section for $p-n$ scattering. At high energies one obtains

$$\sigma_{\text{exchange}} \simeq 8\pi\left(\frac{g_{\rho NN}^2}{4\pi\hbar c}\right)^2\left(\frac{\hbar}{\mu_\rho c}\right)^2 \simeq 0.6 \text{ mb}. \tag{4}$$

Experimentally this cross section is observed to be less than 1.5 mb,[10] but its determination is obviously rather difficult. One could thus conclude that the small ρ-nucleon coupling strength at $\mathbf{k}^2=0$ is responsible for the extremely small high-energy nucleon-nucleon charge-exchange scattering (due primarily to the exchange of a virtual charged ρ meson).

[10] C. H. Tsao, J. G. Parks, and J. J. Lord, Bull. Am. Phys. Soc. **6**, 343 (1961).

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Pion Production in the Low-Energy Limit*

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The low-energy production reaction $\pi+N\rightarrow\pi+\pi+N$ is discussed in the framework of a simplified kinematics. The discussion is based on the Khuri-Treiman representation for processes with a three-particle final state. Under the assumption that the pion-pion rescattering term is dominant, branching ratios are derived for various processes. Under the same assumption, a quantitative analysis is given for the effect of the final-state pion-pion interaction on the three-particle distribution functions.

I. INTRODUCTION

MANY attempts have been made to treat the process of single pion production in pion-nucleon collisions. However, the difficulty in studying kinematics and analytic properties has kept us from going any further than using crude approximations. Among many such attempts, the Chew-Low static model has been used by several authors in calculating the total cross section[1] and, recently, in estimating strength of the pion-pion interaction.[2] But all these authors end up with the unique conclusion that a more satisfactory theory is desired.

In this note, we review the kinematical situation and propose a reasonable approximation scheme for low energies. For a fixed total energy, the distribution function for the three-particle final state is studied. We first replace the transition amplitude by that of the process in which the composite particle with its mass equal to the total energy in the center-of-mass system decays into the final state of our interest. For the decay amplitude of that particle, a Khuri-Treiman type spectral representation is adopted.[3] In this mathematical model, we shall study the effect of the final-particle interactions on the statistical distribution of the three-body final state.

In Sec. II, the kinematical situation is investigated. It is shown that, in the low-energy limit, the incoming beam interacts only through the $p_{\frac{1}{2}}$ channel. It is shown further that the spin structure of the transition amplitude is identical to that of the decay amplitude for a particle with spin $\frac{1}{2}$, even parity, and mass equal to the total center-of-mass energy of the process. In Sec. III, we adopt the view that the production amplitude for the present problem can be replaced by the decay amplitude just introduced. For the decay amplitude, a Khuri-Treiman type spectral representation is adopted. In the approximation of retaining only the lowest-mass intermediate states and under the assumption that the pion-pion interaction is much stronger than that of pion-nucleon, it is shown that the spectral representation corresponds to a rescattering amplitude in which the two final-state pions interact with each other. In Sec. IV, a system of soluble integral equations is derived and solved. Branching ratios are derived for various processes. In Sec. V, scattering-length approximation is made and the decay amplitudes are written in closed forms. In Sec. VI, statistical sum is made over the phase spaces of the three final-state particles. Both the purely statistical distribution and the dynamical deviation are discussed. In Sec. VII, our quantitative results are compared with the low energy data of Batusov *et al.*[4]

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[1] P. Carruthers, thesis, Cornell University, 1960 (unpublished).
[2] C. J. Goebel and H. J. Schnitzer, Phys. Rev. **123**, 1021 (1961).
[3] N. N. Khuri and S. B. Treiman, Phys. Rev. **119**, 1115 (1960).
[4] Yu. A. Barusov, S. A. Bunyatov, V. M. Sidorov, and Y. A. Yarba, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), p. 76.

II. LOW-ENERGY APPROXIMATION OF THE PRODUCTION AMPLITUDES

Let p and p', respectively, be the four-momenta of the initial and final nucleons, and k, k', and k'' be those of the initial and the two final mesons. In the center-of-mass system, where $\mathbf{p}+\mathbf{k}=0$, the momenta of the three outgoing particles are in the same plane. Consider now an orthogonal coordinate system in which the three final momenta are in the xz plane, the direction of the vector $\mathbf{p}'$ being in the z axis. The direction of the incoming beam can now be specified by the conventional polar angles (θ,ϕ). We choose these two angles, the total energy $W=(p_0+k_0)$, and two of the following three quantities:

$$s_a=-(p+k-p')^2, \quad s_b=-(p+k-k')^2, \quad s_c=-(p+k-k'')^2, \tag{1}$$

as the five independent variables.

Near the threshold energy, $p_0+k_0=m+2\mu$, m, and μ, respectively, being nucleonic and pionic masses, the final-state momenta are vanishingly small. In this work, we restrict ourselves to the cases where the total energy is so low that the final-state angular momenta are all in S states and that

$$(p_0+m)|\mathbf{p}'|/(p_0'+m)|\mathbf{p}|\ll 1. \tag{2}$$

In this approximation scheme, the production amplitude does not depend on the angle variables θ and ϕ.

