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Spin Susceptibility of Normal Fermion Systems*

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The calculation of the induced spin density of a normal fermion system, such as the electron gas, in the limit of small wave numbers, is carried beyond the random phase approximation (R.P.A.) by the method of canonical transformation so as to include the first non-R.P.A. corrections. An expression for the induced spin density in the same limit, exact to all orders of particle-particle coupling, is then derived by more general methods of many-body perturbation theory. It depends only on the knowledge, to all orders of interparticle coupling, of the effective mass and the forward scattering of quasi-particles at the Fermi surface, and also yields the results of the extended R.P.A. on appropriate expansion. In the limit of zero wave number, the resulting expression for the magnetic susceptibility is found identical with that deduced by Landau from a phenomenological basis and may be regarded as an additional confirmation of the microscopic validity of the theory of the Fermi liquid.

I. INTRODUCTION

IN this article we study the response of an interacting many-fermion system to an applied, spatially varying, magnetic field, using the methods of many-body perturbation theory developed by the author in prior communications on the electron gas[1] and the problem of the inertial moment.[2,3] Initially, the results of the random phase approximation (R.P.A.) (for simplicity we have assumed a gas of interacting electrons) in the calculation of the induced spin density $\langle\sigma_z(\mathbf{q})\rangle$ for small momentum transfers,[4] $\mathbf{q}$, are reproduced, since they provide the key to the calculation of the first non-random-phase corrections. These latter corrections are then presented in some detail as an indication of the existence of a more general result for the induced spin density of normal fermion systems in the limit $q/k_F\ll 1$. We finally derive this general result, an expression for $\langle\sigma_z(\mathbf{q})\rangle$, for $q/k_F\ll 1$, correct to all orders of particle-particle coupling. For small wave numbers, the induced spin density is found to depend only on the knowledge, although to all orders of particle-particle coupling, of two quantities, the exact effective mass and the forward scattering amplitude of quasi-particles at the Fermi surface (the last quantity is understood in the limit derived by Landau[5]). Further, the magnetic susceptibility obtained from the induced spin density, by considering the limit $q\rightarrow 0$, is identical with that deduced by Landau[6,7] from semiphenomenological arguments, so that this last result may be regarded as an additional confirmation of the microscopic validity of his theory of the Fermi liquid.[6,7]

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[1] R. M. Rockmore, Phys. Rev. **114**, 941 (1959).
[2] R. M. Rockmore, Phys. Rev. **116**, 469 (1959).
[3] R. M. Rockmore, Phys. Rev. **120**, 1933 (1960) and **124**, 27 (1961). These papers are referred to as I and II in the text.
[4] P. A. Wolff, Phys. Rev. **120**, 814 (1960).
[5] L. D. Landau, J. Exptl. Theoret. Phys. (USSR) **35**, 97 (1958) [translation: Soviet Phys.—JETP **8**, 70 (1959)]. See also footnote 17 of II.
[6] L. D. Landau, J. Exptl. Theoret. Phys. (USSR) **30**, 1058 (1956) [translation: Soviet Phys.—JETP **3**, 920 (1956)].
[7] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), pp. 207–213.

II. R.P.A. OR "PAIR" APPROXIMATION

As the first step in the calculation of the non-random-phase corrections to the induced spin density of an electron gas in the limit of low wave numbers, it will be useful to exhibit the R.P.A. for this problem[4] as a suitable pair approximation[1,2] to the exact Hamiltonian, which, in the notation of second quantization, has the form,[3]

$$H=H_T+\lambda H_I, \tag{1}$$

$$H_T=\sum_{\mathbf{k}\sigma}\epsilon_{\mathbf{k}}c_{\mathbf{k}\sigma}{}^{\dagger}c_{\mathbf{k}\sigma}+\frac{\xi}{2\Omega}\sum_{\mathbf{kk'q}(q\neq 0)}\sum_{\sigma\sigma'}v(q)c_{\mathbf{k+q}\sigma}{}^{\dagger}c_{\mathbf{k'-q}\sigma'}{}^{\dagger}c_{\mathbf{k'}\sigma'}c_{\mathbf{k}\sigma}, \tag{2}$$

