several an hour is to be expected. There is a serious problem of background but our rough estimates indicate that it is a solvable problem.

It is not absolutely certain, though quite probable, that the pion-pion resonance recently observed[2] will show up in electron pair production. If it does, the evidence would be overwhelming that this is indeed a $J=1$, $T=1$ resonance and plays an important role in the electromagnetic structure of nucleons and pions. Even if the single pion exchange approximation turns out to be poor, we should still observe this resonance for center of mass energies of the electron positron system close to the pion-pion resonant energy. The recent observation of a three-pion resonance[12] at almost the same energy as for the two-pion resonance could alter our results. It is probable, however, that the effect of the three-pion resonance on the electron pair mass distribution would only arise for larger momentum transfer. It would also be worth while to examine this reaction for values of the electron-positron center-of-mass energy below the two-pion resonant energy to look for further structure effects.

[12] B. C. Maglić, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. Letters **7**, 178 (1961).

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Mass Quantization and Lepton Theory*

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This article discusses some features of mass quantization obtained by the introduction of a continuous inner degree of freedom into a free field. The usual particle interpretation, with discrete mass values, is applicable, as shown in the case of a second-quantized scalar field. A simple class of fermion field equations with unexpected lepton-like properties is also presented and studied in some detail.

1. INTRODUCTION

THE setting up of an eigenvalue problem has for a long time been a tempting method of introducing a mass spectrum into a quantized field theory.[1] Such an approach could be based on the hope that the current Lagrangian models are, by virtue of their interactions, truly consistent only for certain mass ratios; the decoupled fields, of course, are consistent with any mass spectrum.

Alternatively (and this is the approach underlying the present remarks), those models might be modified in such a way that the eigenvalue problem persists even after the coupling constants have been set equal to zero; in this noninteracting case it should be entirely equivalent to the usual attribution of chosen masses to the various decoupled fields. The only interest of formulating this simple assignment of mass values in a less conventional language lies in the possibility that the two languages may become nonequivalent as soon as the "usual" interaction is turned on.

The present discussion is restricted to free fields and is based on the intuitive idea that, if a field ϕ which depends on a continuous parameter λ (in addition to its space-time coordinate x) satisfies the wave equation

$$(\Box+\Lambda)\phi(x,\lambda)=0, \tag{1.1}$$

where Λ is a functional operator acting on λ and having a discrete spectrum, then one should be able to decompose ϕ into free fields whose squared masses are the eigenvalues of Λ. The exact meaning of such a statement will be discussed. In order to illustrate the heuristic value of the approach described here, the example will be presented of a fermion field exhibiting some properties similar to those sought in the theory of leptons.

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[1] See, in particular, E. Minardi, Nuovo cimento **3**, 968 (1956); **7**, 715, 898 (1958), and the bibliography therein contained.

2. NEUTRAL SCALAR FIELD

The simplest example of mass quantization is provided by the neutral scalar field $\phi(x,\lambda)$ for which a Lagrangian density

$$\mathcal{L}=-\frac{1}{2}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu}+\tfrac{1}{2}m^2\left[\left(\frac{\partial\phi}{\partial\lambda}\right)^2-V(\lambda)\phi^2\right], \tag{2.1}$$

and a Lagrangian

$$L=\int d^3x\int d\lambda\ \mathcal{L}$$

(with V a real dimensionless function of the dimensionless variable λ, and m some fixed reference mass) may be postulated. At this stage there is no point in considering more complicated variants of the mass term, although there seems to be no obvious objection to more general real symmetric forms. The particular one chosen in (2.1) is only a convenient illustration. It is local in λ and of second degree in $\partial\phi/\partial\lambda$ in analogy with the space-time structure of $\mathfrak{L}$.