Now consider the following four independent four-vectors.

$$V_1=(p+p'), \quad V_2=(p-p'), \quad V_3=(k'+k''), \quad V_4=(k'-k''). \tag{3}$$

Using the above quantities together with standard forms of the sixteen independent γ matrices, one can write a general and invariant form of the production amplitude. However, according to our basic assumptions, we can rule out the terms which are dependent on θ and/or ϕ in the center-of-mass system. Then the production amplitude takes the form

$$M=\bar{u}(p')\{A'-\tfrac{1}{2}i\gamma\cdot(k'+k'')B'\}\gamma_5 u(p). \tag{4}$$

Using the inequality of Eq. (2), one can show very easily that

$$\bar{u}(p')\{[-i\gamma\cdot(k+p)+W]/2W\}\gamma_5 u(p) = \bar{u}(p')\gamma_5 u(p). \tag{5}$$

Using the above relation, one can show easily that the B' term in Eq. (4) is not necessary and M can, in fact, be written as

$$M=\bar{u}(p')\gamma_5 u(p)A'. \tag{6}$$

Again using Eq. (5), we can show

$$M=\bar{u}(p')A'\left\{-\frac{1}{2}\left[\frac{(W-m)^2-\mu^2}{mW}\right]^{\frac{1}{2}}\right\}u(p^*), \tag{7}$$

where $p^*=p+k$, and

$$u(p^*)=-2\{mW/[(W-m)^2-\mu^2]\}^{\frac{1}{2}} \times\{[-i\gamma\cdot(p+k)+W]/2W\}\gamma_5 u(p). \tag{8}$$

The quantity $u(p^*)$ satisfies all the properties of the spinor of a particle with mass W and four-momentum p^*, that is,

$$(i\gamma\cdot p^*+W)u(p^*)=0, \quad \bar{u}(p^*)u(p^*)=1. \tag{9}$$

Next we consider the decay of the above particle with spin $\frac{1}{2}$ and even parity into the three-particle final state of the present problem. It can be shown easily that the decay amplitude T takes the following form in our approximation scheme.

$$T=\bar{u}(p')Au(p^*), \tag{10}$$

where A is a function of any two of the three invariants s_a, s_b, and s_c.

Now it is seen that both M and T have the same spin structure and that, in the center-of-mass system, they have the form

$$\chi_f^\dagger\mathbf{p}\cdot\boldsymbol{\sigma}\chi_i, \tag{11}$$

where the spin operator $\mathbf{p}\cdot\boldsymbol{\sigma}$ is entirely contained in the spinor $u(p^*)$; χ_i and χ_f are, respectively, the Pauli spinors for the initial and final nucleons. We shall replace M by T and attack the problem in the framework of the Khuri-Treiman type dispersion relation.[3]

III. SPECTRAL REPRESENTATION

Following the line adopted by Khuri and Treiman[3] and investigated in perturbation theory by Barton and Kacser,[5] we write the following spectral representation,

$$T(s_a,s_b,s_c) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} ds_a'\frac{\Phi_a(s_a',s_c)}{s_a'-s_a-i\epsilon}+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds_b'\frac{\Phi_b(s_b',s_c)}{s_b'-s_b+i\epsilon} +\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds_c'\frac{\Phi_b(s_c',s_b)}{s_c'-s_c+i\epsilon}+\frac{G_b}{s_b-m^2}+\frac{G_c}{s_c-m^2}, \tag{12}$$

where

$$\begin{aligned}\Phi_a&=\tfrac{1}{2}(2k_0''p_0^*/W)^{\frac{1}{2}}(2\pi)^4\textstyle\sum_n\langle k''|j(0)|n\rangle\bar{u}(p')\times\langle n|f(0)|p^*\rangle\delta(p_n-k'-k''),\\ \Phi_b&=\tfrac{1}{2}(2k_0''p_0^*/W)^{\frac{1}{2}}(2\pi)^4\textstyle\sum_n\bar{u}(p')\langle k''|f(0)|n\rangle\times\langle n|j(0)|p^*\rangle\delta(p_n-p^*+k'),\\ \Phi_c&=\tfrac{1}{2}(2k_0'p_0^*/W)^{\frac{1}{2}}(2\pi)^4\textstyle\sum_n\bar{u}(p')\langle k'|f(0)|n\rangle\times\langle n|j(0)|p^*\rangle\delta(p_n-p^*+k'');\end{aligned} \tag{13}$$

G_b and G_c are constants. $j(0)$ and $f(0)$ are, respectively, pion and nucleon currents. In order to maintain the proper reality conditions at all stages of approximation, it is convenient to write the above sum over states

[5] G. Barton and C. Kacser (to be published).

$|n\rangle$ as one-half the sum over "out" plus "in" states. Throughout the following discussion, we shall understand this without further mention.

Now we study the analytic property of the amplitude T as a function of s_a for fixed s_c or s_b. In each case, the poles generated by the last two terms are located around $s_a = s\mu^2 + 2m\mu$, which is, in the present approximation scheme, far above the region of interest. Thus we ignore those poles.