$$H_I=\sum_{\mathbf{kq}\sigma}p(\sigma)B(\mathbf{q})c_{\mathbf{k+q}\sigma}{}^{\dagger}c_{\mathbf{k}\sigma}, \tag{3}$$

with $p(\pm)=\pm 1$.[8] The model pair-Hamiltonian which yields the appropriate nonvanishing graphs[9] of the R.P.A. may be constructed by inspection or by simple mutilation of the exact Hamiltonian.[10] One may write the model Hamiltonian as the sum of three parts,[11] the kinetic energy of pairs (which also includes the exchange self-energy of pairs in lowest order[4]),

$$H_K=\tfrac{1}{2}\sum_{\mathbf{pq}\sigma}(\pi_{\mathbf{p,q}}{}^{\sigma\dagger}\pi_{\mathbf{p,q}}{}^{\sigma}+\omega_{\mathbf{p,q}}{}^{*2}\varphi_{\mathbf{p,q}}{}^{\sigma\dagger}\varphi_{\mathbf{p,q}}{}^{\sigma}), \tag{4}$$

with

$$\omega_{\mathbf{p,q}}{}^{*}=\{(\mathbf{p}+\mathbf{q})^2-p^2\}/2M-\frac{\xi}{\Omega}\sum_{p'<P_F}\{v(\mathbf{p}+\mathbf{q}-\mathbf{p}')-v(\mathbf{p}-\mathbf{p}')\}, \tag{5}$$

[8] We take $\hbar=1$.
[9] See for example Figs. 1–3 of reference 4.
[10] This is done, for example, in footnote 13 of reference 2.
[11] We have chosen to transcribe them directly in terms of canonical variables.

pair-pair interaction terms (which describe the *exchange* scattering, annihilation, and creation of pairs),[12]

$$H_{P-P}=-\frac{\xi}{\Omega}\sum_{\mathbf{pp'q}(q\neq 0)\sigma}\{v(\mathbf{p}-\mathbf{p'})(\omega_{\mathbf{p},\mathbf{q}}^{*}\omega_{\mathbf{p'},\mathbf{q}}^{*})^{\frac{1}{2}}[\varphi_{\mathbf{p},\mathbf{q}}^{\sigma\dagger}\varphi_{\mathbf{p'},\mathbf{q}}^{\sigma}+(\omega_{\mathbf{p},\mathbf{q}}^{*}\omega_{\mathbf{p'},\mathbf{q}}^{*})^{-1}\pi_{\mathbf{p},\mathbf{q}}^{\sigma\dagger}\pi_{\mathbf{p'},\mathbf{q}}^{\sigma}]$$
$$+v(\mathbf{p}+\mathbf{q}+\mathbf{p'})(\omega_{\mathbf{p},\mathbf{q}}^{*}\omega_{\mathbf{p'},\mathbf{q}}^{*})^{\frac{1}{2}}[\varphi_{\mathbf{p},\mathbf{q}}^{\sigma\dagger}\varphi_{\mathbf{p'},\mathbf{q}}^{\sigma}-(\omega_{\mathbf{p},\mathbf{q}}^{*}\omega_{\mathbf{p'},\mathbf{q}}^{*})^{-1}\pi_{\mathbf{p},\mathbf{q}}^{\sigma\dagger}\pi_{\mathbf{p'},\mathbf{q}}^{\sigma}]\}, \quad (6)$$

and, finally, the pair approximation H_I,[13]

$$H_{P,I}=\sum_{\mathbf{pq}\sigma}p(\sigma)B(\mathbf{q})(\omega_{\mathbf{p},\mathbf{q}}^{*})^{\frac{1}{2}}\varphi_{\mathbf{p},\mathbf{q}}^{\sigma\dagger}. \quad (7)$$

