Acnodes for Pion-Nucleon Scattering

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An implicit expression is found for the occurrence of acnodes on the Landau curves of a certain fifth-order Feynmann diagram. These acnodes are investigated in cases in which the lines of the graph are either pions or nucleons and in these cases are shown to give no singularities on the physical sheet. To find the Landau curves the internal momentum vectors of the diagram have been parameterized.

If the contribution to the scattering amplitude of a single perturbation diagram is studied as a function of a set of independent scalar invariants, it can be shown that all of the singularities occur on a curve, the Landau curve.[1] It was originally supposed that the Landau curves were themselves free of cusps and nodes. Recently an example has been discovered in which the Landau curve has a pair of isolated real points (acnodes) connected by a complex surface.[2] The existence of the acnodes and the complex surface can for certain masses lead to a failure of the Mandelstam representation. The example (Fig. 1) was only studied under the assumptions that $m_1=m_3$, $m_2=m_4$, $M_1=M_2$, and $m=\tilde{m}$. In the present work, the existence of acnodes for the general case of nine distinct masses is explored and an examination is made of the validity of the Mandelstam representation for some pion-nucleon processes.

In the dual diagram of Fig. 1 we parameterize the internal vectors as follows:

$$\begin{aligned}
q_1&=m_1(\cos\phi_\alpha\cos(\alpha+\psi_\alpha),\ \sin\phi_\alpha\cos(\alpha+\psi_\alpha),\ \sin(\alpha+\psi_\alpha)),\\
q_2&=-m_2(\cos\phi_\alpha\cos(\alpha-\psi_\alpha),\ \sin\phi_\alpha\cos(\alpha-\psi_\alpha),\ \sin(\alpha-\psi_\alpha)),\\
q_3&=m_3(\cos\phi_\beta\cos(\beta+\psi_\beta),\ \sin\phi_\beta\cos(\beta+\psi_\beta),\ \sin(\beta+\psi_\beta)),\\
q_4&=-m_4(\cos\phi_\beta\cos(\beta-\psi_\beta),\ \sin\phi_\beta\cos(\beta-\psi_\beta),\ \sin(\beta-\psi_\beta)),\\
q_5&=m_5(0,0,1);
\end{aligned}\tag{1}$$

where

$$\begin{aligned}
\sin\psi_\alpha&=\left(\frac{(m_1+m_2)^2-m^2}{4m_1m_2}\right)^{\frac{1}{2}},\\
\sin\psi_\beta&=\left(\frac{(m_3+m_4)^2-\tilde{m}^2}{4m_3m_4}\right)^{\frac{1}{2}},\\
\cos\psi_\alpha&=\left(\frac{m^2-(m_1-m_2)^2}{4m_1m_2}\right)^{\frac{1}{2}},\\
\cos\psi_\beta&=\left(\frac{\tilde{m}^2-(m_3+m_4)^2}{4m_3m_4}\right)^{\frac{1}{2}}.
\end{aligned}\tag{2}$$

With this parametrization all the conditions for the Landau curves are fulfilled except

$$M_1^2=(q_2+q_3+q_5)^2 \quad\text{and}\quad M_2^2=(q_1+q_4+q_5)^2. \tag{3}$$

There are three independent parameters α, β, and $\phi=\phi_\alpha-\phi_\beta$ of which two are fixed to satisfy Eq. (3). The conventional energy and momentum transfer variables s and t are given by

$$\begin{aligned}
s&=(q_1+q_3+q_5)^2,\\
t&=(q_2+q_4+q_5)^2,
\end{aligned}\tag{4}$$

and as functions of the one remaining parameter specify the Landau curve.

To find an acnode connected to a complex surface we permit α to become $\alpha+i\gamma$, β to become $\beta+i\delta$, and $\cos\phi$ (which is the only ϕ dependence) to become $x+iy$, and require that M_1^2, M_2^2, s, and t be real. The imposition of these requirements results in the equations

$$y=0, \tag{5a}$$

$$A\sinh\gamma+B\sinh\delta=0, \tag{5b}$$

$$C(\sinh^2\delta-\sinh^2\gamma)=D\sinh^2\gamma\cosh\delta-E\sinh^2\delta\cosh\gamma, \tag{5c}$$

$$\begin{aligned}
x&=\frac{\sinh\gamma\cosh\delta}{\cosh\gamma\sinh\delta}+\frac{D}{C}\frac{\sinh\gamma}{\cosh\gamma\sinh\delta}\\
&=\frac{\cosh\gamma\sinh\delta}{\sinh\gamma\cosh\delta}+\frac{E\sinh\delta}{C\sinh\gamma\cosh\delta};
\end{aligned}\tag{5d}$$

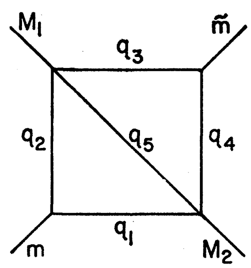

FIG. 1. The scattering diagram. The mass of the line q_i is m_i.

