves a number of divergent subdiagrams. Let us assume that these divergences due to the subdiagrams have already been removed by the subtraction method. Then, from Dyson's criterion it follows that $b_{n,m,l}$ is linearly divergent for $m=0$ (no external photon lines), or logarithmically divergent for $m=1$ (electromagnetic vertex), or finite for $m\geq 2$.

Now Σ^* should satisfy the following relations:

For the coordinate transformations,

$$\kappa\left(\xi^\mu(x)\frac{\partial}{\partial x^\mu}+\xi^\mu(y)\frac{\partial}{\partial y^\mu}\right)\Sigma^*(x,y)$$

$$+\int\left\{\frac{\delta\Sigma^*}{\delta\langle A_\rho(z)\rangle}\bar{\delta}\langle A_\rho(z)\rangle+\frac{\delta\Sigma^*}{\delta a_{k\mu}(z)}\bar{\delta}a_{k\mu}(z)\right\}dz$$

$$+\kappa\frac{\partial\xi^\mu(y)}{\partial y^\mu}\Sigma^*(x,y)=0;\quad(5.4)$$

for the Vierbein rotations,

$$\tfrac{1}{8}\{\epsilon_{kl}(x)[\gamma^k,\gamma^l]\Sigma^*-\epsilon_{kl}(y)\Sigma^*[\gamma^k,\gamma^l]\}$$

$$=\int\frac{\delta\Sigma^*}{\delta a_{k\mu}}\{\epsilon_{k\mu}(z)+\kappa\epsilon_{kl}(z)a_\mu{}^l(z)\}dz;\quad(5.5)$$

for the gauge transformations,

$$-ie[\lambda(x)-\lambda(y)]\Sigma^*(x,y)=\int\frac{\delta\Sigma^*(x,y)}{\delta\langle A_\mu(z)\rangle}\partial_\mu\lambda(z)dz.\quad(5.6)$$

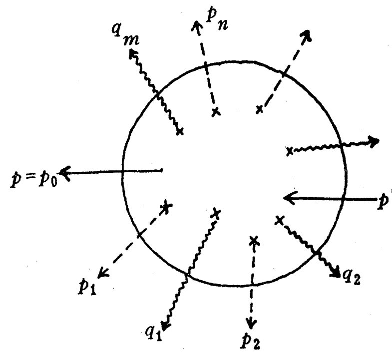

FIG. 3. Feynman Diagram representing $b_{n,m,l}(p_0\cdots p_n, q_1\cdots q_m)$. Wavy lines—external photon lines; dashed lines—g lines; and solid lines—electron lines.

In (5.4), the last term is due to the factor $[-g(y)]^{\frac{1}{2}}$ in Σ^*, and $\bar{\delta}A$ and $\bar{\delta}a$ are given by

$$\bar{\delta}A_\mu=-\kappa(\partial\xi^\nu/\partial z^\mu)A_\nu(z)-\kappa\xi^\nu(z)\partial_\nu A_\mu(z),$$

$$\bar{\delta}a_{k\mu}(z)=a'_{k\mu}(z)-a_{k\mu}(z)$$

$$=-\partial_\mu\xi_k(z)-\kappa(\partial_\mu\xi^\nu)a_{k\nu}-\kappa\xi^\nu\partial_\nu a_{k\mu}.$$

The derivation of (5.5) and (5.6) was given in Sec. 3.

In the momentum representation of (5.4), the coefficient of the arbitrary function $\xi_\nu(k)$ yields the relation

$$\sum_{j=0}^n p_{(j)}{}^\nu\frac{\partial}{\partial p_{(j)\lambda}}b_{n,m,l}\{(k_1,\cdots,k_n),(\mu_1,\cdots,\mu_n);$$

$$(\nu_1,\cdots,\nu_m),p_0,\cdots,p_n,q_1,\cdots,q_m\}$$

$$+\sum_{j=1}^m q_{(j)}{}^\nu\frac{\partial}{\partial q_{(j)\lambda}}b_{n,m,l}\{(k_1\cdots),(\mu_1\cdots);$$

$$(\nu_1\cdots),p_0,\cdots,q_m\}$$

$$+(2\pi)^2b_{n+1,m,l}\{(k_1,\cdots,k_n,\nu),(\mu_1,\cdots,\mu_n,\lambda);$$

$$(\nu_1,\cdots,\nu_m),p_0,\cdots,p_{n+1},q_1,\cdots,q_m\}_{p_{n+1}}=0,$$

$$+\sum_{j=1}^n\eta^{\nu\mu_j}b_{n,m,l}\{(k_1,\cdots,k_n),(\mu_1,\cdots,\lambda,\cdots,\mu_n);$$

$$(\nu_1,\cdots,\nu_m),p_0,\cdots,p_n,q_1,\cdots,q_m\}$$

$$+\sum_{j=1}^m\eta^{\nu\nu_j}b_{n,m,l}\{(k_1\cdots),(\mu_1\cdots);$$

$$(\nu_1,\cdots,\lambda,\cdots,\nu_m)p_0,\cdots p_n,q_1,\cdots q_m\}=0,\quad(5.7)$$

where

$q_{(j)}{}^\nu=\nu$th component of the jth momentum $q_{(j)}$, $(j=1,\,2,\,\cdots,\,m)$,

$$\xi_\nu(k)=(2\pi)^{-2}\int\xi_\nu(x)e^{ikx}dx,$$

and the limit $k\to 0$ has been taken in the end.