The model problem as set forth in Eqs. (4), (6), and (7) is then easily solved via the unitary transformation

$$U=U_{+}U_{-}, \quad (8)$$

with[14]

$$U_{\sigma}=\exp[-i\lambda\sum_{\mathbf{pq}}(2/\omega_{p,q}^{*})^{\frac{1}{2}}F_{\mathbf{p},\mathbf{q}}^{\sigma}\pi_{\mathbf{p},\mathbf{q}}^{\sigma}, \quad (9)$$

which brings about the appropriate translation in φ space.[15] That is, we require the vanishing of the terms in $U^{\dagger}(H_K+H_{P-P}+\lambda H_{P,I})U$ linear in the canonical variables φ. One then finds that this requirement,

$$\lambda\sum_{\mathbf{pq}\sigma}(2/\omega_{\mathbf{p},\mathbf{q}}^{*})^{\frac{1}{2}}\omega_{\mathbf{p},\mathbf{q}}^{*2}F_{\mathbf{p},\mathbf{q}}^{\sigma}\varphi_{\mathbf{p},\mathbf{q}}^{\sigma\dagger}$$
$$-\frac{\lambda\xi}{2\Omega}\sum_{\mathbf{pp'q}\sigma}[v(\mathbf{p}-\mathbf{p'})+v(\mathbf{p}+\mathbf{q}+\mathbf{p'})](\omega_{\mathbf{p},\mathbf{q}}^{*}\omega_{\mathbf{p'},\mathbf{q}}^{*})^{\frac{1}{2}}$$
$$\times\{(2/\omega_{\mathbf{p},\mathbf{q}}^{*})^{\frac{1}{2}}F_{\mathbf{p},\mathbf{q}}^{\sigma*}\varphi_{\mathbf{p'},\mathbf{q}}^{\sigma}+(2/\omega_{\mathbf{p'},\mathbf{q}}^{*})^{\frac{1}{2}}F_{\mathbf{p'},\mathbf{q}}^{\sigma}\varphi_{\mathbf{p},\mathbf{q}}^{\sigma\dagger}\}$$
$$+\lambda\sum_{\mathbf{pq}\sigma}p(\sigma)B(\mathbf{q})(\omega_{\mathbf{p},\mathbf{q}}^{*})^{\frac{1}{2}}\varphi_{\mathbf{p},\mathbf{q}}^{\sigma\dagger}=0, \quad (10)$$

yields an integral equation for $F_{\mathbf{p},\mathbf{q}}^{\sigma}$,

$$\omega_{\mathbf{p},\mathbf{q}}^{*}F_{\mathbf{p},\mathbf{q}}^{\sigma}-\frac{\xi}{\Omega}\sum_{\mathbf{p'}\,(p'<P_F;\,|\mathbf{p'}+\mathbf{q}|>P_F)}[v(\mathbf{p}-\mathbf{p'})+v(\mathbf{p}+\mathbf{p'}+\mathbf{q})]F_{\mathbf{p'},\mathbf{q}}^{\sigma}+p(\sigma)B(\mathbf{q})=0. \quad (11)$$

This integral equation is essentially[16] the same as that derived in reference 4 by the method of equations of motion. It is solved by the ansatz,

$$F_{\mathbf{p},\mathbf{q}}^{\sigma}=\frac{p(\sigma)B(\mathbf{q})}{\omega_{\mathbf{p},\mathbf{q}}^{*}}\phi(p)\eta(1-p)\eta(|\mathbf{p}+\mathbf{q}|-1), \quad (12)$$

which, in the limit $q\rightarrow 0$, in turn, yields the auxiliary integral equation,[17]

$$\phi(p)-\frac{\xi}{\Omega}\sum_{\mathbf{p'}}[v(\mathbf{p}-\mathbf{p'})+v(\mathbf{p}+\mathbf{p'})]\frac{M^*}{\mathbf{p'}\cdot\mathbf{q}}\hat{p}'\cdot\mathbf{q}\eta(\hat{p}'\cdot\mathbf{q})$$
$$\times\delta(p'-P_F)\phi(p')+1=0, \quad (p=P_F). \quad (13)$$

