

requiring the exact conservation, in all physical processes, of a quantum number, distinguishing between muon-like and electron-like leptons. The value of q_e cannot be used in this manner, since for a given mass number it distinguishes only between lepton and antilepton. There remain q_A and q_a for this purpose; q_A has been chosen for the scheme of Table I. Then $\frac{1}{2}(q_V + q_A)$ might be called the electron number, while $\frac{1}{2}(q_V - q_A)$ would be the muon number. If ν_1 and ν_2 are left- and right-handed neutrinos ($q_V = +1$), respectively, then e^- and μ^+ must be associated with these as shown in the table, in view of the experimental data. It now follows that the electric charge q is given by

$$q = -\frac{1}{2}(q_A + q_a). \quad (8.1)$$

One obtains the curious (but not *a priori* objectionable) result that the usual formal charge conjugation does not change the sign of q , while space reflection does.

To obtain a realistic theory, one eventually will have to remove the μ - e mass degeneracy through a parity nonconserving interaction². Lepton conservation (or electric charge conservation, if one prefers) will prevent

² In this connection, the large value of m_μ/m_e raises an interesting problem.

the radiative decay of a muon into an electron and photons. The quantum numbers, M and q_e , should not remain exactly conserved, as can be seen for example from the process

$$\pi^+ \rightarrow \mu^+ + \bar{\nu}_2.$$

An equally plausible scheme results from using q_a rather than q_A in order to distinguish between μ and e . In Table I this only switches the labels e^- , μ^+ , and simultaneously the labels μ^- , e^+ . It interchanges the physical roles of q_a and q_A , as well as the sign of (8.1). Further rearrangements can be made if one gives up some conservation laws among the new quantum numbers.³

ACKNOWLEDGMENTS

I am much indebted to Professor R. G. Newton for some illuminating remarks. It is a pleasure to thank Professor E. J. Konopinski and Professor R. B. Curtis for their helpful comments on the manuscript.

³ The recent "neutrino-flip" suggestion of G. Feinberg, F. Gürsey, and A. Pais, *Phys. Rev. Letters* **7**, 208 (1961) seems incompatible with the schemes discussed here. Underlying that suggestion is the assumption that μ^- is a lepton rather than an antilepton.

Exact Numerical Solution of a Three-Body Ground-State Problem*

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The appropriate Schrödinger equation is solved numerically to give the wave function for the ground-state H^3 problem. An ordinary, Gaussian, two-body force without a hard core is used. We outline how our method can be applied (including the Pauli exclusion principle) to the zero-energy scattering problem.

I. INTRODUCTION

THE purpose of this paper is to describe our calculation of the H^3 ground-state energy and wave function. We calculate them by using an IBM 7090 to solve the appropriate Schrödinger equation for the three-body wave function. The only unrealistic aspect of this calculation is the simplified potential used. The two-body interaction is taken to be an ordinary, central force without a hard core. The inclusion of more complex forces results¹ in coupled sets of equations of the type we solve herein. As the speed of computing machines increase, it should be

possible to solve three-body problems with realistic forces exactly numerically.

We outline how our methods may be applied to the calculation of the three-body scattering lengths. This calculation is much more extensive than the ground-state problem, and while it appears feasible on existing computing machines, we have not performed it. If this calculation were performed, even using a simplified potential, one might expect, in view of the arguments² that n - d scattering is not sensitive to the refined details of the two-nucleon potential, that at least the quartet scattering length would be good enough to indicate unambiguously the correct set of experimental scattering lengths.³

The exact wave functions calculated herein should be useful in testing the applicability of variational techniques to the solution of three-body Schrödinger equations.

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¹ M. Verde, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957), Vol. 39, p. 144.

² R. S. Christian and J. L. Gammel, *Phys. Rev.* **91**, 100 (1953).

³ E. O. Wollan, C. G. Shull, and W. E. Koehler, *Phys. Rev.* **83**, 700 (1951).

The methods employed in this paper are also applicable to e^- -H and e^+ -H problems.