Similar relations can be derived from (5.5) and (5.6).

For example, (5.6) gives, for the case of $m=0$,

$$\frac{\partial}{\partial p_{(0)\lambda}} b_{n,0,l}\{(k_1,\cdots),(\mu_1,\cdots\mu_n),p_0,\cdots p_n\}$$
$$=(2\pi)^2 b_{n,1,l}\{(k_1,\cdots k_n),(\mu_1,\cdots\mu_n);$$
$$\lambda,p_0,\cdots p_n,q\}_{q=0}. \quad (5.8)$$

In order to segregate the singular part of each $b_{n,m,l}$ ($m=0$ and $m=1$), let us put

$$b_{n,m,l}\{(k_1\cdots),(\mu_1,\cdots);(\nu_1\cdots),p_0,\cdots p_n,q_1,\cdots q_m\} = {}^{\text{sing}}b_{n,m,l}+{}^{\text{finite}}b_{n,m,l}$$

where

$${}^{\text{sing}}b_{n,0,l}\{(k_1\cdots),(\mu_1,\cdots\mu_n),p_0,\cdots p_n\}$$
$$=b_{n,0,l}\{(k_1\cdots),(\mu_1\cdots),p_0,\cdots p_n\}_{\text{all } p=0}$$
$$+\sum_{j=0}^{n} p_{(j)\lambda}\left[\frac{\partial}{\partial p_{(j)\lambda}} b_{n,0,l} \times\{(k_1\cdots),(\mu_1\cdots),p_0,\cdots p_n\}\right]_{\text{all } p=0}$$
$$+\alpha_{n,0,l}\{(k_1\cdots),(\mu_1\cdots)\}$$
$$+\sum_{j=0}^{n} p_{(j)\lambda}\beta_{n,0,l}\{j,\lambda,(k_1\cdots),(\mu_1\cdots)\}, \quad (5.9)$$

and

$${}^{\text{sing}}b_{n,1,l}\{(k_1\cdots),(\mu_1\cdots);\nu,p_0,\cdots p_n,q\}$$
$$=[b_{n,1,l}\{(k_1,\cdots),(\mu_1,\cdots);\nu,p_0,\cdots p_n,q]_{p,q=0}$$
$$+\gamma_{n,1,l}\{(k_1,\cdots),(\mu,\cdots);\nu\}, \quad (5.10)$$

α, β, and γ = undetermined finite constants.

Because of the particular structure of $b_{n,m,l}$ (*viz.*, a product of S^F and D^F), we see that the first term of the right-hand side in (5.9) diverges linearly while the second term in (5.9) and the first term in (5.10) are logarithmically divergent.

For the sake of simplicity, let us put[15]

$${}^{\text{sing}}b_{n,0,l}\{(k_1\cdots),(\mu_1\cdots),p_0,\cdots p_n\}$$
$$=A_{n,l}\{(k_1,\cdots k_n),(\mu_1,\cdots\mu_n)\}$$
$$+\sum_{j=0}^{n} p_{(j)\lambda}B_{n,l}\{j,(k_1,\cdots k_n),(\mu_1,\cdots\mu_n);\lambda\}, \quad (5.11)$$

$${}^{\text{sing}}b_{n,1,l}\{(k_1\cdots),(\mu_1\cdots);\nu\}$$
$$=C_{n,l}\{(k_1,\cdots k_n),(\mu_1,\cdots\mu_n);\nu\},$$

where $A=\infty$, B and $C=\log\infty$.

These undetermined infinite constants A, B, and C are fixed by the requirement that the singular function ${}^{\text{sing}}\Sigma^*$ should satisfy the relations (5.4)–(5.6).[15]

[15] A, B, and C are independent of p's. The letters appearing in { }'s of these constants represent contravariant suffixes and take values ranging from 0 to 3.

For example, (5.8) gives the relation B and C, i.e.,

$$B_{n,l}\{0,(k_1\cdots),(\mu_1\cdots);\lambda\}$$
$$=(2\pi)^2 C_{n,l}\{(k_1\cdots),(\mu_1\cdots);\lambda\}. \quad (5.12)$$

In particular, for $n=0$ (no gravitational interaction) we have

$$B_{0,l}\{0;\lambda\}=(2\pi)^2 C_{0,l}\{0;\lambda\},$$

which is just Ward's identity to order e^l. The method of derivation of Ward's identity developed above seems to be the most elegant one compared with those so far published.