Since for $q\rightarrow 0$, $\phi(p)$, $\phi(p')\simeq\phi(P_F)$, we have

$$\phi-\frac{\xi}{\Omega}\frac{M^*}{P_F}\phi\sum_{\mathbf{p'}}v(\mathbf{p}-\mathbf{p'})\delta(p'-P_F)\bigg|_{(p=P_F)}+1=0, \quad (14)$$

and, hence,

$$F_{p,q}^{\sigma}\rightarrow f_{p,q}^{\sigma}=-\frac{p(\sigma)B(\mathbf{q})\eta(\hat{p}\cdot\mathbf{q})\delta(p-P_F)}{P_F\left[(1/M^*)-(\xi P_F/(2\pi)^3)\int d\Omega_{p'}\,v(\mathbf{p}-\mathbf{p'})\right]}, \quad (15a)$$

and

$$|F_{\mathbf{p},\mathbf{q}}^{\sigma}|^2\rightarrow g_{\mathbf{p},\mathbf{q}}^{\sigma}=\frac{|B(\mathbf{q})|^2}{P_F^2}\times\left[(1/M^*)-(\xi P_F/(2\pi)^3)\int d\Omega_{p'}\,v(\mathbf{p}-\mathbf{p'})\right]^{-2}\times\frac{\eta(\hat{p}\cdot\mathbf{q})\delta(p-P_F)}{\hat{p}\cdot\mathbf{q}}. \quad (15b)$$

III. NON-RANDOM-PHASE CORRECTIONS

In order to exhibit the first non-random-phase corrections to the induced spin density, in the limit of low wave numbers, it is necessary to transform away the pair diagrams of the R.P.A. *in the exact Hamiltonian.* We do this in the manner of I. As was the case there, the unitary transformation appropriate to the pair approximation [Eq. (9)] furnishes the proper guide. That is, we make the correspondence,[18]

$$U_{\sigma}=\exp[-\lambda\sum_{\mathbf{pq}}F_{\mathbf{p},\mathbf{q}}^{\sigma}(c_{-\mathbf{p},-\mathbf{q}}^{\sigma}-c_{\mathbf{p},\mathbf{q}}^{\sigma\dagger})]\rightarrow$$
$$\bar{U}_{\sigma}=\exp[-\lambda\sum_{\mathbf{pq}}F_{\mathbf{p},\mathbf{q}}^{\sigma}(b_{-\mathbf{p}\sigma}a_{-\mathbf{p}-\mathbf{q}\sigma}-a_{\mathbf{p}+\mathbf{q}\sigma}^{\dagger}b_{\mathbf{p}\sigma}^{\dagger})$$
$$+\tfrac{1}{2}\sum_{\mathbf{pq}}G_{\mathbf{p},\mathbf{q}}^{\sigma}(a_{\mathbf{p}+\mathbf{q}\sigma}^{\dagger}a_{\mathbf{p}\sigma}-a_{-\mathbf{p}\sigma}^{\dagger}a_{-\mathbf{p}-\mathbf{q}\sigma})$$
$$+\tfrac{1}{2}\sum_{\mathbf{pq}}H_{\mathbf{p},\mathbf{q}}^{\sigma}(b_{-\mathbf{p}-\mathbf{q}\sigma}^{\dagger}b_{-\mathbf{p}\sigma}-b_{\mathbf{p}\sigma}^{\dagger}b_{\mathbf{p}+\mathbf{q}\sigma})]$$
$$\equiv\exp(-\lambda S_{\sigma}). \quad (16)$$