For the rest, we make an approximation of retaining contributions only from the lowest-mass states, that is, the two pion intermediate state for the first expansion and the state of a single nucleon and one pion for the second and third terms. Each of these three has a concrete physical interpretation. In the sense of dispersion diagram, the first term represents the decay process with two interacting final mesons. The second term, on the other hand, represents the final-state interaction of the nucleon and the pion k''. Similarly, the third expansion refers to the interaction of the nucleon and meson k''. The amplitude T, therefore, is a sum of the three competing final-state-interaction terms. We now assume that the pion-pion rescattering term dominates in the low-energy limit and ignore the pion-nucleon terms completely, that is, we write the dispersion representation of Eq. (12) as

$$T = \frac{1}{\pi} \int_{4\mu^2} ds_a' \frac{\Phi_a(s_a', s_c)}{s_a' - s_a - i\epsilon}. \tag{14}$$

In applying this form, one has to consider a possible need for subtraction.

In the Appendix, a detailed discussion is given for isospin formulation of the production process. In the following section, we shall derive, from the preceding formalism, a system of coupled integral equations for the s- and p-wave amplitudes.

IV. INTEGRAL EQUATIONS

We assign isospin indices α, β and γ, respectively, to the mesons denoted by their four-momenta k, k', and k'' and label the transition amplitude T as

$$T_{\beta\gamma\alpha} = \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds_a' \frac{\Phi_{\beta\gamma\alpha}(s_a')}{s_a' - s_a - i\epsilon}, \tag{15}$$

where

$$\begin{aligned}\Phi_{\beta\gamma\alpha} = \tfrac{1}{2}(2k_0''p_0^*/W)^{\frac{1}{2}}(2\pi)^4 &\times \sum_{\beta'\gamma'} \int \frac{d^3q'd^3q''}{(2\pi)^6} \langle k''(\gamma) | j_\beta(0) | q'(\beta')q''(\gamma')\rangle \\ &\times \bar{u}(p')\langle q'(\beta')q''(\gamma') | f(0) | p^*(\alpha)\rangle \\ &\times \delta(q'+q''-k'-k''); \end{aligned} \tag{16}$$

$q'(\beta')$ and $q''(\gamma')$ are the four-momenta (isospin) of the intermediate-state mesons.

Now $\Phi_{\beta\gamma\alpha}$ can be written as

$$\begin{aligned}\Phi_{\beta\gamma\alpha} = (4\pi^2)^{-1}(k_0''/2q_0'q_0'')^{\frac{1}{2}} &\times \sum_{\beta'\gamma'} \int d^3q'd^3q''\, \delta(q'+q''-k'-k'') \\ &\times \mathrm{Re}[\langle k''(\gamma) | j_\beta(0) | q'(\beta')q''(\gamma')\mathrm{out}\rangle \\ &\times T_{\beta'\gamma'\alpha}(s_a', s_b', s_c')], \end{aligned} \tag{17}$$

where

$$s_a' = -(q'+q'')^2, \quad s_b' = -(p^*-q')^2, \quad s_c' = -(p^*-q'')^2.$$

We evaluate the above integral in the Lorentz frame where

$$\mathbf{k}' + \mathbf{k}'' = \mathbf{q}' + \mathbf{q}'' = 0. \tag{18}$$

In this frame

$$\begin{aligned}\langle k''(\gamma) | j_\beta(0) | q'(\beta')q''(\gamma')\mathrm{out}\rangle &= 16\pi k_0'(8k_0''q_0'q_0'')^{\frac{1}{2}} f_{\beta\gamma;\beta'\gamma'}^* \\ &= 16\pi k_0'(8k_0''q_0'q_0'')^{\frac{1}{2}} \\ &\times [\tfrac{1}{3}(f_0^* - f_2^*)\delta_{\beta\gamma}\delta_{\beta'\gamma'} + \tfrac{1}{2}f_2^*(\delta_{\beta'\beta}\delta_{\gamma'\gamma} + \delta_{\beta'\gamma}\delta_{\gamma'\beta}) \\ &+ \tfrac{1}{2}f_1^*(\delta_{\beta'\beta}\delta_{\gamma'\gamma} - \delta_{\beta'\gamma}\delta_{\gamma'\beta})], \end{aligned} \tag{19}$$

where f_t is the pion-pion scattering amplitude for the total isospin t. Let us notice that, according to the Appendix, the amplitude T takes the form

$$T_{\beta\gamma\alpha} = T_1\delta_{\beta\gamma}\tau_\alpha + T_3\delta_{\alpha\gamma}\tau_\beta + T^{(}\delta_{\alpha\beta}\tau_\gamma + iT_4\epsilon_{\alpha\beta\gamma}. \tag{20}$$

We define an orthogonal coordinate system in which the vector $\mathbf{p}'$ is in the z direction and the vector $\mathbf{k}'$ in the xz plane making an angle θ with the z axis. The direction of the vector $\mathbf{q}'$ can be specified by the polar angles (θ', ϕ'), the direction cosines being

$$(\sin\theta' \cos\phi', \ \sin\theta' \sin\phi', \ \cos\theta').$$

Retaining only the s- and p-wave amplitudes for the pion-pion scattering, one has

$$f_t = f_t^{(s)} + (3)^{\frac{1}{2}} f_t^{(p)} (\sin\theta \sin\theta' \cos\phi' + \cos\theta \cos\theta'). \tag{21}$$