Inserting (5.11) into (5.7), we find that the terms independent of p's satisfy the relation

$$(2\pi)^2 A_{n+1,l}\{(k_1,\cdots k_n,\nu),(\mu_1,\cdots\mu_n,\lambda)\}$$
$$+\sum_{j=1}^{n}\eta^{\nu\mu_j}A_{n,l}\{(k_1,\cdots k_n),(\mu_1,\cdots\lambda,\cdots\mu_n\}=0, \quad \text{for } n\geq 1,$$

and

$$A_{1,l}\{(\nu),(1)\}=0.$$

The latter relation, which comes from (5.7) for $n=0$, combined with the former recurrence formula leads to the conclusion

$$A_{0,l}\neq 0,$$
$$A_{n,l}\{(k_1,\cdots k_n),(\mu_1,\cdots\mu_n)\}=0, \ (n\geq 1). \quad (5.13)$$

The singular part of Σ^* corresponding to $A_{n,l}$ turns out, in the coordinate representation, to be

$${}^{\text{sing}}\Sigma_1{}^*(x,y)=\frac{1}{(2\pi)^4}\int e^{ipx-ip'y}\delta^4(p-p') \times(2\pi)^2\left(\sum_{l=0}^{\infty}e^lA_{0,l}\right)dpdp'$$
$$=(2\pi)^2\delta^4(x-y)\left(\sum_{l=0}^{\infty}e^lA_{0,l}\right). \quad (5.14)$$

The simplicity of this result is due to the fact that we have made use of Σ^* instead of Σ.

The relation satisfied by $B_{n,l}$ is

$$\eta^{\nu\rho}B_{n,l}\{j;(k_1,\cdots k_n),(\mu_1,\cdots\mu_n);\lambda\}$$
$$+\sum_{i=1}^{n}\eta^{\nu\mu_i}B_{n,l}\{j;(k_1,\cdots k_n),(\mu_1,\cdots\lambda,\cdots\mu_n);\rho\}$$
$$+(2\pi)^2B_{n+1,l}\{j;(k_1,\cdots k_n,\nu),(\mu_1,\cdots\mu_n,\rho)\}=0$$
$$(j=0,1,\cdots n). \quad (5.15)$$

This relation gives $B_{n+1,l}\{j;(\cdots),(\cdots),\rho\}$ ($j=0$, $1,\cdots n$) in terms of $B_{n,l}$ but does not give any information about $B_{n+1,l}\{j;(\cdots),(\cdots);\rho\}$, $j=n+1$. This coefficient is given by a relation which follows from (5.5)

together with the expression (5.11), i.e.,

$$\tfrac{1}{4}[\gamma^i,\gamma^k]B_{n,l}\{0;\,(k_1,\cdots k_n),(\mu_1,\cdots\mu_n);\lambda\}$$
$$=[(2\pi)^2B_{n+1,l}\{n+1,\,(k_1,\cdots k_n,i),\,(\mu_1,\cdots\mu_n,k);\lambda\}$$
$$+\sum_{j=1}^{n}B_{n,l}\{j;\,(k_1,\cdots i,\cdots k_n),(\mu_1,\cdots\mu_n);\lambda\}\eta^{k,k_j}]$$
$$-[i\text{ and }k\text{ interchanged}].\quad(5.16)$$

Relations (5.15) and (5.16) show that all the $B_{n,l}$ are proportional to $B_{0,l}$ and are completely determined up to this factor.

Let us now consider the second singular part of Σ^*, namely,

$${}^{\mathrm{sing}}\Sigma_2{}^*(x,y)=(2\pi)^{-4}\int e^{i(px-p'y)}\,{}^{\mathrm{sing}}\Sigma_2{}^*(p,p')d^4pd^4p',$$

with

$${}^{\mathrm{sing}}\Sigma_2{}^*(p,p')$$
$$=(2\pi)^2\sum_{n,l=0}^{\infty}\frac{e^l\kappa^n}{n!}\int d^4p_1,\cdots d^4p_n\,\delta^4[p'-\sum_{j=0}^{n}p_j]$$
$$\times\sum_{j=0}^{m}p_{(j),\lambda}B_{n,l}\{j;\,(k_1,\cdots k_n),(\mu_1,\cdots\mu_n);\lambda\}$$
$$\times\prod_{i=1}^{n}a_{k_i,\mu_i}(p_i),$$

which tends to

$$\lim_{\kappa\to0}{}^{\mathrm{sing}}\Sigma_2{}^*(x,y)$$
$$=\frac{1}{(2\pi)^2}\sum_{l=0}^{\infty}e^lB_{0,l}\{0;\lambda\}\int p_\lambda\delta^4(p'-p)e^{ip(x-y)}d^4pd^4p'$$
$$=(2\pi)^2(\sum_{l=0}^{\infty}e^lB_{0,l}\{0;\lambda\})\frac{1}{i}\frac{\partial}{\partial x^\lambda}\delta^4(x-y).\quad(5.17)$$

