We may solve for it, however, by requiring that the solution be self-consistent. To do this we note that, asymptotically $u(r,q)$ *must* be of the form (3.2) except that $(-\Delta a)$ may be replaced by $(\epsilon q_{\text{exch}}-\Delta a)$. We seek to choose Δa to use in (3.2) such that ϵ vanishes. This requires us to solve (2.7) twice with different values of Δa [say 0, $(Q-a)$, for instance]. We normalize by taking

$$u(r,Q)=\varphi(r). \tag{3.3}$$

Then it is easy to show, by using $Hu=0$ and some integrations by parts, that if u has the asymptotic form for (3.2) and

$$S=\sum_{l=0}^{\infty}\int_s\left(\frac{\partial u}{\partial n}\varphi_l-u\frac{\partial \varphi_l}{\partial n}\right)ds,$$
$$T=\sum_{l=0}^{\infty}\int_s\left(\frac{\partial u}{\partial n}\phi_l-u\frac{\partial \phi_l}{\partial n}\right)ds, \tag{3.4}$$

where s is arc length and $\int_s$ denotes integration over the edge of the box, n is the normal direction to the surface, and

$$\varphi_0=q\varphi(r),\quad \varphi_l=0,\quad l>0$$
$$\phi_l=\tfrac{1}{2}rq\int_{-1}^{+1}\frac{\varphi(r_{\text{exch}})}{r_{\text{exch}}}P_l(x)dx, \tag{3.5}$$

then

$$a=QS/(1+S),\quad \Delta a=-QT/(1+S). \tag{3.6}$$

In general, the Δa used in (3.2) will not equal the Δa of (3.6); however we may solve for the correct linear combination of the two solutions so that the two Δa's are equal and (3.3) is maintained. As (2.7) is a linear equation we know that if we solve using that set of boundary conditions, (3.6) will be consistent with (3.2). When the appropriate exchange combination[1] of the solution to (2.7) is formed, we find

$$a_4=a+\Delta a,\quad a_2=a-\tfrac{1}{2}\Delta a, \tag{3.7}$$

for the quartet and doublet scattering lengths, respectively.

We estimate that of the order of 10 different l values are necessary in the expansion of u in order to obtain a good representation of the wave function in the exchanged channels. The larger the (r,q) box taken, the more l values are required.

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Ground-State Energy of the Nucleon*

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Halpern's method of moments has been applied to the intermediate-coupling reduced Hamiltonian, whose lowest eigenvalue is a variational upper bound to the ground-state energy of the nucleon in the fixed-source model of meson theory. The results compare favorably with an earlier intermediate-coupling calculation of Friedman, Lee, and Christian, and agree with direct moment-method results. A discussion of the relationship between the present work and a Tamm-Dancoff approximation for the reduced Hamiltonian is included.

THEORY

THE (somewhat overworked) Chew[1] model of meson theory, consisting of pseudoscalar mesons gradient-coupled to a static, extended nucleon, leads to a Hamiltonian of the form

$$H=\int_0^{\infty}dk[\omega_k a_{i\alpha}^{\dagger}(k)a_{i\alpha}(k)+V(k)\sigma_i\tau_\alpha\{a_{i\alpha}(k)+a_{i\alpha}^{\dagger}(k)\}]. \tag{1}$$

Here $\omega_k=(k^2+1)^{\frac{1}{2}}$, $V(k)=fk^2U(k)/(3\pi\omega_k)^{\frac{1}{2}}$, and summation over repeated indices i, $\alpha=1$, 2, 3, referring to components of angular momenta and isotopic spin, is assumed. As usual, $[a_{i\alpha}(k),a_{j\beta}^{\dagger}(k')]=\delta_{ij}\delta_{\alpha\beta}\delta(k-k')$, and we have set $\hbar=c=m=1$. $U(k)$ is the conventional cutoff function.

Halpern *et al.*[2] have solved for the lowest eigenvalue of the Hamiltonian above by the method of moments, whose nth order approximation is the lowest root E_0 of the determinantal equation

$$\begin{vmatrix} 1 & E & E^2 & \cdots & E^n \\ H_0 & H_1 & H_2 & \cdots & H_n \\ H_1 & H_2 & \cdots & \cdots & H_{n+1} \\ H_{n-1} & \cdots & & \cdots & H_{2n-1} \end{vmatrix}=0, \tag{2}$$

where $H_j=\langle 0|H^j|0\rangle$, and where $|0\rangle$ is the "bare"

* This work was done in part at the Computation Center at Massachusetts Institute of Technology, Cambridge, Massachusetts.