II. CALCULATION OF THE GROUND-STATE WAVE FUNCTION

The specific problem which we solve is the single bound state of a system of three particles. The interaction between all three is the same and it is an ordinary, central force. It is a Gaussian with a standard range parameter $\beta=1.60$ f and a depth of 51.5 Mev. Using $\beta=1.60$ f and fitting the deuteron binding energy (2.226 Mev) we obtain a triplet potential depth of 63.71 Mev ($S_t=1.466$), and a triplet effective range⁴ of about 1.8 f. Assuming that $S_s \approx 0.9$, we obtain a singlet effective range of about 2.5 f. We have picked our well depth in the three-body problem to be about $\frac{1}{2}(V_t+V_s)$.⁵

In the ground-state problem the space wave function is symmetric and hence the Pauli exclusion principle does not effect the solution with the simplified potential we are using. In the scattering problem this is not so even with the simplified potential. As we are considering the ground-state problem, only S states will occur. Thus we may treat the much simpler three-body problem in a plane. This simplification is also valid for zero-energy scattering. The Schrödinger equation for this problem is¹

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial q^2} + (r^2 + q^2) \frac{\partial}{\partial x} (1-x^2) \frac{\partial u}{\partial x} - V(r, q, x) u + i \frac{\partial u}{\partial t} = 0, \quad (2.1)$$

where $r=(\sqrt{3}/2)$ times the distance between particles 1 and 2, q is the distance between particle 3 and the center of mass of 1 and 2, x is the cosine of the angle θ between q and the line joining 1 and 2, t is time, and V the potential energy. The quantity u is rq times the wave function Ψ . To obtain a physically acceptable solution, Ψ must be regular everywhere. Hence u is zero if r or q is zero and u is finite on the singular planes $x=\pm 1$.

In order to impose regularity on the $x=\pm 1$ planes, we expand u in terms of the *regular* Legendre polynomials, $P_l(x)$. It is to be noted that only the even ones occur in our problem as $V(r, q, x)=V(r, q, -x)$.

$$\begin{aligned} \frac{\partial^2 u_l}{\partial r^2} + \frac{\partial^2 u_l}{\partial q^2} - l(l+1)(r^2 + q^2) u_l + i \frac{\partial u_l}{\partial t} \\ - \sum_{l'=0}^{\infty} \sum_{k=|l-l'|}^{l+l'} \left\{ \frac{(2l+1)A_{\frac{1}{2}(l-k-l)} A_{\frac{1}{2}(l'+k-l)} A_{\frac{1}{2}(k+l-l')}}{(l+k+l+1)A_{\frac{1}{2}(l'+l+k)}} \right\} V_k(r, q) u_{l'}(r, q) = 0, \end{aligned} \quad (2.7)$$

⁴ J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).

⁵ See, for instance, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

⁶ E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1927), 4th ed., p. 331, Example 11.

⁷ This expansion was first used in some unpublished work by G. Breit and S. Shore at the University of Wisconsin a decade ago. See also, P. J. Luke, R. E. Meyerott, W. W. Clendenin, and S. Geltman, Phys. Rev. **85**, 393, 401 (1952), who have demonstrated the practicability of a procedure very similar to the one described in the present paper.

If one considers a potential with a hard core, then clearly the expansion of the potential does not exist. In that case one could impose regularity on the $x=\pm 1$ planes by using

$$v = -rqx^2 \Psi / \ln(1-x^2), \quad (2.2)$$

instead of u . The regularity of Ψ then becomes simply $v=0$ for $x=\pm 1$. The criterion as to whether or not (2.2) is useful computationally is the smallness of the shift in the eigenvalues of the difference approximation to

$$\begin{aligned} (1-x^2) \frac{\partial^2 w}{\partial x^2} - \frac{4}{x} \left[1 - \frac{x^2}{2} + \frac{x^2}{\ln(1-x^2)} \right] \frac{\partial w}{\partial x} \\ + \frac{6}{x^2} \left[1 - \frac{x^2}{3} + \frac{x^2}{\ln(1-x^2)} \right] w + l(l+1)w = 0 \end{aligned} \quad (2.3)$$

from the analytical result, l an integer. This equation arises from transforming Legendre's equation through the use of (2.2). We find, for instance, that the lowest value of l is shifted from zero to approximately 0.012 when a mesh spacing of $\Delta x=0.04$ is used. We expect that results using this mesh spacing would be good to about 1%.