In the case of no gravitational interaction, the mass operator of an electron in the momentum representation has, in the lth order, the form

$${}^0\Sigma_l(p,p')=\{X_l(p^2)+\gamma^\lambda p_\lambda Y_l(p^2)\}\delta(p-p'),$$

where X_l and Y_l are Lorentz-invariant functions of p^2 and diverge linearly and logarithmically, respectively. The singular part of ${}^0\Sigma$ is defined by

$${}^{\mathrm{sing}\,0}\Sigma_l(p,p')=\delta(p-p')\{M_l+(\gamma^\lambda p_\lambda-im)N_l\},$$

where M_l and N_l are independent of p and are given by

$$M_l=X_l(-m^2)+imY_l(-m^2),$$
$$N_l=Y_l(-m^2)+2im\left(\frac{d}{ds}X_l(s)\right)_{s=-m^2}-2m^2\left(\frac{d}{ds}Y_l(s)\right)_{s=-m^2}.$$

Transforming the above expression to the coordinate representation, we have

$${}^{\mathrm{sing}\,0}\Sigma(x,y)=(\sum_{l=0}^{\infty}e^lM_l)\delta^4(x-y)$$
$$-i(\sum_{l=0}^{\infty}e^lN_l)\left(\gamma^\lambda\frac{\partial}{\partial x^\lambda}+m\right)\delta^4(x-y).\quad(5.18)$$

The comparison of (5.18) with (5.14) and (5.17) leads to

$$A_{0,l}=(2\pi)^{-2}(M_l-imN_l),\qquad B_{0,l}\{0;\lambda\}=(2\pi)^{-2}N_l\gamma^\lambda,\quad(5.19)$$

because of the relation

$$\lim_{\kappa\to0}\{{}^{\mathrm{sing}}\Sigma_1{}^*+{}^{\mathrm{sing}}\Sigma_2{}^*\}={}^{\mathrm{sing}\,0}\Sigma^*(x,y).\quad(5.20)$$

As was already mentioned, the "initial values" of $B_{n,l}$ given by (5.19) are sufficient for the determination of ${}^{\mathrm{sing}}\Sigma_2{}^*$. However, the following facts give us the explicit expression of ${}^{\mathrm{sing}}\Sigma^*$ even though we do not solve actually the recurrence formulas (5.15) and (5.16):

(a) relation (5.20);
(b) ${}^{\mathrm{sing}}\Sigma^*$ is linear with respect to the first derivative (or linear with respect to $p_{(j)}$ $(j=0,1,\cdots n)$;
(c) ${}^{\mathrm{sing}}\Sigma^*$ should satisfy the relations (5.4) and (5.5).

The result obtained in this way is a simple generalization of (5.18) in a curved manifold, and is given by

$${}^{\mathrm{sing}}\Sigma_1{}^*(x,y)+{}^{\mathrm{sing}}\Sigma_2{}^*(x,y)$$
$$=(\sum_{l=0}^{\infty}e^lM_l)\delta^4(x-y)-i(\sum_{l=0}^{\infty}e^lN_l)$$
$$\times\left\{\gamma^\lambda(x)\left(\frac{\partial}{\partial x^\lambda}+B_\lambda(x)\right)+m\right\}\delta^4(x-y).$$

The third singular part of ${}^{\mathrm{sing}}\Sigma^*$, namely that which comes from the $C_{n,l}$'s, can be also transformed into a compact form if we take into consideration Ward's identity (5.12) or the requirement of gauge invariance (5.6). Thus we arrive at the final expression for the singular part of Σ^*:

$${}^{\mathrm{sing}}\Sigma^*(x,y)$$
$$=(\sum_{l=0}^{\infty}e^lM_l)\delta(x-y)-i(\sum_{l=0}^{\infty}e^lN_l)\{\gamma^\lambda(x)[\partial_\lambda+B_\lambda(x)$$
$$+ie\langle A_\lambda(x)\rangle]+m\}\delta(x-y),\quad(5.20\text{a})$$

where the divergent constants M_l and N_l *are identical with those evaluated by the perturbation method in the case of no gravitational interaction.*

The argument we have made so far is also applicable to the segregation of the singular part of the electro-

magnetic vertex function:

$$\Gamma^\nu(x,y;z)=\frac{\gamma^\nu(x)\delta(x-y)\delta(x-z)}{[-g(y)]^{\frac{1}{2}}[-g(z)]^{\frac{1}{2}}}+e^2\Lambda^\nu(x,y;z), \quad (2.11)$$

where

$$\Lambda^\nu(x,y;z)=-\frac{i}{e}\frac{\delta\Sigma(x,y)}{[-g(z)]^{\frac{1}{2}}\delta\langle A_\nu(z)\rangle}$$
$$=-\frac{i}{e}\frac{\delta\Sigma^*(x,y)}{[-g(z)]^{\frac{1}{2}}[-g(y)]^{\frac{1}{2}}\delta\langle A_\nu(z)\rangle}. \quad (5.21)$$