Retaining only the constant and linear terms in $\cos\theta$, one can write the amplitude T_i as

$$T_i = T_i^{(s)}(s_a) + (3)^{\frac{1}{2}} T_i^{(p)}(s_a) \cos\theta. \tag{22}$$

Thus

$$\begin{aligned}\Phi_{\beta\gamma\alpha} = (4\pi)^{-1} \sum_{\beta'\gamma'} \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' &\times \mathrm{Re}\{[f_{\beta\gamma;\beta'\gamma'}^{(s)*}(s_a) + (3)^{\frac{1}{2}} f_{\beta\gamma;\beta'\gamma'}^{(p)*}(s_a) \\ &\times (\sin\theta \sin\theta' \cos\phi' + \cos\theta \cos\theta')] \\ &\times [T_{\beta'\gamma'\alpha}^{(s)}(s_a) + (3)^{\frac{1}{2}} T_{\beta'\gamma'\alpha}^{(p)}(s_a) \cos\theta']\}. \end{aligned} \tag{23}$$

For simplicity we denote s_a by s. Now the above integral can be evaluated easily.

$$\begin{aligned}\Phi_{\beta\gamma\alpha} = \sum_{\beta'\gamma'} \mathrm{Re}[&f_{\beta\gamma;\beta'\gamma'}^{(s)*}(s) T_{\beta'\gamma'\alpha}^{(s)}(s) \\ &+ f_{\beta\gamma;\beta'\gamma'}^{(p)*}(s) T_{\beta'\gamma'\alpha}^{(p)}(s) \cos\theta]. \end{aligned} \tag{24}$$

According to Bose statistics,

$$
\begin{aligned}
&T_1{}^{(p)}(s)=T_4{}^{(s)}(s)=0, \quad T_2{}^{(s)}(s)=T_3{}^{(s)}(s),\\
&T_2{}^{(p)}(s)=-T_3{}^{(p)}(s).
\end{aligned}
\tag{25}
$$

Thus the dispersion relation of Eq. (14) can be written as

$$
\begin{aligned}
T_1{}^{(s)}(s)&=\frac{1}{\pi}\int_{4\mu^2}^{\infty} ds' \frac{\mathrm{Re}[f_0{}^{(s)*}(s')T_1(s')+\frac{2}{3}[f_0{}^{(s)*}(s')-f_2{}^{(s)*}(s')]T_2{}^{(s)}(s')]}{s'-s-i\epsilon},\\
T_2{}^{(s)}(s)&=\frac{1}{\pi}\int_{4\mu^2}^{\infty} ds' \frac{\mathrm{Re}[f_2{}^{(s)*}(s')T_2{}^{(s)}(s')]}{s'-s-i\epsilon},\\
T_2{}^{(p)}(s)&=\frac{1}{3^{\frac{1}{2}}\pi}\int_{4\mu^2}^{\infty} ds' \frac{\mathrm{Re}[f_1{}^{(p)*}(s')T_2{}^{(p)}(s')]}{s'-s-i\epsilon},\\
T_4{}^{(p)}(s)&=\frac{1}{3^{\frac{1}{2}}\pi}\int_{4\mu^2}^{\infty} ds' \frac{\mathrm{Re}[2f_1{}^{(p)*}(s')T_2{}^{(p)}(s')]}{s'-s-i\epsilon}.
\end{aligned}
\tag{26}
$$

Although the p-wave equations can be and have been derived, they are quite academic in this low-energy limit. We henceforth ignore them and suppress the superscript "s" with an understanding that only the s-wave amplitudes are retained.

As a consequence of the relation in Eq. (25) and the above remark, the reaction amplitudes for the following five processes take relatively simple forms. See Table I.

TABLE I. Isospin amplitudes for the five processes.

Reactions	Isospin amplitudes
(I) $\pi^-+p\rightarrow\pi^-+\pi^++n$	$(\sqrt{2}/3)A_{\frac{1}{2}(0)}-(2/15)^{\frac{1}{2}}A_{\frac{3}{2}(2)} = -\sqrt{2}T_1+[2(3-2\sqrt{15})/3\sqrt{3}]T_2$
(II) $\pi^-+p\rightarrow n^-+\pi^0+p$	$(1/\sqrt{10})A_{\frac{3}{2}(2)}=-T_2$
(III) $\pi^-+p\rightarrow\pi^0+\pi^0+n$	$-(\sqrt{2}/3)A_{\frac{1}{2}(0)}-(2/3\sqrt{5})A_{\frac{3}{2}(2)} =\sqrt{2}T_1+(2\sqrt{2}/3)(1+\sqrt{10})T_2$
(IV) $\pi^++p\rightarrow\pi^++\pi^0+p$	$(1/\sqrt{10})A_{\frac{3}{2}(2)}=-T_2$
(V) $\pi^++p\rightarrow\pi^++\pi^++n$	$(2/\sqrt{5})A_{\frac{3}{2}(2)}=-2\sqrt{2}T_2$

$$
\begin{aligned}
&(\mathrm{I}) \quad \pi^-+p\rightarrow\pi^-+\pi^++n,\\
&(\mathrm{II}) \quad \pi^-+p\rightarrow\pi^-+\pi^0+p,\\
&(\mathrm{III}) \quad \pi^-+p\rightarrow\pi^0+\pi^0+n,\\
&(\mathrm{IV}) \quad \pi^++p\rightarrow\pi^++\pi^0+p,\\
&(\mathrm{V}) \quad \pi^++p\rightarrow\pi^++\pi^++n.
\end{aligned}
\tag{27}
$$

It is clear from Table I that branching ratios for the reactions (II), (III), (V) are 1:1:8. This result is a corollary of our assumption on the pion-pion scattering and is quite independent of the solutions of the above integral equations.