[12] In anticipation of R.P.A. results, we have judiciously omitted the direct matrix elements of v.
[13] The Hermiticity of $H_{P,I}$ follows from the relations $B^*(\mathbf{q})=B(-\mathbf{q})$ and $\varphi_{\mathbf{p},\mathbf{q}}^{\sigma}=\varphi_{-\mathbf{p},-\mathbf{q}}^{\sigma\dagger}$.
[14] The symmetry $F_{\mathbf{p},\mathbf{q}}^{\sigma*}=F_{-\mathbf{p},-\mathbf{q}}^{\sigma}$ follows from the unitarity of U_{σ}.
[15] Note that $U_{\sigma}^{\dagger}\varphi_{\mathbf{p},\mathbf{q}}^{\sigma\dagger}U_{\sigma}=\varphi_{\mathbf{p},\mathbf{q}}^{\sigma\dagger}+\lambda(2/\omega_{\mathbf{p},\mathbf{q}}^{*})^{\frac{1}{2}}F_{\mathbf{p},\mathbf{q}}^{\sigma*}$.
[16] The precise correspondence is $p(\sigma)\phi(\mathbf{k}+\mathbf{q},-\mathbf{q})=\lambda F_{\mathbf{k},\mathbf{q}}^{\sigma*}$, where $\lambda=\mu g$.

[17] R. M. Rockmore, Phys. Rev. **118**, 1645 (1960).
[18] We have $c_{\mathbf{p},\mathbf{q}}^{\sigma}-c_{-\mathbf{p},-\mathbf{q}}^{\sigma\dagger}=i(2/\omega_{\mathbf{p},\mathbf{q}}^{*})^{\frac{1}{2}}\pi_{\mathbf{p},\mathbf{q}}^{\dagger}$, with the conventional correspondence, $c_{\mathbf{p},\mathbf{q}}^{\sigma}\rightarrow b_{\mathbf{p}\sigma}a_{\mathbf{p}+\mathbf{q}\sigma}$.

Since[19]

$$\frac{2\Omega}{\lambda}\langle\sigma_z(\mathbf{q})\rangle = -\frac{1}{\lambda}\frac{\delta E(\lambda)}{\delta[\lambda B(-\mathbf{q})]}\bigg|_{(\lambda=0)} = -\frac{1}{\lambda}\frac{\delta E^{(2)}(\lambda)}{\delta[\lambda B(-\mathbf{q})]}, \quad (17)$$

it will be sufficient to calculate the low-wave-number contribution to the second-order magnetic energy. This is done *using the transformed Hamiltonian*, $\bar{U}(\lambda)^\dagger H\bar{U}(\lambda)$, which, after neglecting terms $O(\lambda^3)$, takes the form,

$$\bar{U}^\dagger H\bar{U} = H_T + \lambda H_1 + \lambda^2 H_2, \quad (18)$$

with[20]

$$H_1 = H_I + [\mathcal{S}, H_T], \qquad H_2 = \tfrac{1}{2}[\mathcal{S}, H_I + H_1]. \quad (19)$$

We may then express the magnetic energy $O(\lambda^2)$ in terms of the effective interactions H_1 and H_2 as

$$E_0{}^\xi(\lambda) - E_0{}^\xi(0) = -\lambda^2 W, \quad (20)$$

with[21]

$$\begin{aligned} W &= \langle\mathbf{0}|H_1[H_T(\xi) - E_0{}^\xi(0)]^{-1}H_1|\mathbf{0}\rangle - \langle\mathbf{0}|H_2|\mathbf{0}\rangle \\ &= \langle 0|H_1H_0{}^{-1}H_1|0\rangle + \langle 0|\xi H_v H_0{}^{-1}[\mathcal{S}, H_1]|0\rangle \\ &\quad - \tfrac{1}{2}\langle\mathbf{0}|[\mathcal{S}, H_I]|\mathbf{0}\rangle. \end{aligned} \quad (21)$$