Let us now expand⁶

$$u = \sum_{l=0}^{\infty} u_l(r, q) P_l(x). \quad (2.4)$$

steps of two

Substituting (2.4) into (2.1), multiplying by $P_l(x)$, and integrating over x from -1 to $+1$, we obtain

$$\begin{aligned} \frac{\partial^2 u_l}{\partial r^2} + \frac{\partial^2 u_l}{\partial q^2} - l(l+1)(r^2 + q^2) u_l + i \frac{\partial u_l}{\partial t} \\ - \int_{-1}^{+1} P_l(x) V(r, q, x) \sum_{l'=0}^{\infty} u_{l'}(r, q) \\ \times P_{l'}(x) dx \Big/ \int_{-1}^{+1} P_l^2(x) dx = 0. \end{aligned} \quad (2.5)$$

steps of two

If we also expand the potential as

$$V(r, q, x) = \sum_{k=0}^{\infty} V_k(r, q) P_k(x), \quad (2.6)$$

steps of two

then we get⁷

where

$$A_0=1, \quad A_m=[1 \cdot 3 \cdot 5 \cdots (2m-1)]/m!. \quad (2.8)$$

In the process of imposing the boundary condition of regularity at $x=\pm 1$, we have reduced the original three-dimensional partial differential equation (2.1) to a system of coupled two-dimensional partial differential equations. As many fewer values of l will be required than values of x would have been, this transformation also effects a considerable saving in the amount of computational labor.

As the ground-state wave function falls to zero very quickly, we may solve our problem in a box in which we assume $u(R, q)=0$ for R the maximum value of r and $u(r, Q)=0$ for Q the maximum value of q . If the box is chosen sufficiently large, there will be only a negligible effect on the energy eigenvalue.

In order to solve Eqs. (2.7) we use a modification of a scheme previously developed by one of us.⁸ We consider $t=-i\alpha^2\tau$ for real positive value of τ , that is imaginary time. If we guess a solution to (2.7) and then expand it in eigenfunctions $\varphi_i(r, q, x)$ with corresponding eigenvalues E_i , then each error component changes in "time" like $\exp(-\alpha^2 E_i \tau)$. Thus every state of positive energy tends to zero as t tends to infinity. The ground state (single state of negative energy in this problem), on the other hand, increases exponentially in "time." As a result of this energy separation between the ground state and the next level, we may easily obtain as large a fraction of ground state relative to the other states as we wish.⁹ In our solution the u_l are advanced simultaneously in time by the method of reference 8. The $u_{l'}$ with $l' \neq l$ are treated as source terms. We solve for u_l starting with $l=0$ and going to $l=L$. We assume that $u_l=0$ for $l>L$. We use linear

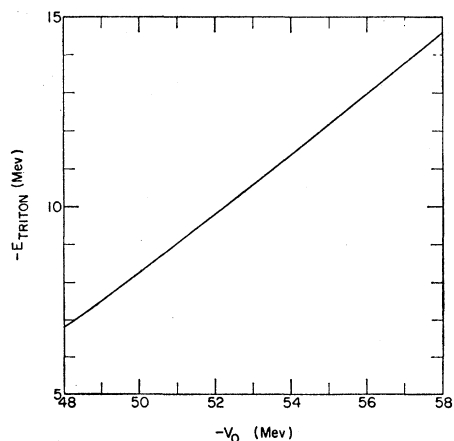


FIG. 1. Dependence of the triton binding energy on the Gaussian depth parameter.

⁸ G. A. Baker, Jr., *Quart. Appl. Math.* **17**, 314 (1959); **17**, 440 (1960); G. A. Baker, Jr. and T. A. Oliphant, *ibid.* **17**, 361 (1960).

⁹ For a general discussion of the error analysis of methods of this type, see, for instance, G. E. Forsythe and W. R. Wasow, *Finite-Difference Methods for Partial Differential Equations* (John Wiley & Sons, New York, 1960).