We now make one subtraction for the s-wave dispersion relation at the threshold $s=4\mu^2=s_0$.

$$
T_1(s)=g_1+\frac{(s-s_0)}{\pi}\int_{4\mu^2}^{\infty} ds' \frac{\mathrm{Re}[f_0{}^*(s')T_1(s')+\frac{2}{3}(f_0{}^*(s')-f_2{}^*(s')T_2(s')]}{(s'-s_0)(s'-s-i\epsilon)},
$$

and

$$
T_2(s)=g_2+\frac{(s-s_0)}{\pi}\int_{4\mu^2}^{\infty} ds' \frac{\mathrm{Re}[f_2{}^*(s')T_2(s')]}{(s'-s_0)(s'-s-i\epsilon)}. \tag{28}
$$

The quantities g_1 and g_2 are the amplitudes at the threshold and, therefore, are well-defined constants.

The second integral equation can be solved by an Omnes-like method,[6] and the solution is

$$
T_2(s)=g_2 \exp\left[\frac{(s-s_0)}{\pi}\int_{4\mu^2}^{\infty} ds' \frac{\delta_2(s')}{(s'-s_0)(s'-s-i\epsilon)}\right]. \tag{29}
$$

where $f_t=e^{i\delta_t}\sin\delta_t$.

[6] R. Omnes, Nuovo cimento **8**, 316 (1958).

In order to solve the first equation, we take the following linear combinations of T_1 and T_2:

$$
T_1(s)+\tfrac{2}{3}T_2(s)=(g_1+\tfrac{2}{3}g_2)+\frac{(s-s_0)}{\pi} \times\int_{4\mu^2}^{\infty} ds' \frac{\mathrm{Re}\{f_0{}^*(s')[T_1(s')+\frac{2}{3}T_2(s')]\}}{(s'-s_0)(s'-s-i\epsilon)}.
$$

This immediately leads to the solution for $T_1(s)$. Using Eq. (29), we obtain

$$T_1(s)=(g_1+\tfrac{2}{3}g_2)\times\exp\left[\frac{s-s_0}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{\delta_0(s')}{(s'-s_0)(s'-s-i\epsilon)}\right]-\tfrac{2}{3}g_2\times\exp\left[\frac{s-s_0}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{\delta_2(s')}{(s'-s_0)(s'-s-i\epsilon)}\right]. \quad (30)$$

V. SCATTERING-LENGTH APPROXIMATION

The amplitudes T_1 and T_2 have been expressed in terms of the pion-pion phase shifts δ_0 and δ_2. Since, however, the pion-pion amplitudes are not completely known, we cannot expect any quantitative result unless we make adequate approximations. In the following discussions, we shall make an approximation of retaining only the lowest-order terms in the variable s. In this scheme, the scattering-length parametrization is quite adequate for the pion-pion amplitudes.

Let us introduce two scattering lengths a_0 and a_2, defined as

$$f_t=ka_t/(\omega-ika_t), \quad (31)$$

where $\omega=(\mu^2+k^2)^{\frac{1}{2}}$, $k=|\mathbf{k}'|=|\mathbf{k}''|$.

Using the following parametrization, one can easily evaluate the integrals in Eqs. (29) and (30).

$$\frac{s-s_0}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{\delta_t(s')}{(s-s_0)(s'-s-i\epsilon)}=-\frac{2a_tk^2}{\pi}+ia_tk^2,$$

where only the lowest-order terms in k are retained in both the real and imaginary parts of the integrals. The momentum k is measured in μ.

Again retaining only the lowest-order terms in k, we obtain the following forms of T_1 and T_2:

$$T_1=g_1(1+ia_0k-2a_0k^2/\pi)+\tfrac{2}{3}g_2(a_0-a_2)(ik-2k^2/\pi), \quad (32)$$

$$T_2=g_2(1+ia_2-2a_2k^2/\pi). \quad (33)$$

The amplitudes T_1 and T_2 are functions of k and the pion-pion scattering length a_0 and a_2. If there were not the final-state interactions, that is, $a_0=a_2=0$, each amplitude would be reduced to the constant equaling that at the threshold, and the three-particle distribution function would be purely statistical. Now we are ready to examine how these amplitudes cause deviation from the purely statistical distribution.

VI. DISTRIBUTION FUNCTIONS

In order to evaluate the total rate of transition, we have to sum over the phase spaces of the three final-state particles. We shall first evaluate the phase-space integration in the Lorentz frame where $\mathbf{k}'+\mathbf{k}''=0$ and, then, Lorentz-generalize the distribution function. Throughout the discussion, all constant factors will be suppressed. Thus the rate R can be simply wrriten as

$$R=\int\frac{d^3p'd^3k'd^3k''}{p_0{}^*p_0{}'k_0{}'k_0{}''}|T|^2\delta(k'+k''+p'-p^*). \quad (34)$$

The rate R, of course, is dependent on the polarization of the initial and final nucleons and also on the direction of the initial beam. But these can be regarded as constant factors as far as the s dependence is concerned. Keeping this in mind, we calculate the distribution function $F(s)=dR/ds$.