H_1 and H_2 are next determined by requiring that the new $F_{\mathbf{p},\mathbf{q}}{}^\sigma$ satisfy the same integral equation as the old [Eq. (11)],[22] and also taking for $G_{\mathbf{p},\mathbf{q}}{}^\sigma$ and $H_{\mathbf{p},\mathbf{q}}{}^\sigma$ the solutions of those algebraic equations which remove the bilinear terms in H_1 proportional to $a^\dagger a'$ and $b^\dagger b'$. For example the requirement,

$$\sum_{\mathbf{p}\mathbf{q}\sigma} a_{\mathbf{p}+\mathbf{q}\sigma}{}^\dagger a_{\mathbf{p}\sigma}\{p(\sigma)B(\mathbf{q}) + G_{\mathbf{p},\mathbf{q}}{}^\sigma(\tilde{\epsilon}_\mathbf{p} - \tilde{\epsilon}_{\mathbf{p}+\mathbf{q}}) + \frac{2\xi}{\Omega}\sum_{\mathbf{p}'} F_{\mathbf{p}',\mathbf{q}}{}^\sigma[(\mathcal{V}_{\mathbf{p},-\mathbf{p}';\mathbf{p}+\mathbf{q},-\mathbf{p}'-\mathbf{q}})_{\text{exchange}} + (\mathcal{V}_{\mathbf{p}+\mathbf{q},\mathbf{p}';\mathbf{p},\mathbf{p}'+\mathbf{q}})_{\text{exchange}}]\} = 0,$$

yields the solution (in the limit $q \to 0$),[23]

$$G_{\mathbf{p},\mathbf{q}}{}^\sigma = -\frac{p(\rho)B(\mathbf{q})}{\tilde{\epsilon}_\mathbf{p} - \tilde{\epsilon}_{\mathbf{p}+\mathbf{q}}} - \frac{1}{\tilde{\epsilon}_\mathbf{p} - \tilde{\epsilon}_{\mathbf{p}+\mathbf{q}}}\frac{\xi}{\Omega}\sum_{\mathbf{p}'} F_{\mathbf{p}',\mathbf{q}}{}^\sigma[v(\mathbf{p}+\mathbf{p}') + v(\mathbf{p}-\mathbf{p}')],$$

although, for our purposes, it will be enough to take as solutions,

$$G_{\mathbf{p},\mathbf{q}}{}^\sigma = -\frac{p(\sigma)B(\mathbf{q})}{\epsilon_\mathbf{p} - \epsilon_{\mathbf{p}+\mathbf{q}}}, \quad (p, |\mathbf{p}+\mathbf{q}| > P_F) \quad (22\text{a})$$

and

$$H_{\mathbf{p},\mathbf{q}}{}^\sigma = -\frac{p(\sigma)B(\mathbf{q})}{\epsilon_\mathbf{p} - \epsilon_{\mathbf{p}+\mathbf{q}}}; \quad (p, |\mathbf{p}+\mathbf{q}| < P_F) \quad (22\text{b})$$

we thereby remove terms $O(\lambda\xi^0)$ from H_1 and H_2.[24] One has, for example,[25]