[1] G. F. Chew, Phys. Rev. **94**, 1748 (1954).

[2] F. R. Halpern, L. Sartori, K. Nishimura, and R. Spitzer, Ann. Phys. **7**, 154 (1959).

TABLE I. Ground-state energy as a function of coupling constant f and order of calculation n. The minimizing values of λ are shown when they are less than 6, and the second value of E_0 represents the results of Halpern *et al.*

$f^2 \backslash n$	2 $-E_0$	3 $-E_0$	λ	4 $-E_0$	λ	5 $-E_0$	λ	6 $-E_0$	λ
0.2	5.71 5.7	7.3487 7.3	3.5	7.8335 7.8	2.5	7.9816 7.96	2	8.0204 8.02	2
0.4	8.81 8.8	11.720 11.7	5.5	12.826 12.8	4	13.324 13.3	3	13.537 13.5	2.5
0.6	11.22 11.2	15.16 15.1	$\cdots$	16.835 16.8	5	17.723 17.7	4	18.198 18.2	3

nucleon state vector. As n increases, E_0 approaches the exact eigenvalue of the Hamiltonian.

Friedman, Lee, and Christian[3] had earlier solved the same problem, using Tomonaga's intermediate-coupling approximation.[4] This is a variational method, leading to a reduced Hamiltonian

$$H_r=\Omega[a_{i\alpha}^{\dagger}a_{i\alpha}+I^{\frac{1}{2}}(A+A^{\dagger})], \tag{3}$$

whose lowest eigenvalue is an upper bound to the lowest eigenvalue of H. Here,

$$\Omega=\int_0^\infty dk\,\omega_k f^2(k), \qquad I^{\frac{1}{2}}=\frac{f}{\Omega(3\pi)^{\frac{1}{2}}}\int_0^\infty dk\,\frac{f(k)k^2}{\omega_k^{\frac{1}{2}}},$$

$$A=\sigma_i\tau_\alpha a_{i\alpha}, \qquad f(k)=\frac{NU(k)k^2}{\omega_k^{\frac{1}{2}}(\omega_k+\lambda)},$$

$$\int_0^\infty f^2(k)dk=1, \quad [a_{i\alpha},a_{j\beta}^{\dagger}]=\delta_{ij}\delta_{\alpha\beta},$$

and λ is a variational parameter. To solve this problem, FLC used a coordinate representation for H_r, and solved approximately the resulting differential equations. Although the lowest eigenvalue of H_r agrees exactly with that of H in the two limits of $f\rightarrow 0$ and $f\rightarrow\infty$, only the former is still correct in the approximation used by FLC. It is thus of some interest to apply the method of moments to H_r, to compare the results with those of Halpern and FLC, for intermediate values of f.

We have made use of the results of Halpern[2] to evaluate the various moments of $H_r/\Omega=h$ for various values of λ, and have then numerically solved the appropriate determinantal equations, for various orders of approximation.[5]

To illustrate the procedure, we examine the first nontrivial approximation ($n=2$) for both H and h:

$$H_0=h_0=1;$$

$$H_1=h_1=0;$$

$$H_2=9\int_0^\infty V^2(k)dk, \quad h_2=9I; \tag{4}$$

$$H_3=9\int_0^\infty V^2(k)\omega_k dk, \quad h_3=9I.$$

It can be easily seen that, for each of the integrals $I_n=\int_0^\infty V^2(k)\omega_k{}^n dk$ appearing in Halpern's calculations for H_j, we can substitute the single integral I in our evaluation of the various moments h_j. This is because the intermediate-coupling approximation essentially averages over the momenta of the virtual mesons, but treats the operators $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ correctly.

RESULTS

Table I shows a comparison between the results of the present calculation and that of Halpern. [We also use his form for $U(k)$: $U(k)=1$ $k<6$, $U(k)=0$ $k>6$.] As an example of the dependence on λ, Table II examines the case of $f^2=0.6$. Clear-cut minimizing values of λ are obtained for the higher values of n.

These diagnostic results encourage us to re-examine the case reported by FLC: $f^2=0.712$, $K=6.13$. Our

TABLE II. Ground-state energy as a function of λ and n, for $f^2=0.6$. Asterisks indicate minimizing values of λ for each n.