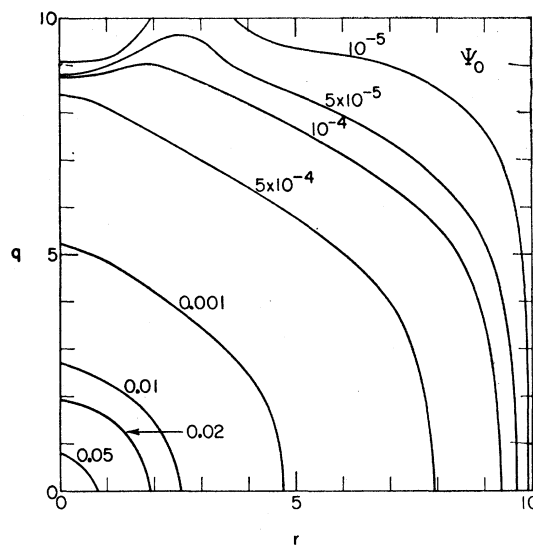


FIG. 2. $l=0$ component of the bound-state wave function, Ψ_0 . This is the dominant contribution to the wave function.

extrapolation from the two previous "times" to calculate the $u_{l'}$, $l'>l$, in the source terms and we use the $u_{l'}$ just calculated for $l'<l$. One could iterate at each time step to obtain a completely consistent set of u_l , thus avoiding reliance on linear extrapolation, but the results of reference 8, indicate that this added refinement of the numerical method is not necessary here for the accuracy we seek. We find that a mesh spacing of $\Delta r=7/64$ f and $\Delta q=7/64$ f for r and q less than about $5\frac{1}{4}$ f and $\Delta r=\frac{1}{4}$ f and $\Delta q=\frac{1}{4}$ f for r and q greater than about $5\frac{1}{4}$ f is adequate to reduce the error in the energy of the ground state to about 0.05 Mev. We find that enclosing the problem in a box $R=Q=10$ f does not change the eigenvalue by more than 0.01 Mev, or so.

In order to calculate the ground-state energy we note that

$$\lambda(\tau) = \sum_{r,q,l} [u_l(\tau - \Delta\tau) - u_l(\tau - 2\Delta\tau)]^2 / \sum_{r,q,l} [u_l(\tau) - u_l(\tau - \Delta\tau)]^2 \rightarrow \exp(\alpha^2 E_0 \Delta\tau) \quad (2.9)$$

rapidly for $\Delta\tau$ small. For $\Delta\tau$ not zero, the exponential is only approximate, but⁸ the proper function is known so we may solve for E_0 from the asymptotic value of $\lambda(\tau)$. Requiring $\lambda(\tau)$ to converge to about one part in 10^5 , we obtained (for $V_0=-51.5$ Mev):

L	0	2	4
E_0	-9.080	-9.396	-9.416.

Approximately 30 iterations were required to obtain this accuracy. The rapid convergence in L is to be

noted. These values are probably too low rather than too high due to Δr , and Δq not being small enough.

In Fig. 1 we have plotted E_0 for an l_{\max} of 4 for various nucleon-nucleon well depths. In Figs. 2-4 we have given contour plots of the triton ground-state wave function.

III. THE ZERO-ENERGY SCATTERING PROBLEM

The zero-energy scattering problem may be solved by an extension of the method described in the previous section for the ground-state problem. We have not carried it out, but shall describe how it can be done. The procedure for finding the wave function requires one basic modification from that for finding the ground-state wave function. When we are solving the zero-energy scattering problem, any small amount of the ground-state wave function in our solution must be regarded as an error term. However as we saw, this error term grows in "time" and does not decay. We may correct this situation in the following way. We allow the ground-state error term to grow until it dominates the whole error term. When successive $\lambda(\tau)$, Eq. (2.9), agree to, for instance, 5 decimal places we fit the wave function u_i with

$$A_i(r, q) + B_i(r, q)\lambda^{-\tau/\Delta\tau}, \quad (3.1)$$

and replace u_i by A_i , its "asymptotic" value. This procedure approximately removes the divergent state. We have tried it and find that it is sufficiently good that the occurrence of one such state is not troublesome.

We remark that, since we obtain an exact solution by this method we may antisymmetrize the wave function in accordance with the Pauli exclusion principle *after* we have found it, rather than trying to maintain antisymmetry at every step of the solution as is necessary with approximate solutions.