$$\begin{aligned} H_1 = &\sum_{\mathbf{p}\mathbf{p}'\mathbf{q}\sigma} F_{\mathbf{p}',\mathbf{q}}{}^\sigma\frac{2\xi}{\Omega}v(q)c_{\mathbf{p}+\mathbf{q}\sigma}{}^\dagger c_{\mathbf{p}\sigma} - \sum_{\mathbf{p}\mathbf{q}\sigma} P_{\mathbf{p},\mathbf{q}}{}^\sigma\sum_{\mathbf{q}'}(a_{\mathbf{p}+\mathbf{q}\sigma}{}^\dagger a_{\mathbf{q}'\sigma}{}^\dagger\mathfrak{F}_{\mathbf{p},\mathbf{q}}{}^{\sigma\dagger} + a_{\mathbf{p}+\mathbf{q}\sigma}{}^\dagger\mathfrak{F}_{\mathbf{p},\mathbf{q}'}{}^{\sigma\dagger}b_{\mathbf{q}'\sigma}) \\ &- \sum_{\mathbf{p}\mathbf{q}\sigma} Q_{\mathbf{p},\mathbf{q}}{}^\sigma\sum_{\mathbf{q}'}(\mathfrak{F}_{-\mathbf{p}-\mathbf{q},\mathbf{q}'}{}^{\sigma\dagger}b_{\mathbf{q}'\sigma}b_{-\mathbf{p}\sigma} + a_{\mathbf{q}'\sigma}{}^\dagger\mathfrak{F}_{-\mathbf{p}-\mathbf{q},\mathbf{q}'}{}^{\sigma\dagger}b_{-\mathbf{p}\sigma}) - \sum_{\mathbf{p}'\mathbf{q}'\mathbf{p}''\mathbf{q}''\mathbf{q}}\sum_{\sigma\neq\sigma'}\frac{\xi}{\Omega}\mathcal{V}'_{\mathbf{p}'\mathbf{q}';\mathbf{q}''\mathbf{p}''}c_{\mathbf{q}'\sigma'}{}^\dagger c_{\mathbf{p}''\sigma'} \\ &\times[P_{\mathbf{p}',\mathbf{q}}{}^\sigma a_{\mathbf{p}'+\mathbf{q}\sigma}{}^\dagger - Q_{-\mathbf{p}'-\mathbf{q},\mathbf{q}}{}^\sigma b_{\mathbf{p}'+\mathbf{q}\sigma}]c_{\mathbf{q}''\sigma} + \text{H.c.}, \end{aligned} \quad (23)$$

where

$$\begin{aligned} P_{\mathbf{p},\mathbf{q}}{}^\sigma &= F_{\mathbf{p},\mathbf{q}}{}^\sigma, \quad (p < P_F;\ |\mathbf{p}+\mathbf{q}| > P_F) \\ &= -G_{\mathbf{p},\mathbf{q}}{}^\sigma, \quad (p, |\mathbf{p}+\mathbf{q}| > P_F) \end{aligned} \quad (24\text{a})$$

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[19] Equation (17) follows trivially from Eqs. (22) and (4) of reference 4. Note that

$$H_I = \int d\mathbf{r}\, B(\mathbf{r})(\psi^\dagger(\mathbf{r})\sigma_z\psi(\mathbf{r})) = \int d\mathbf{r}\, B(\mathbf{r})\sigma_z(\mathbf{r}),$$

with $\sigma_z(-\mathbf{q}) = \sum_{\mathbf{k}\sigma} p(\sigma)c_{\mathbf{k}+\mathbf{q}\sigma}{}^\dagger c_{\mathbf{k}\sigma}$.

[20] The requirement of unitarity yields the symmetry $\mathcal{S}^\dagger = -\mathcal{S}$ with the additional relations, $F_{\mathbf{p},\mathbf{q}}{}^{\sigma*} = F_{-\mathbf{p},-\mathbf{q}}{}^\sigma$, $G_{\mathbf{p},\mathbf{q}}{}^{\sigma*} = G_{-\mathbf{p},-\mathbf{q}}{}^\sigma$, and $H_{\mathbf{p},\mathbf{q}}{}^{\sigma*} = H_{-\mathbf{p},-\mathbf{q}}{}^\sigma$.

[21] Our notation follows that of reference 3 closely. Note that the last term in (21) includes propagator changes which were inadvertently omitted in the calculation in I.

[22] This procedure has the effect of removing those terms in H_1 proportional to the bilinear combinations of fermion operators (the pair terms, ba and $a^\dagger b^\dagger$) which give rise to the R.P.A. graphs of reference 4.