It is easily seen from (5.20) and (5.21) that

$$\begin{aligned}{}^{\text{sing}}\Gamma^\nu(x,y;z) &= e^2\,{}^{\text{sing}}\Lambda^\nu(x,y;z)\\ &= -ie\delta\,{}^{\text{sing}}\Sigma^*(x,y)/[-g(y)]^{\frac{1}{2}}[-g(z)]^{\frac{1}{2}}\delta\langle A_\nu(z)\rangle\\ &= -ie^2\Big(\sum_{l=0}^{\infty}e^lN_l\Big)\gamma^\nu(x)\frac{\delta(x-y)\delta(x-z)}{[-g(y)]^{\frac{1}{2}}[-g(z)]^{\frac{1}{2}}}.\end{aligned} \quad (5.22)$$

Accordingly we have

$$\Gamma^\nu(x,y;z)=\Big(1-ie^2\sum_{l=0}^{\infty}e^lN_l\Big)\frac{\gamma^\nu(x)\delta(x-y)\delta(x-z)}{[-g(y)]^{\frac{1}{2}}[-g(z)]^{\frac{1}{2}}}+e^2\Lambda^\nu(x,y;z)_{\text{finite}}. \quad (5.23)$$

The remaining singular function which must be considered is the polarization operator ${}^{\text{sing}}P^{\mu\nu}(x,y)$. For the segregation of ${}^{\text{sing}}P^{\mu\nu}$ in a generally covariant fashion, use may be made of relation (3.12) or its equivalent:

$$\begin{aligned}&\kappa\frac{\partial\xi^\mu(x)}{\partial x^\rho}P^{\rho\nu}(x,y)+\kappa\frac{\partial\xi^\nu(y)}{\partial y^\rho}P^{\mu\rho}(x,y)\\ &=\kappa\Big[\xi^\rho(x)\frac{\partial}{\partial x^\rho}+\xi^\rho(y)\frac{\partial}{\partial y^\rho}\Big]P^{\mu\nu}(x,y)\\ &\quad+\int\Big\{\frac{\delta P^{\mu\nu}}{\delta\phi_{\rho\sigma}(z)}\bar{\delta}\phi_{\rho\sigma}(z)+\frac{\delta P^{\mu\nu}}{\delta\langle A_\rho(z)\rangle}\bar{\delta}\langle A_\rho(z)\rangle\Big\}dz,\end{aligned} \quad (5.24)$$

which follows from the fact that P is a functional of $g_{\mu\nu}$ and hence depends only on the symmetric combination

$$\phi_{\rho\sigma}=a_{\rho\sigma}+a_{\sigma\rho}.$$

The following relation is also needed:

$$(\partial/\partial x^\mu)\{[-g(x)]^{\frac{1}{2}}P^{\mu\nu}(x,y)\}=0, \quad (5.25)$$

which can be derived as follows. The Lorentz condition

$$\nabla_\rho\langle A^\rho(x)\rangle=0,$$

when substituted into (2.6), leads to

$$e(\partial/\partial x^\mu)\{[-g(x)]^{\frac{1}{2}}\,\mathrm{Tr}[\gamma^\mu(x)G(x,x)]\}=\partial_\mu J^\mu(x),$$

which vanishes owing to the definition of J^μ introduced in section 1. Relation (5.25) then follows by a functional differentiation with respect to $J^\nu(y)$, taking note of the definitions of Γ and $P^{\mu\nu}$.

By following the line of thought developed in the case of ${}^{\text{sing}}\Sigma^*$, and making use of the relations (5.24) and (5.25), the singular part of $P^{\mu\nu}$ turns out to be

$$\begin{aligned}&[-g(x)]^{\frac{1}{2}}\,{}^{\text{sing}}P^{\mu\nu}(x,y)[-g(y)]^{\frac{1}{2}}\\ &=\Big(\sum_{l=0}^{\infty}e^lL_l\Big)\partial_\rho\Big\{[-g(x)]^{\frac{1}{2}}\\ &\qquad\times[g^{\mu\lambda}(x)g^{\nu\rho}(x)-g^{\mu\nu}g^{\rho\lambda}]\frac{\partial}{\partial x^\lambda}\Big\}\delta(x-y).\end{aligned} \quad (5.26)$$

The constants L_l's which diverge logarithmically are, like the other divergent constants, the same as those obtained in the absence of a classical gravitational field.