After some elementary algebra, we obtain

$$R=\int_{4\mu^2}^{(W-m)2}dsF(s), \quad (35)$$

where

$$F(s)=|T(s)/s|^2\times[(s-4\mu^2)[s-(W-m)^2][s-(W+m)^2]]^{\frac{1}{2}}.$$

In the above expression for $F(s)$, as has been remarked before, the dynamical factor $|T(s)|^2$ introduces deviations from the purely statistical distribution.

To lowest order in k, the dynamical factor for the reactions (II), (IV), and (V) takes the form

$$|T|^2=[1-(a_2/\pi)(s-4\mu^2)]=[1+(b_1/4)(s-4\mu^2)],$$

or

$$b_1=-(4/\pi)a_2. \quad (36)$$

In the preceding expressions, the subtraction constant g_2 is completely absorbed by the multiplicative constants which are being ignored. In a similar way, we can write the dynamical factors for the reactions $\pi^-+p\rightarrow\pi^-+\pi^++n$ and $\pi^-+p\rightarrow\pi^0+\pi^0+n$, respectively, as $[1+(b_2/4)(s-4\mu^2)]$ and $[1+(b_3/4)(s-4\mu^2)]$, where

$$\begin{aligned}b_2=&(1.82+1.41f)^{-2}[(-0.56a_0{}^2-3.12a_2{}^2+3.68a_0a_2\\&+3.26a_2-7.51a_0)+(-0.56a_0{}^2-2.56a_2{}^2\\&+3.12a_0a_2-5.80a_0-0.72a_2)f-2.54a_0f^2],\\ b_3=&(3.92+1.41f)^{-2}[(-16.98a_0+7.05a_2-3.54a_0{}^2\\&-9.07a_2{}^2+1.54a_0a_2)+(-5.80a_0-3.25a_2\\&-3.54a_0{}^2-5.54a_2{}^2+9.08a_0a_2)f-2.55a_0f^2],\end{aligned} \quad (37)$$

and $f=3g_1/(2g_2)$.

Now one can measure the parameters b_1, b_2 and b_3 by inspecting the final-state spectra of any three reactions including II and III. It is then a matter of algebra to determine the three parameters a_0, a_2 and f. If $b=0$, then the spectral function is reduced to a purely statistical distribution. Figure 1 illustrates the b dependence of the distribution function at the total energy $W=m+(5/2)\mu$.

VII. COMPARISON WITH EXPERIMENT

As has been stated repeatedly, the present treatment is expected to be good only in the low-energy limit

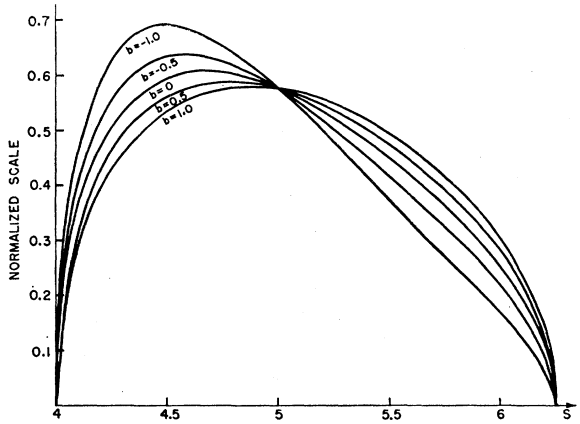

FIG. 1. Distribution functions for various values of b at the total energy $W=m+(5/2)\mu$.

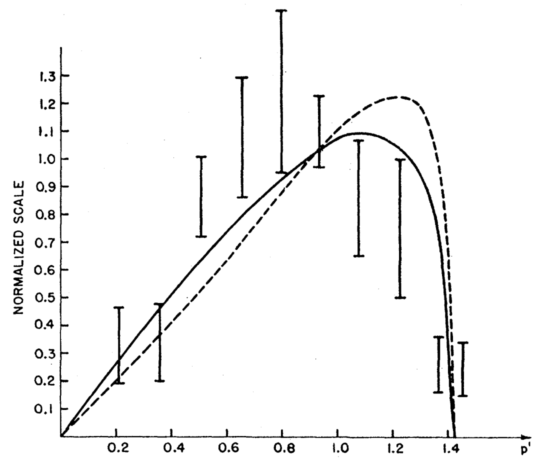

FIG. 3. Momentum distribution for the recoil nucleon. The solid line indicates the $b_2=1$ distribution and the dashed line the purely statistical distribution. The experimental data are from reference 4.

where the laboratory kinetic energy of the incoming pion is slightly above 171 Mev. However, the lowest energy for which the experimental data are available today is 280 Mev, where the total center-of-mass kinetic energy available in the final state is 0.6μ. Although the final-state particles cannot be regarded as completely nonrelativistic at this energy, the kinematical deviation is still insignificant. We shall compare here the calculated results of the present work with the experimental data of Batusov *et al.*[4] on the reaction $\pi^-+p\rightarrow\pi^-+\pi^++n$. The following comparisons are made in the center-of-mass system.