From (2.1) there follows the field equation,

$$\Box\phi+m^2[-\partial^2/\partial\lambda^2+V(\lambda)]\phi=0, \qquad (2.2)$$

and the commutation rules,

$$[\phi(x,\lambda),\phi(y,\mu)]=0, \qquad (x^0=y^0), \qquad (2.3)$$

$$[\phi(x,\lambda),\dot{\phi}(y,\mu)]=i\delta(\mathbf{x}-\mathbf{y})\delta(\lambda-\mu) \qquad (x^0=y^0). \qquad (2.4)$$

The operator,

$$\Lambda=m^2[-\partial^2/\partial\lambda^2+V(\lambda)], \qquad (2.5)$$

is assumed, for suitable boundary conditions, to have a wholly discrete spectrum without negative eigenvalues. Equations (2.2)–(2.4), constituting a fully determined Cauchy problem, show that the two-field commutators are c numbers for all values of x^0 and y^0. Knowledge of the commutator (2.3) for all x^0 and y^0 thus summarizes the whole contents of the theory.

Let $f_k(\lambda)$ be a solution of the eigenvalue problem,

$$\Lambda f_k=m_k^2f_k, \qquad (2.6)$$

where we shall assume for simplicity that the boundary conditions have the form

$$f_k(+\infty)=f_k(-\infty)=0. \qquad (2.7)$$

Normalizations are then chosen such that orthonormality and completeness read

$$\int f_kf_ld\lambda=\delta_{kl}, \qquad (2.8)$$

$$\sum_k f_k(\lambda)f_k(\mu)=\delta(\lambda-\mu). \qquad (2.9)$$

It can now be verified by inspection that the solution of equations (2.2)–(2.4) reads

$$[\phi(x,\lambda),\phi(y,\mu)]=-i\sum_k \Delta(x-y;m_k)f_k(\lambda)f_k(\mu), \qquad (2.10)$$

where $\Delta(x;m)$ is the homogeneous Δ function corresponding to a mass m.

The theory is fully reduced to that of ordinary particles of masses m_k if one notes that the fields

$$\phi_k(x)=\int f_k(\lambda)\phi(x,\lambda)d\lambda \qquad (2.11)$$

satisfy the relations

$$(\Box+m_k^2)\phi_k(x)=0, \qquad (2.12)$$

$$[\phi_k(x),\phi_l(y)]=-i\delta_{kl}\Delta(x-y;m_k), \qquad (2.13)$$

and that, in fact, the Lagrangian density contributes to the Lagrangian in the following manner:

$$\int \mathfrak{L}d\lambda=\sum_k \frac{1}{2}\left(\frac{\partial\phi_k}{\partial x^\mu}\frac{\partial\phi_k}{\partial x_\mu}-m_k^2\phi_k^2\right). \qquad (2.14)$$

3. FERMION FIELD

Some interesting symmetries, suggestive of leptons, are found to arise in a natural way if one quantizes the mass of a fermion, while the same symmetries would have to be introduced *ad hoc* in a theory having a separate field for each mass. In the following sections second-quantized features will be ignored unless otherwise mentioned, as they are in no essential way different from those discussed in Sec. 2. The basic symmetries, except charge conjugation, are equally well exhibited by the c-number theory. The remainder of this article will deal with fermions.

The Dirac equation is now to be considered, with the mass term replaced by a linear, first-order differential operator in λ. Let the term in $\partial/\partial\lambda$ have a constant coefficient, and let the boundary conditions consist in having the eigenfunctions vanish at $\lambda=\pm\infty$. These choices are motivated by simplicity and by the analogy of λ with the spatial variables. Since the mass operator must be Hermitian, one arrives at the form $i\partial/\partial\lambda+V(\lambda)$ for it. This, however, can only give rise to a continuous spectrum, as shown by explicit solution. (This conclusion is not essentially modified by the fact that only the square of the mass operator would occur in the Klein-Gordon equation which results from this Dirac equation.)