6. REMOVAL OF DIVERGENCES

The method we are now going to develop is essentially a repetition of the argument given in a previous paper by the author.[16]

Let us regard all the quantities so far considered as unrenormalized ones and make the transformation[17]

$$\left.\begin{aligned}\psi&\to\psi'=Z^{-\frac{1}{2}}\psi,\\ \eta&\to\eta'=Z^{\frac{1}{2}}\eta,\\ m&\to m'=m-\delta m,\\ A_\mu&\to A'_\mu=(ZZ_3)^{-\frac{1}{2}}A_\mu,\\ e&\to e'=(ZZ_3)^{\frac{1}{2}}e,\\ J^\mu&\to J'^\mu=Z_3^{\frac{1}{2}}Z^{-\frac{1}{2}}J^\mu,\end{aligned}\right\} \quad (6.1)$$

Z and Z_3 are renormalization constants.

Applying this transformation to the Lagrangian (1.1), we obtain

$$\begin{aligned}L'=&-Z(-g)^{\frac{1}{2}}\bar{\psi}'\{\gamma^\mu(x)(\partial_\mu+B_\mu+ie'\langle A_\mu'\rangle)+m'+\delta m\}\psi'\\ &-\tfrac{1}{4}Z_3Z(-g)^{\frac{1}{2}}f'_{\mu\rho}\cdot f'_{\nu\sigma}g^{\mu\nu}g^{\rho\sigma}-(Z/2)(-g)^{\frac{1}{2}}\\ &\times(\nabla_\mu A'^\mu)^2-iZJ'^\mu A_\mu'-i\bar{\eta}'\psi'-i\bar{\psi}'\eta'.\end{aligned} \quad (6.2)$$

The new Lagrangian L' is not numerically identical with the original L given by (1.1) because of the term $-Z(-g)^{\frac{1}{2}}(\nabla_\mu A'^\mu)^2/2$. It will be seen that this modification of the Lagrangian is essential for the removal of the singularity of the vacuum polarization.

The transformation (6.1) gives the relationship between unrenormalized and renormalized Green's functions

$$\left.\begin{aligned}G'(x,y)&=\delta\langle\psi'(x)\rangle/\delta\eta'(y)=Z^{-1}G(x,y),\\ \mathcal{G}'_{\mu\nu}(x,y)&=\delta\langle A_\mu'(x)\rangle/\delta J'^\nu(y)=Z_3^{-1}\mathcal{G}_{\mu\nu}(x,y),\end{aligned}\right\} \quad (6.3)$$

[16] R. Utiyama, S. Sunakawa and T. Imamura, Prog. Theor. Phys. (Kyoto) 8, 77 (1952).

[17] The primed letters denote the renormalized quantities.

and consequently gives

$$G'^{-1}(x,y)=ZG^{-1}(x,y),$$

$$\Gamma'^{\mu}(x,y;z)=-\frac{1}{e'[-g(z)]^{\frac{1}{2}}}\frac{\delta G'^{-1}(x,y)}{\delta\langle A_{\mu}'(z)\rangle}=Z\Gamma^{\mu}(x,y;z), \quad (6.4)$$

$$e'^2\Sigma'(x,y)=ie'^2\gamma^{\mu}(x)\int G'(x,u)[-g(u)]^{\frac{1}{2}}\Gamma'^{\nu}(u,y;z)\times[-g(z)]^{\frac{1}{2}}\mathcal{G}'_{\nu\mu}(z,x)dudz$$
$$=Ze^2\Sigma(x,y), \quad (6.5)$$

$$e'^2P'^{\mu\nu}(x,y)=-ie'^2\operatorname{Tr}\int\{\gamma^{\mu}(x)G'(x,u)[-g(u)]^{\frac{1}{2}}\times\Gamma'^{\nu}(u,v;y)[-g(v)]^{\frac{1}{2}}G'(v,x)\}dudv$$
$$=Z_3e^2P^{\mu\nu}(x,y). \quad (6.6)$$

The new Lagrangian (6.2) together with the relations (6.3)–(6.5) gives the equation for G'

$$i[-g(x)]^{\frac{1}{2}}\{\gamma^{\mu}(\partial_{\mu}+B_{\mu}+ie'\langle A_{\mu}'\rangle)+m\}G'(x,y)$$
$$+i(-g)^{\frac{1}{2}}\int[e'^2\Sigma'(x,u)[-g(u)]^{\frac{1}{2}}+Z\delta m\delta(x-u)$$
$$+(Z-1)\{\gamma^{\mu}(\partial_{\mu}+B_{\mu}+ie'\langle A_{\mu}'\rangle)+m\}\delta(x-u)]$$
$$\times G'(u,y)du=\delta(x-y), \quad (6.7)$$

and the new expression for Γ'

$$\Gamma'^{\mu}(x,y;z)=\frac{\gamma^{\mu}(x)\delta(x-y)\delta(x-z)}{[-g(y)]^{\frac{1}{2}}[-g(z)]^{\frac{1}{2}}}-ie'\frac{\delta\Sigma'(x,y)}{[-g(z)]^{\frac{1}{2}}\delta\langle A_{\mu}'(z)\rangle}$$
$$+(Z-1)\frac{\gamma^{\mu}(x)\delta(x-y)\delta(x-z)}{[-g(y)]^{\frac{1}{2}}[-g(z)]^{\frac{1}{2}}}. \quad (6.8)$$