First of all, the angular distributions of the final-state particles in this model are completely isotropic with respect to the incoming beam if polarization is not considered. According to the experimental data, the final-state pions show a strong isotropy, but the recoil nucleon seems to have a significant contribution from anisotropic components. However, the present data are not accurate enough to allow us to determine strength of any partial-wave component.

As has been noticed before, the reaction of Batusov *et al.* alone cannot determine all three quantities b_1, b_2, and b_3. However, their result is to be a measure of b_2. As in reference 3, we regard the values of the pion-pion scattering lengths to be $a_0=-1.0$ and $a_2=-0.3$ in units of the pion Compton wavelength and compute the value of b_2 for a wide range of f. It is shown in Fig. 2 that, except at and near the discontinuous point, the value of b_2 is about 1. For this value of b_2, we have plotted the momentum distribution (see Fig. 3). Also shown in the same graph is the purely statistical distribution. Here again, no definite conclusion can be drawn either for or against the present theoretical result.

Also available from the Batusov experiment is the angular distribution of the final-state pions with respect to the recoil nucleon. Both the purely statistical distribution and that for $b_2=1$ have been plotted in Fig. 4. The experimental data, however, are not accurate enough for a definite conclusion.

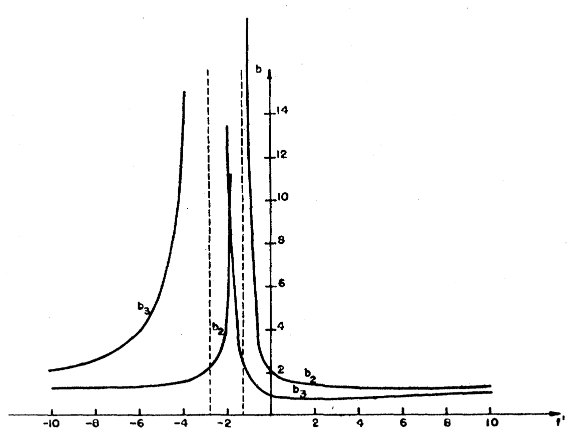

FIG. 2. b_2 and b_3 vs f. Except at and near the discontinuity point, the value of b_2 is about 1.

VII. CONCLUSIONS

The Khuri-Treiman representation has been used in the study of the pion production in the low-energy limit. As a corollary of our physical assumption that the pion-pion rescattering term dominates the low-energy behavior, the branching ratios have been derived for the reactions (II), (IV), and (V).

A quantitative analysis has been carried out with the result that the low-energy amplitudes can be written as functions of the pion-pion scattering parameters. Effect of the pion-pion interaction has been discussed.

We have made some quantitative comparisons with the experimental results of Batusov *et al.*[4] But no definite information can be obtained from the existing data. The predictions of the present work, nevertheless, can be tested in principle.

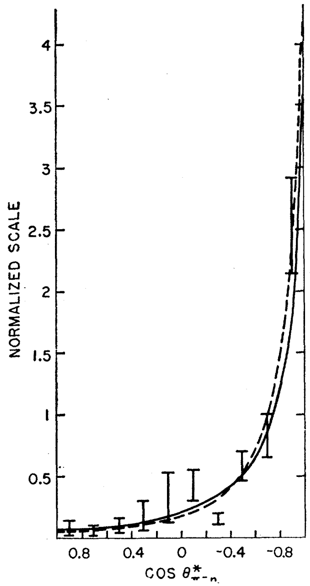

FIG. 4. Angular distribution of the final-state pions with respect to the recoil nucleon in the center-of-mass system. The solid line corresponds to the $b_2=1$ distribution and the dashed line to the purely statistical distribution. The experimental data are from reference 4.

ACKNOWLEDGMENTS

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APPENDIX

Addition of isospin can certainly be handled by the use of Glebsch-Gordan coefficients. However, we can simplify the algebra by extending the conventional method to the projection operator formalism.

In this problem, the total isospin of the initial state, I_i, is either $\frac{1}{2}$ or $\frac{3}{2}$. Thus that of the final state, I_f, is either $\frac{1}{2}$ or $\frac{3}{2}$. But the complete description of the three-particle system requires an intermediate isospin value of two of the three particles. We first adopt the representation where the two pions are coupled to give the intermediate isospin $\mathbf{t}$, which, then, is coupled to the nucleon isospin.

The initial state with $I_i=\frac{3}{2}$ goes to the $I_f=\frac{3}{2}$ final state, which has two possibilities for t: $t=1$ or 2. We denote these amplitudes by $A_{\frac{3}{2}(1)}$ and $A_{\frac{3}{2}(2)}$ respectively. Similarly, we introduce the amplitudes $A_{\frac{1}{2}(0)}$ and $A_{\frac{1}{2}(1)}$. The total amplitude, then, takes the form

$$\begin{aligned} A=\sum_{m=-\frac{3}{2}}^{\frac{3}{2}} &[A_{\frac{3}{2}(1)}|I_f=\tfrac{3}{2}, m, t=1\rangle\langle I_i=\tfrac{3}{2}, m| \\ &+A_{\frac{3}{2}(2)}|I_f=\tfrac{3}{2}, m, t=2\rangle\langle I_i=\tfrac{3}{2}, m|] \\ &+\sum_{m=-\frac{1}{2}}^{\frac{1}{2}}[A_{\frac{1}{2}(0)}|I_f=\tfrac{1}{2}, m, t=0\rangle\langle I_i=\tfrac{1}{2}, m| \\ &+A_{\frac{1}{2}(1)}|I_f=\tfrac{1}{2}, m, t=1\rangle\langle I_i=\tfrac{1}{2}, m|]. \end{aligned} \tag{A1}$$