In order to obtain the form postulated above, while securing a discrete spectrum, one is led to double the number of spinor components. This allows the introduction of the Pauli matrices τ_1, τ_2, τ_3 (formally the isospin matrices: Hermitian and anticommuting with each other, but commuting with the Dirac γ's). The eight-component wave function ψ now satisfies

$$(i\gamma\cdot\partial-M)\psi=0, \qquad (3.1)$$

where

$$M=m[i\tau_2\partial/\partial\lambda+\tau_1V(\lambda)], \qquad (3.2)$$

m again being some fixed reference mass, and V being real. In order to achieve a high degree of symmetry (as discussed presently), terms in γ^5, τ_3, or the unit matrix have been omitted. It will also be assumed that

$$V(-\lambda)=-V(\lambda). \qquad (3.3)$$

The assignment of the subscripts 1 and 2 to the Pauli matrices occurring in (3.2) is of course arbitrary.

4. CONSERVED CURRENTS

It follows directly from (3.1) and from the boundary conditions at $\lambda=\pm\infty$ that the following quantities are

differentially conserved four-currents:

$$j_V{}^\mu=\int\bar{\psi}(\lambda)\gamma^\mu\psi(\lambda)d\lambda, \tag{4.1}$$

$$j_A{}^\mu=i\int\bar{\psi}(\lambda)\gamma^\mu\gamma^5\tau_3\psi(\lambda)d\lambda, \tag{4.2}$$

$$j_v{}^\mu=\int\bar{\psi}(\lambda)\gamma^\mu\tau_3\psi(-\lambda)d\lambda, \tag{4.3}$$

$$j_a{}^\mu=i\int\bar{\psi}(\lambda)\gamma^\mu\gamma^5\psi(-\lambda)d\lambda, \tag{4.4}$$

$(\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3;\ \bar{\psi}=\psi^\dagger\gamma^0)$.

The fourth components ("charges") are respectively the expection values of

$$q_V=1, \tag{4.5}$$

$$q_A=i\gamma^5\tau_3, \tag{4.6}$$

$$q_v=\tau_3P_\lambda, \tag{4.7}$$

$$q_a=i\gamma^5P_\lambda, \tag{4.8}$$

P_λ being the parity operator in λ. These operators commute with each other and with the Hamiltonian and, thus, can be diagonalized simultaneously, together with the energy, giving rise to the eigenvalues ± 1 for q_A, q_v, and q_a. The statement $q_V=\pm 1$ will make sense in the second-quantized theory only. Of the four conservation laws, only three are independent, since

$$q_a=q_Aq_v. \tag{4.9}$$

Finally the conserved mass currents,

$$j^\mu{}_{M^\kappa}=\int\bar{\psi}(\lambda)\gamma^\mu M^\kappa\psi(\lambda)d\lambda, \quad (\kappa=1,2,3,\cdots) \tag{4.10}$$

should be mentioned. One can diagonalize M^2, but not M, together with q_A, q_a, and q_v.

5. MASS SPECTRUM

Equation (3.1) is solved in the standard way by setting

$$\psi=(i\gamma\cdot\partial+M)\chi. \tag{5.1}$$

This gives, for the four-dimensional Fourier transform $\omega(p,\lambda)$ of $\chi(x,\lambda)$, the equation

$$M^2\omega=m^2[-\partial^2/\partial\lambda^2+V^2(\lambda)+\tau_3V'(\lambda)]\omega=p^2\omega. \tag{5.2}$$

Setting

$$\omega=\begin{pmatrix}\omega_+\\ \omega_-\end{pmatrix} \tag{5.3}$$

in the representation where

$$i\tau_2\tau_1=\tau_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{5.4}$$

one finds

$$\Lambda_\pm\omega_\pm=p^2\omega_\pm, \tag{5.5}$$

where

$$\Lambda_\pm=m^2[-\partial^2/\partial\lambda^2+V^2(\lambda)\pm V'(\lambda)]. \tag{5.6}$$