$\lambda \backslash n$	2 $-E_0$	3 $-E_0$	4 $-E_0$	5 $-E_0$	6 $-E_0$
0	10.967	14.868	16.566	17.496	18.010
1.0	11.112	15.044	16.739	17.656	18.153
1.5	11.146	15.084	16.776	17.687	18.178
2.0	11.169	15.110	16.799	17.705	18.191
2.5	11.186	15.127	16.814	17.715	18.196
3.0	11.198	15.139	16.823	17.720	18.198*
3.5	11.206	15.148	16.829	17.722	18.197
4.0	11.213	15.154	16.832	17.723*	18.194
5.0	11.222	15.161	16.835*	17.720	18.186
6.0	11.227	15.164	16.834	17.714	18.176

[3] M. Friedman, T. D. Lee, and R. Christian, Phys. Rev. **100**, 1494 (1955) (referred to as FLC).

[4] S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **2**, 6 (1947).

[5] We have defined h this way to simplify the calculations. After solving for the lowest root, ϵ_0, of the determinantal equation we obtain the energy as $E_0=\Omega\epsilon_0$.

best ($n=6$) result gives $E_0=-21.69$, for $\lambda=3-3.5$, compared with the result $E_0=-20.67$, $\lambda=3.39$ previously obtained, and represents an improvement of about 5%.

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APPENDIX

It is interesting to examine the relation between the moment method calculations reported here and the Tamm-Dancoff approximation[6] for the reduced Hamiltonian h. The moment method of order n is a variational approximation employing trial functions of the form

$$|\Psi_n\rangle=\sum_{j=0}^{n-1} C_j H^j|0\rangle. \tag{5}$$

This state vector contains up to $n-1$ virtual mesons in the field. The Tamm-Dancoff method also retains a finite number of mesons, but determines the relative occupation by solving the Schrödinger equation $(h-\epsilon)|\Psi\rangle=0$. For the moment method of order 3, for example, Eq. (5) takes the form

$$|\Psi_3\rangle=\sum_{j=0}^{2} C_j'(A^\dagger)^j|0\rangle, \tag{5$'$}$$

[6] I. Tamm, J. Phys. (USSR) **9**, 449 (1945); S. M. Dancoff, Phys. Rev. **78**, 382 (1950).

where C_j' depends on C_j and I. The 2-meson Tamm-Dancoff state vector has the same form as Eq. (5′), and the Schrödinger equation for $|\Psi_3\rangle$ reduces to a set of linear equations for the C_j', for which the secular equation is

$$\begin{vmatrix} -\epsilon & 9I^{\frac{1}{2}} & 0 \\ I^{\frac{1}{2}} & 1-\epsilon & 10I^{\frac{1}{2}} \\ 0 & I^{\frac{1}{2}} & 2-\epsilon \end{vmatrix}=0. \tag{6}$$

The moment method of order 3 requires the solution of the following cubic determinantal equation:

$$\begin{vmatrix} 1 & \epsilon & \epsilon^2 & \epsilon^3 \\ 1 & 0 & 9I & 9I \\ 0 & 1 & 1 & (1+19I) \\ 1 & 1 & (1+19I) & (1+58I) \end{vmatrix}=0. \tag{7}$$

These two equations are of the same degree, and can be shown directly to be equivalent.

In higher orders, the moment method corresponds to a restricted Tamm-Dancoff approximation. For example, the $n=4$ state vector again has the simple form

$$|\Psi_4\rangle=\sum_{j=0}^{3} C_j''(A^\dagger)^j|0\rangle. \tag{5$''$}$$

Since

$$A(A^\dagger)^3|0\rangle=[11(A^\dagger)^2+8a_{i\alpha}^\dagger a_{i\alpha}^\dagger]|0\rangle, \tag{8}$$

the two-meson part of the Schrödinger equation cannot be exactly satisfied. If we consistently retain terms like

$$(A^\dagger)^2|0\rangle=[\delta_{ij}\delta_{\alpha\beta}-\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\sigma_k\tau_\gamma]a_{i\alpha}^\dagger a_{j\beta}^\dagger|0\rangle, \tag{9}$$

and neglect orthogonal terms of the form

$$[\delta_{ij}\delta_{\alpha\beta}+\tfrac{1}{4}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\sigma_k\tau_\gamma]a_{i\alpha}^\dagger a_{j\beta}^\dagger|0\rangle, \tag{10}$$

we obtain the moment method results.