where m represents the third component of the total isospin. Next, one can simply evaluate the above quantity to obtain

$$A=A_{\beta\gamma\alpha}=A_1\delta_{\beta\gamma}\tau_\alpha+A_2\delta_{\alpha\gamma}\tau_\beta+A_3\delta_{\alpha\beta}\tau_\gamma+iA_4\epsilon_{\alpha\beta\gamma}, \tag{A2}$$

where α, β, and γ, respectively, refer to the isospin states of the mesons labeled by their four-momenta k, k', and k''; τ_α, τ_3, and τ_γ are conventional Pauli isospin matrices. $A_1, \cdots, A_4$ are linearly related to the isospin amplitudes by

$$\begin{aligned} A_1&=(-1/3)A_{\frac{1}{2}(0)}+(2/3)A_{\frac{3}{2}(2)}, \\ A_2&=(1/3\sqrt{2})\{A_{\frac{1}{2}(1)}-A_{\frac{3}{2}(1)}\}+(-1/\sqrt{10})A_{\frac{3}{2}(2)}, \\ A_3&=(-1/3\sqrt{2})\{A_{\frac{1}{2}(1)}-A_{\frac{3}{2}(1)}\}+(-1/\sqrt{10})A_{\frac{3}{2}(2)}, \\ A_4&=(1/3\sqrt{2})\{A_{\frac{1}{2}(1)}+2A_{\frac{3}{2}(1)}\}. \end{aligned} \tag{A3}$$

Next, we adopt the representation where the pion β and the nucleon are coupled to give an intermediate isospin $\mathbf{T}$, which, then, is coupled to the pion γ. In this case, the four amplitudes are $B_{\frac{3}{2}(\frac{3}{2})}$, $B_{\frac{3}{2}(\frac{1}{2})}$, $B_{\frac{1}{2}(\frac{3}{2})}$, and $B_{\frac{1}{2}(\frac{1}{2})}$, where the first subscript refers to the total isospin and the one in parenthesis to the intermediate pion-nucleon isospin. Now we write the projection-operator amplitude as

$$\begin{aligned} A=\sum_{m=-\frac{3}{2}}^{\frac{3}{2}} &[B_{\frac{3}{2}(\frac{3}{2})}|I_f=\tfrac{3}{2}, m, T=\tfrac{3}{2}\rangle\langle I_i=\tfrac{3}{2}, m| \\ &+B_{\frac{3}{2}(\frac{1}{2})}|I_f=\tfrac{3}{2}, m, T=\tfrac{1}{2}\rangle\langle I_i=\tfrac{1}{2}, m|] \\ &+\sum_{m=-\frac{1}{2}}^{\frac{1}{2}}[B_{\frac{1}{2}(\frac{3}{2})}|I_f=\tfrac{1}{2}, m, T=\tfrac{3}{2}\rangle\langle I_i=\tfrac{1}{2}, m| \\ &+B_{\frac{1}{2}(\frac{1}{2})}|I_f=\tfrac{1}{2}, m, T=\tfrac{1}{2}\rangle\langle I_i=\tfrac{1}{2}, m|]. \end{aligned} \tag{A4}$$

The right side of the above expression can also be reduced to the form of Eq. (A2), and

$$\begin{aligned} A_1=(1/3\sqrt{3})B_{\frac{1}{2}(\frac{1}{2})}&-(2/3\sqrt{6})B_{\frac{1}{2}(\frac{3}{2})} \\ &-(1/2\sqrt{3})B_{\frac{3}{2}(\frac{1}{2})}-(1/3\sqrt{15})B_{\frac{3}{2}(\frac{3}{2})}, \\ A_2=(1/3\sqrt{3})B_{\frac{1}{2}(\frac{1}{2})}&+(1/3\sqrt{6})B_{\frac{1}{2}(\frac{3}{2})} \\ &+(1/\sqrt{3})B_{\frac{3}{2}(\frac{1}{2})}-(1/3\sqrt{15})B_{\frac{3}{2}(\frac{3}{2})}, \\ A_3=(-1/3\sqrt{3})B_{\frac{1}{2}(\frac{1}{2})}&-(1/3\sqrt{6})B_{\frac{1}{2}(\frac{3}{2})} \\ &+(1/2\sqrt{3})B_{\frac{3}{2}(\frac{1}{2})}+(4/3\sqrt{15})B_{\frac{3}{2}(\frac{3}{2})}, \\ A_4=(1/3\sqrt{3})B_{\frac{1}{2}(\frac{1}{2})}&+(1/3\sqrt{6})B_{\frac{1}{2}(\frac{3}{2})} \\ &-(1/2\sqrt{3})B_{\frac{3}{2}(\frac{1}{2})}+(5/3\sqrt{15})B_{\frac{3}{2}(\frac{3}{2})}. \end{aligned} \tag{A5}$$