These operators may be factored as follows:

$$\begin{aligned}\Lambda_+&=M_+M_-,\\ \Lambda_-&=M_-M_+,\end{aligned} \tag{5.7}$$

where

$$M_\pm=m[\pm\partial/\partial\lambda+V(\lambda)]. \tag{5.8}$$

Let $|V|$ increase sufficiently fast as $\lambda\to\infty$ to make the spectra of Λ_+ and Λ_- wholly discrete. There is then no loss of generality in assuming $V>0$ as $\lambda\to+\infty$ (the opposite case corresponds to an alternative representation of the τ matrices). It will be seen from (5.7) that the spectra of Λ_+ and Λ_- both consist of nonnegative eigenvalues only. Indeed, for a trial function $f(\lambda)$ one has

$$\int f\Lambda_+fd\lambda=\int(M_-f)^2d\lambda\geqslant 0, \tag{5.9}$$

and a similar relation for Λ_-. Also, the lowest eigenvalue of Λ_- is zero, since the eigenfunction

$$f_0(\lambda)=\exp\left[-\int^\lambda V(\lambda')d\lambda'\right], \tag{5.10}$$

satisfying

$$M_+f_0=0, \tag{5.11}$$

obeys the boundary conditions. On the other hand, the lowest eigenvalue of Λ_+ must be positive, since a similar explicit solution of

$$M_+M_-f=0 \tag{5.12}$$

shows that f cannot obey the boundary conditions. It must finally be noted that every eigenvalue except zero is common to both Λ_+ and Λ_-. Indeed, if f_k is a well behaved solution of

$$M_-M_+f_k=m_k{}^2f_k, \tag{5.13}$$

then multiplication from the left by M_+ shows that M_+f_k is a (well behaved) eigenfunction of M_+M_- corresponding to the same eigenvalue $m_k{}^2$ *unless* $m_k=0$ in view of (5.11). A similar argument shows that, conversely, every eigenvalue of M_+M_- is also one of M_-M_+, without exception since for every k one has $M_-f_k\neq 0$, as seen before.

In conclusion: A zero-mass fermion exists and has half the degeneracy of the fermions of higher mass.

The most general positive-energy solution of (3.1) in momentum space is thus a linear combination of terms

that can be written [in the τ_3 representation, (5.4)]

$$\varphi_k=(p_k\cdot\gamma-M)\begin{pmatrix} aM_+f_k \\ bf_k \end{pmatrix}=\begin{pmatrix} a'M_+f_k \\ -p_k\cdot\gamma a'f_k \end{pmatrix}, \quad (5.14)$$

where the subscript $k=0, 1, 2, \cdots$ labels the eigenvalues m_k^2 of p^2 in ascending order; it is understood that

$$p_k^2=m_k^2, \quad p_k^0>0. \quad (5.15)$$

The quantities a and b (or, alternatively, just a') are arbitrary four-component spinors independent of λ. A special case of (5.14) is the most general zero-mass solution,

$$\varphi_0=p_0\cdot\gamma\begin{pmatrix} 0 \\ b \end{pmatrix}f_0. \quad (5.16)$$

There are only four arbitrary components left in (5.14), because $(p_k\cdot\gamma-M)$ is, as usual, essentially a projection operator onto the positive-energy solutions. For a similar reason, there are only two arbitrary components in (5.16).

6. PSEUDOCHARGE AND HELICITY OF THE ZERO-MASS FERMION

Since the pseudocharge q_A is a good quantum number, one can now restrict oneself to the consideration of solutions for which, say, $q_A=+1$. This is accomplished by multiplying the solutions (5.14), (5.16) from the left by the projection operator $(i\gamma^5\tau_3+1)$. There are now two arbitrary components left in $(i\gamma^5\tau_3+1)\varphi_k$ if $k>0$, and only one if $k=0$.