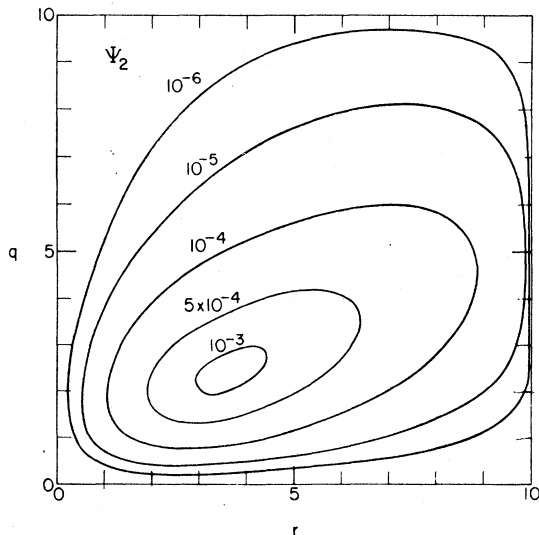


FIG. 3. $l=2$ component of the bound-state wave function, Ψ .

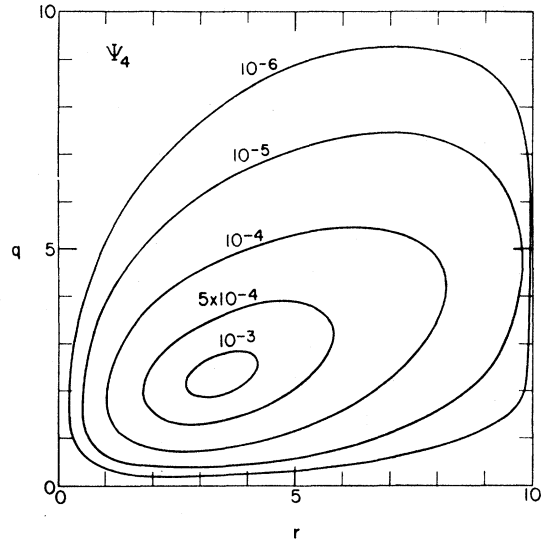


FIG. 4. $l=4$ component of the bound-state wave function, Ψ . It can be seen that the $l=2$ and $l=4$ wave functions are quite similar which indicates that the wave function is not well determined; however, this was not found to be critical in the determination of the binding energy.

The following boundary conditions are appropriate for the zero-energy, three distinguishable-particle scattering problem:

$$\begin{aligned} u(r, q) &\propto (q-a)\varphi(r) \quad \text{as } q \rightarrow \infty, r \text{ finite;} \\ u(r, q) &\propto r q (-\Delta a) \varphi(r_{\text{exch}}) / q_{\text{exch}} r_{\text{exch}} \quad \text{as } q_{\text{exch}} \rightarrow \infty, \\ &\quad r_{\text{exch}} \text{ finite;} \end{aligned} \quad (3.2)$$

here r_{exch} and q_{exch} are the r and q that result from exchanging particle 3 with 1 or 2, a is the scattering length, and $\varphi(r)$ is the ground-state wave function for the two-particle bound system (deuteron). The quantity Δa will be seen to be related to the difference in doublet and quartet scattering lengths when the appropriately exchanged solutions of (2.7) are considered. The proper, ordinary central potential for this problem is the triplet potential as any smaller potential would close the entrance and exit channels and thereby disallow any n - d type scattering.

The procedure for solving the three-body scattering problem with ordinary central forces is to guess a wave function and advance it in "time" until, taking appropriate measures with regard to divergent state, all the error components in the original guess have been eliminated. One can solve the problem in a large box (we estimate about $R=15$, $Q=15$ f) subject to boundary conditions (3.2). If the strength of the triplet potential is adjusted so that $\varphi(R)=0$ instead of $\varphi(\infty)=0$, the boundary conditions are somewhat simplified. If one knew $(\Delta a)/(Q-a)$, one could at once impose consistent boundary conditions and solve for the wave function. We, however, do not, as this is part of the solution we seek,