[1] L. D. Landau, Nuclear Phys. **13**, 181 (1959).
[2] R. J. Eden, P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor, J. Math. Phys. **2**, 656 (1961).

TABLE I. A list of the eight pion-nucleon graphs having the structure of Fig. 1. M is the nucleon mass and μ the meson mass. Blank spaces are indicative of ignorance at the present time.

									Acnodes		
m_1	m_2	m_3	m_4	m_5	M_1	M_2	m	$\tilde{m}$	General	$r=\pm 1$	$A=B=0$
M	μ	μ	M	μ	μ	μ	M	M	None	$s=t=3(\mu^2-M^2)$	None
M	M	M	M	M	M	M	μ	μ	None	$s=t=0$ and $s^2+t^2=2(M^2+\mu^2)$	None
μ	M	μ	M	M	μ	μ	M	M	None		None
M	μ	M	M	μ	M	μ	M	μ	Probably none	None	
μ	M	M	M	M	M	μ	M	μ	Probably none	None	
μ	M	M	μ	M	M	M	M	M	None		
μ	M	μ	M	μ	M	M	M	M	None		None
M	M	M	M	μ	μ	μ	μ	μ	$s=t=2\mu^2$		If $M_1{}^2=M_2{}^2=(1+\mu^2/8M^2)\mu^2$ there is an acnode

where

$$
\begin{aligned}
A&=\sin 2\psi_\alpha[\sin(\beta+\psi_\beta)/m_4+\sin(\beta-\psi_\beta)/m_3],\\
B&=\sin 2\psi_\beta[\sin(\alpha+\psi_\alpha)/m_2+\sin(\alpha-\psi_\alpha)/m_1],\\
C&=\sin 2\psi_\alpha \sin 2\psi_\beta,\\
D&=m_5 \sin 2\psi_\alpha[\cos(\beta+\psi_\beta)/m_4+(\beta-\psi_\beta)/m_3],\\
E&=m_5 \sin 2\psi_\beta[\cos(\alpha+\psi_\alpha)/m_2+(\alpha-\psi_\alpha)/m_1].
\end{aligned}
\tag{6}
$$

Unless $A=\pm B$, or $A=B=0$. It is possible to use Eqs. (5b), (5c), and (5d) to solve for $\cosh\gamma$, $\cosh\delta$, and x in terms of α and β. Then by varying α and β between 0 and 2π to find the values of $M_1{}^2$, $M_2{}^2$, s, and t for which acnodes exist. If the ratio $r=A/B$ is equal to 0, ± 1, or $\pm\infty$ we shall speak of special acnodes. If all the lines in Fig. 1 are restricted to be pion or nucleon lines and baryon and parity conservation are imposed there are eight possible graphs that are listed in Table I, together with the types of acnodes that they have.

We first consider the general acnodes. These occur only for the case of the pion-pion scattering graph at s and t equal two meson masses squared. This acnode is not singular on the physical sheet. The $r=\pm 1$ acnodes exist for most of the symmetric diagrams. Of these, the first two cases of Table I have been completely worked out analytically and seem to indicate what type of situation may arise, both have acnodes; the first is at $s=t=3(\mu^2-M^2)$, which is in a region in which the denominator of the relevant integral is negative and hence this acnode is not singular on the physical sheet. The second diagram has an acnode at $s=t=0$. It also has a new type of phenomenon in the Landau surface which could conceivably be troublesome by introducing singularities into the enlarged Symanzik or Euclidean region.[3] This piece of the Landau curve is a line $s+t=2M^2+2\mu^2$ with a gap from

$$s=(M-\mu),\quad t=(M+\mu)^2$$

to

$$s=(M+\mu)^2,\quad t=(M-\mu)^2.$$

Along this line s and t are parameterized by complex parameters although they are real. In the present case this line is not singular on the physical sheet. Finally, the $A=B=0$ acnodes are singular as discussed by Eden *et al.*,[1] but not for the pion-nucleon mass ratio.

The fourth and fifth diagrams on the list are unsymmetrical and thus the general acnodes were treated numerically. Within the limits of numerical accuracy, there are no acnodes for the pion-nucleon mass ratio. The remaining $r=0$, ± 1, cases have been briefly investigated and do not appear to be qualitatively different from those reported on. We thus conclude that although the Landau curve has a number of acnodes, none of them are singular on the physical sheet, and the Mandelstam representation remains valid to this order.

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[3] Richard J. Eden, Phys. Rev. **121**, 1567 (1961).