One may now use the fact that the helicity $\boldsymbol{\sigma}\cdot\mathbf{p}/|\mathbf{p}|$ itself is a good quantum number and commutes with q_A. Thus, if $q_A=+1$, φ_0 must have a well-defined helicity, which turns out on explicit calculation to be

$$\boldsymbol{\sigma}\cdot\mathbf{p}/|\mathbf{p}|=-1; \quad (6.1)$$

the higher-mass solutions can have both helicities. The sign of (6.1) is reversed if $q_A=-1$, as can be seen from the fact that the two sets of solutions (i.e., with $q_A=\pm1$) are carried into each other by space reflection, a transformation under which the theory is invariant and whose form is precisely the usual one.

7. CHARGE CONJUGATION AND THE CLASSIFICATION OF SOLUTIONS

Symmetries of the theory under time reversal and charge conjugation hold separately and are most simply considered in that representation of the Pauli matrices where τ_1 and $i\tau_2$ are both real. These transformations then assume their usual form. One should note in particular that, under charge conjugation, q_V and q_v change sign, while q_A and q_a do not. (In this section we are dealing with the quantized field: anticommutation properties of ψ are relevant to the changes in sign discussed here.) Also, the dynamical variables such as momentum, angular momentum, and helicity are invariant under charge conjugation.

TABLE I. The solutions of (3.1), classified according to their quantum numbers. The horizontal bars represent qualitative mass levels, the lowest one in each group standing for zero mass. On the right in each group are the right-handed $(\boldsymbol{\sigma}\cdot\mathbf{p}/|\mathbf{p}|=+1)$ solutions; on the left, the left-handed ones. The values of q_a correspond to $q_A=+1$ on the left and to $q_A=-1$ on the right. The mass levels are numbered by k. One possible lepton interpretation is indicated in parentheses next to the levels.

	$q_A=+1$	$q_A=-1$	q_v	q_a		k
$q_V=+1$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	— —	— —	-1	-1	$+1$	2
	— —(e^-)	— —(μ^+)	$+1$	$+1$	-1	1
	— (ν_1)	—(ν_2)	-1	-1	$+1$	0
$q_V=-1$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	— —	— —	$+1$	-1	$+1$	2
	— —(μ^-)	— —(e^+)	-1	$+1$	-1	1
	— $(\bar{\nu}_2)$	—$(\bar{\nu}_1)$	$+1$	-1	$+1$	0

Additional invariances, under the transformations

$$\psi\rightarrow\tau_3P_\lambda\psi, \quad (7.1)$$

and

$$\psi\rightarrow i\gamma^5\tau_3\psi, \quad (7.2)$$

should also be mentioned. They are connected with the conservation laws of Sec. 4.

The good quantum numbers q_v and q_a commute with q_A, and either one of those can be used to distinguish between solutions of alternate masses. Indeed, since V is odd, alternate values of m_k also correspond to alternate parities (with respect to λ) of the eigenfunctions of $\Lambda_\pm$, even parity corresponding to the lowest eigenvalue. It is then readily verified that, for the zero-mass particle $(m_0=0)$ one has

$$\begin{aligned} q_v=q_a&=-1, \quad (q_A=1,\ k=0) \\ q_v=-q_a&=-1, \quad (q_A=-1,\ k=0) \end{aligned} \quad (7.3)$$

and that the sign of q_v, as well as that of q_a, alternates as masses increase.

The classification of the various solutions is summarized in Table I.