We may solve for it, however, by requiring that the solution be self-consistent. To do this we note that, asymptotically $u(r, q)$ must be of the form (3.2) except that $(-\Delta a)$ may be replaced by $(\epsilon q_{\text{exch}} - \Delta a)$. We seek to choose Δa to use in (3.2) such that ϵ vanishes. This requires us to solve (2.7) twice with different values of Δa [say 0, $(Q-a)$, for instance]. We normalize by taking

$$u(r, Q) = \varphi(r). \quad (3.3)$$

Then it is easy to show, by using $Hu=0$ and some integrations by parts, that if u has the asymptotic form for (3.2) and

$$S = \sum_{l=0}^{\infty} \int_s \left(\frac{\partial u}{\partial n} \varphi_l - u \frac{\partial \varphi_l}{\partial n} \right) ds, \quad (3.4)$$

$$T = \sum_{l=0}^{\infty} \int_s \left(-\phi_l - u \frac{\partial \phi_l}{\partial n} \right) ds,$$

where s is arc length and \int_s denotes integration over the edge of the box, n is the normal direction to the

surface, and

$$\begin{aligned} \varphi_0 &= q\varphi(r), \quad \varphi_l = 0, \quad l > 0 \\ \phi_l &= \frac{1}{2}rq \int_{-1}^{+1} \frac{\varphi(r_{\text{exch}})}{r_{\text{exch}}} P_l(x) dx, \end{aligned} \quad (3.5)$$

then

$$a = QS/(1+S), \quad \Delta a = -QT/(1+S). \quad (3.6)$$

In general, the Δa used in (3.2) will not equal the Δa of (3.6); however we may solve for the correct linear combination of the two solutions so that the two Δa 's are equal and (3.3) is maintained. As (2.7) is a linear equation we know that if we solve using that set of boundary conditions, (3.6) will be consistent with (3.2). When the appropriate exchange combination¹ of the solution to (2.7) is formed, we find

$$a_4 = a + \Delta a, \quad a_2 = a - \frac{1}{2}\Delta a, \quad (3.7)$$

for the quartet and doublet scattering lengths, respectively.

We estimate that of the order of 10 different l values are necessary in the expansion of u in order to obtain a good representation of the wave function in the exchanged channels. The larger the (r, q) box taken, the more l values are required.

Ground-State Energy of the Nucleon*

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Halpern's method of moments has been applied to the intermediate-coupling reduced Hamiltonian, whose lowest eigenvalue is a variational upper bound to the ground-state energy of the nucleon in the fixed-source model of meson theory. The results compare favorably with an earlier intermediate-coupling calculation of Friedman, Lee, and Christian, and agree with direct moment-method results. A discussion of the relationship between the present work and a Tamm-Dancoff approximation for the reduced Hamiltonian is included.

THEORY

THE (somewhat overworked) Chew¹ model of meson theory, consisting of pseudoscalar mesons gradient-coupled to a static, extended nucleon, leads to a Hamiltonian of the form

$$H = \int_0^\infty dk [\omega_k a_{i\alpha}^\dagger(k) a_{i\alpha}(k) + V(k) \sigma_i \tau_\alpha \{ a_{i\alpha}(k) + a_{i\alpha}^\dagger(k) \}]. \quad (1)$$

Here $\omega_k = (k^2 + 1)^{1/2}$, $V(k) = f k^2 U(k) / (3\pi \omega_k)^{1/2}$, and summation over repeated indices $i, \alpha = 1, 2, 3$, referring to

components of angular momenta and isotopic spin, is assumed. As usual, $[a_{i\alpha}(k), a_{j\beta}^\dagger(k')] = \delta_{ij} \delta_{\alpha\beta} \delta(k - k')$, and we have set $\hbar = c = m = 1$. $U(k)$ is the conventional cutoff function.

Halpern *et al.*² have solved for the lowest eigenvalue of the Hamiltonian above by the method of moments, whose n th order approximation is the lowest root E_0 of the determinantal equation

$$\begin{vmatrix} 1 & E & E^2 & \cdots & E^n \\ H_0 & H_1 & H_2 & \cdots & H_n \\ H_1 & H_2 & \cdots & \cdots & H_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{n-1} & \cdots & \cdots & \cdots & H_{2n-1} \end{vmatrix} = 0, \quad (2)$$

where $H_j = \langle 0 | H^j | 0 \rangle$, and where $|0\rangle$ is the "bare"

* This work was done in part at the Computation Center at Massachusetts Institute of Technology, Cambridge, Massachusetts.

¹ G. F. Chew, Phys. Rev. 94, 1748 (1954).

² F. R. Halpern, L. Sartori, K. Nishimura, and R. Spitzer, Ann. Phys. 7, 154 (1959).