In deriving (6.7) the following relation has been used:

$$Z\langle A_{\mu}'(x)\psi'(x)\bar{\psi}'(y)\rangle$$
$$=Z\langle A_{\mu}'(x)\rangle G'(x,y)+\frac{\delta}{\delta J'^{\mu}}G'(x,y). \quad (6.9)$$

The Z factor in the last term of (6.9) disappears because of the term

$$-iZJ'^{\mu}A_{\mu}'$$

in (6.2).

In an analogous way we can obtain the equations for $\langle A_{\mu}'\rangle$ and $\mathcal{G}'_{\mu\nu}$:

$$-i[-g(x)]^{\frac{1}{2}}\{\Box_c g^{\mu\rho}+R^{\mu\rho}\}\langle A_{\rho}'(x)\rangle+i(Z_3-1)\partial_{\lambda}$$
$$\times\{[-g(x)]^{\frac{1}{2}}(g^{\mu\tau}g^{\lambda\rho}-g^{\mu\rho}g^{\lambda\tau})\partial_{\tau}\}\langle A'_{\rho}\rangle$$
$$+e'(-g)^{\frac{1}{2}}\operatorname{Tr}[\gamma^{\mu}(x)G'(x,x)]=J'^{\mu}(x), \quad (6.10)$$

and

$$-i[-g(x)]^{\frac{1}{2}}\{\Box_c g^{\mu\rho}+R^{\mu\rho}\}\mathcal{G}'_{\rho\nu}(x,y)$$
$$+i\int\{e'^2[-g(x)]^{\frac{1}{2}}P'^{\mu\rho}(x,u)[-g(u)]^{\frac{1}{2}}$$
$$+(Z_3-1)\partial_{\lambda}[(-g)^{\frac{1}{2}}(g^{\mu\tau}g^{\rho\lambda}-g^{\mu\rho}g^{\lambda\tau})\partial_{\tau}]\delta(x-u)\}$$
$$\times\mathcal{G}'_{\rho\nu}(u,y)du=\delta_{\nu}{}^{\mu}\delta(x-y). \quad (6.11)$$

In order to show that G', $\mathcal{G}'$, and $\langle A'\rangle$ are free of divergences, let us expand all the quantities in power series in e:

$$G'=\sum_{n=0}^{\infty}e'^nG_n',\quad \mathcal{G}'=\sum_{n=0}^{\infty}e'^n\mathcal{G}_n',\quad Z=1+\sum_{n=2}^{\infty}e'^nZ_n,$$

$$\delta m=\sum_{n=2}^{\infty}e'^n(\delta m)_n,\quad Z_3=1+\sum_{n=2}^{\infty}e'^n(Z_3)_n,$$

and make use of mathematical induction.

[A]: G_k', $\mathcal{G}_k'$, Γ_k', and A_k' are free of singularity for $k=0,1,\cdots n$, and consider the terms of (6.7) of order e^{n+1} satisfying

$$i[-g(x)]^{\frac{1}{2}}\{\gamma^{\mu}(x)(\partial_{\mu}+B_{\mu})+m\}G'_{n+1}(x,y)$$
$$-[-g(x)]^{\frac{1}{2}}\gamma^{\mu}(x)\sum_{k=0}^{n}\langle A'_{\mu}\rangle_k G'_{n-k}(x,y)$$
$$+i[-g(x)]^{\frac{1}{2}}\Bigg[\sum_{k=0}^{n-1}\int\Sigma'^*(x,u)_k G'_{n-k-1}(u,y)du$$
$$+\sum_{k=0}^{n-1}(Z\delta m)_{k+2}G'_{n-k-1}(x,y)$$
$$+i\gamma^{\mu}(x)\sum_{i+j+l=n-2}(Z-1)_{i+2}\langle A'_{\mu}(x)\rangle_j G_l'(x,y)$$
$$+\{\gamma^{\mu}(x)(\partial_{\mu}+B_{\mu})+m\}$$
$$\times\sum_{k=0}^{n-1}(Z-1)_{k+2}G'_{n-k-1}(x,y)\Bigg]=0. \quad (6.12)$$