8. INTERPRETATION IN TERMS OF LEPTONS

The solutions of (3.1) are suggestive of a two-neutrino theory of leptons. Introduction of interaction terms into the theory may be expected to make all those mass levels unstable which are not prevented by exact conservation laws from decaying into lower ones. Thus, in any group of Table I only the lowest two masses (at best) will survive. This is under the assumption (explained below) that, in a fully interacting theory, M is no longer conserved. There are several schemes for attributing leptons to those solutions. One such scheme is shown in Table I. It is reasonable to postulate conservation of leptons, and to call q_V the lepton number. The number of schemes is further reduced by

requiring the exact conservation, in all physical processes, of a quantum number, distinguishing between muon-like and electron-like leptons. The value of q_v cannot be used in this manner, since for a given mass number it distinguishes only between lepton and antilepton. There remain q_A and q_a for this purpose; q_A has been chosen for the scheme of Table I. Then $\frac{1}{2}(q_V+q_A)$ might be called the electron number, while $\frac{1}{2}(q_V-q_A)$ would be the muon number. If ν_1 and ν_2 are left- and right-handed neutrinos ($q_V=+1$), respectively, then e^- and μ^+ must be associated with these as shown in the table, in view of the experimental data. It now follows that the electric charge q is given by

$$q=-\tfrac{1}{2}(q_A+q_a). \tag{8.1}$$

One obtains the curious (but not *a priori* objectionable) result that the usual formal charge conjugation does not change the sign of q, while space reflection does.

To obtain a realistic theory, one eventually will have to remove the $\mu-e$ mass degeneracy through a parity nonconserving interaction[2]. Lepton conservation (or electric charge conservation, if one prefers) will prevent the radiative decay of a muon into an electron and photons. The quantum numbers, M and q_v, should not remain exactly conserved, as can be seen for example from the process

$$\pi^+\rightarrow\mu^++\bar{\nu}_2.$$

An equally plausible scheme results from using q_a rather than q_A in order to distinguish between μ and e. In Table I this only switches the labels e^-, μ^+, and simultaneously the labels μ^-, e^+. It interchanges the physical roles of q_a and q_A, as well as the sign of (8.1). Further rearrangements can be made if one gives up some conservation laws among the new quantum numbers.[3]

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[2] In this connection, the large value of m_μ/m_e raises an interesting problem.

[3] The recent "neutrino-flip" suggestion of G. Feinberg, F. Gürsey, and A. Pais, Phys. Rev. Letters **7**, 208 (1961) seems incompatible with the schemes discussed here. Underlying that suggestion is the assumption that μ^- is a lepton rather than an antilepton.

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Exact Numerical Solution of a Three-Body Ground-State Problem*

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The appropriate Schrödinger equation is solved numerically to give the wave function for the ground-state H^3 problem. An ordinary, Gaussian, two-body force without a hard core is used. We outline how our method can be applied (including the Pauli exclusion principle) to the zero-energy scattering problem.

I. INTRODUCTION

THE purpose of this paper is to describe our calculation of the H^3 ground-state energy and wave function. We calculate them by using an IBM 7090 to solve the appropriate Schrödinger equation for the three-body wave function. The only unrealistic aspect of this calculation is the simplified potential used. The two-body interaction is taken to be an ordinary, central force without a hard core. The inclusion of more complex forces results[1] in coupled sets of equations of the type we solve herein. As the speed of computing machines increase, it should be possible to solve three-body problems with realistic forces exactly numerically.

We outline how our methods may be applied to the calculation of the three-body scattering lengths. This calculation is much more extensive than the ground-state problem, and while it appears feasible on existing computing machines, we have not performed it. If this calculation were performed, even using a simplified potential, one might expect, in view of the arguments[2] that n-d scattering is not sensitive to the refined details of the two-nucleon potential, that at least the quartet scattering length would be good enough to indicate unambiguously the correct set of experimental scattering lengths.[3]

The exact wave functions calculated herein should be useful in testing the applicability of variational techniques to the solution of three-body Schrödinger equations.

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[1] M. Verde, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin-Gottingen-Heidelberg, 1957), Vol. 39, p. 144.

[2] R. S. Christian and J. L. Gammel, Phys. Rev. **91**, 100 (1953).

[3] E. O. Wollan, C. G. Shull, and W. E. Koehler, Phys. Rev. **83**, 700 (1951).