Since $\Sigma_k'^*$ has the structure

$$\{\Sigma'^*(x,y)\}_k$$
$$=i\sum_{i+j+l=k}\gamma^{\mu}(x)\int G_i'(x,u)[-g(u)]^{\frac{1}{2}}\Gamma'^{\nu}(u,y;z)_j$$
$$\times[-g(z)]^{\frac{1}{2}}[\mathcal{G}'_{\nu\mu}(z,x)]_l[-g(y)]^{\frac{1}{2}}dudz,$$

it has no divergences arising from divergent sub-diagrams (or, more precisely, from the factors G_i', Γ_j', and $\mathcal{G}_l'$ involved in Σ') by virtue of our assumption [A] (i.e., i, j, $l\leq n-1$). Accordingly $\Sigma_k'^*$ must have the

following form, in accordance with (5.20)

$$\{\Sigma'^*(x,u)\}_k$$
$$=M_k\delta(x-u)-iN_k\{\gamma^\lambda(x)(\partial_\lambda+B_\lambda)+m\}\delta(x-u)$$
$$+\sum_{i+j=k-1} N_i\gamma^\lambda\langle A_\lambda'\rangle_j\delta(x-u)+({}^{\text{finite}}\Sigma^*)_k.$$

Inserting this expression into (6.12), the third term $i(-g)^{\frac{1}{2}}[\cdots]$ becomes

$$i[-g(x)]^{\frac{1}{2}}\Bigg[\sum_{k=0}^{n-1}\int[{}^{\text{finite}}\Sigma'^*(x,u)]_k G'_{n-k-1}(u,y)du$$
$$+\sum_{k=0}^{n-1}[M_k+(Z\delta m)_{k+2}]G'_{n-k-1}(x,u)$$
$$+\sum_{k=0}^{n-1}[(Z-1)_{k+2}-iN_k][\gamma^\lambda(\partial_\lambda+B_\lambda)+m]G'_{n-k-1}$$
$$+i\sum_{a+b+c=n-2}[(Z-1)_{a+2}-iN_a]\gamma^\lambda\langle A_\lambda'\rangle_b G_c'(x,y)\Bigg].$$

Therefore, if we choose the indefinite constants Z_1, and δm_1 as follows:

$$Z_{k+2}=iN_k,$$
$$(Z\delta m)_{k+2}=\sum_{l=0}^{k} Z_l(\delta m)_{k+2-l} \tag{6.13}$$
$$=-M_k,\quad (k=0,1,\cdots n-1),$$

then the third term $i(-g)^{\frac{1}{2}}[\cdots]$ turns out to be finite:

$$i(-g)^{\frac{1}{2}}\sum_{k=0}^{n-1}\int\{{}^{\text{finite}}\Sigma'^*(x,u)\}_k G_{n-1-k}(u,y)du.$$

Consequently (6.12) becomes

$$i(-g)^{\frac{1}{2}}\{\gamma^\mu(\partial_\mu+B_\mu)+m\}G'_{n+1}(x,y)$$
$$=\sum_{k=0}^{n}(-g)^{\frac{1}{2}}\gamma^\mu\langle A_\mu'\rangle_k G'_{n-k}$$
$$-i(-g)^{\frac{1}{2}}\sum_{k=0}^{n-1}\{{}^{\text{finite}}\Sigma'^*(x,u)\}_k G'(u,y)_{n-1-k}du, \tag{6.14}$$

which shows that G'_{n+1} is actually free of divergences.

In a similar way the terms of order e^{n+1} in the expression (6.8) are written

$$(\Gamma'^\mu)_{n+1}=\{(Z-1)_{n+1}-iN_{n-1}\}\frac{\gamma^\mu\delta(x-y)\delta(x-z)}{[-g(y)]^{\frac{1}{2}}[-g(z)]^{\frac{1}{2}}}$$
$$+\{{}^{\text{finite}}\Lambda'^\mu(x,y;z)\}_{n-1},$$

which becomes

$$(\Gamma'^\mu)_{n+1}=({}^{\text{finite}}\Lambda'^\mu)_{n-1}=\text{finite},$$

owing to the choice of (6.13).

It is not hard to prove that $(\mathcal{G}_{\mu\nu}')_{n+1}$ is likewise free of divergences if the renormalization constant Z_3 is defined as

$$(Z_3)_{k+2}=-iL_k\quad (k=0,1,\cdots,n-1),$$

where L_k is given in (5.26). This completes the proof.

The success of the renormalization in the present case seems not surprising if one takes account of the following situations.

As was emphasized in Sec. 4, the singularities of the Green's functions G^0 and $\mathcal{G}^0$ in a curved manifold are exactly the same as those of S^F and D^F. Consequently the singular parts of the kernels Σ and P of the integral equations (4.2) and (4.4) have the same nature as those in a flat space-time. In other words, these singular parts are proportional to the δ function or its derivatives (of, at most second order). This local character of the singular parts enables one to apply the principle of equivalence which, by a suitable general coordinate transformation, can reduce the present problem to the conventional one with no gravitational effect in the vicinity of the world point concerned. The result that the mass renormalization δm in the curved manifold is exactly the same as that in a flat space-time shows that the identity of the inertial mass with the gravitational mass is also true for the renormalization part of the mass.

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