[23] Note the additional symmetries, $G_{\mathbf{p},\mathbf{q}}{}^\sigma = -G_{\mathbf{p}+\mathbf{q},-\mathbf{q}}{}^{\sigma*}$ and $H_{\mathbf{p},\mathbf{q}}{}^\sigma = -H_{\mathbf{p}+\mathbf{q},-\mathbf{q}}{}^{\sigma*}$.

[24] These become essentially "esthetic" considerations in the limit $q \to 0$, since the contribution to W from the region of vanishingly small q is independent of the functions G and H.

[25] The terms $\Sigma F_{\mathbf{p}',\mathbf{q}}{}^\sigma(2\xi/\Omega)v(q)c_{\mathbf{p}+\mathbf{q}\sigma}{}^\dagger c_{\mathbf{p}\sigma}$ may be identified as plasma terms; they make no contribution to W from the region of small q.[4] For translation-invariant potentials $v_{\mathbf{p}+\mathbf{q},\mathbf{p}';\mathbf{p},\mathbf{p}'+\mathbf{q}} = v(q)$.

and

$$Q_{\mathbf{p},\mathbf{q}}{}^{\sigma}=F_{\mathbf{p},\mathbf{q}}{}^{\sigma}, \quad (p<P_F;\ |\mathbf{p}+\mathbf{q}|>P_F)$$
$$=-H_{\mathbf{p},\mathbf{q}}{}^{\sigma}. \quad (p,\ |\mathbf{p}+\mathbf{q}|<P_F) \tag{24b}$$

We shall omit the rather tedious, though straightforward, calculation of the low-wave-number contribution to W, here, and content ourselves merely with a discussion of the structure of $W(\mathbf{q})$, for small q, where,

$$W=\sum_{\mathbf{q}} W(\mathbf{q}). \tag{25}$$

One has

$$\begin{aligned}
\lim_{q\to 0} W(\mathbf{q})=&-\tfrac{1}{2}\sum_{\mathbf{p}\sigma} p(\sigma)B(\mathbf{q})(f_{\mathbf{p},-\mathbf{q}}{}^{\sigma}+f_{-\mathbf{p},-\mathbf{q}}{}^{\sigma})\\
&-\tfrac{1}{2}\sum_{\mathbf{p}\sigma} g_{\mathbf{p},\mathbf{q}}{}^{\sigma}\frac{\xi^2}{\Omega^2}\Bigg[\sum_{(|\mathbf{p}+\mathbf{q}|,s>P_F;\,l,m<P_F)}\frac{(2\mathcal{V}^2+\mathcal{V}'^2)_{\mathrm{lm};\,\mathbf{p}+\mathbf{q},\mathbf{s}}}{\epsilon_{\mathbf{p}+\mathbf{q}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}}}-\sum_{(p,s>P_F;\,l,m<P_F)}\frac{(2\mathcal{V}^2+\mathcal{V}'^2)_{\mathrm{lm};\,\mathrm{ps}}}{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}}}\Bigg]\\
&+\tfrac{1}{2}\sum_{\mathbf{p}\sigma} g_{\mathbf{p},-\mathbf{q}}{}^{\sigma}\frac{\xi^2}{\Omega^2}\Bigg[\sum_{(p,s>P_F;\,l,m<P_F)}\frac{(2\mathcal{V}^2+\mathcal{V}'^2)_{\mathrm{lm};\,\mathrm{ps}}}{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}}}-\sum_{(|\mathbf{p}-\mathbf{q}|,s>P_F;\,l,m<P_F)}\frac{(2\mathcal{V}^2+\mathcal{V}'^2)_{\mathrm{lm};\,\mathbf{p}-\mathbf{q},\mathbf{s}}}{\epsilon_{\mathbf{p}-\mathbf{q}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}}}\Bigg]\\
&+\tfrac{1}{2}\sum_{\mathbf{m}\sigma} g_{\mathbf{m},\mathbf{q}}{}^{\sigma}\frac{\xi^2}{\Omega^2}\Bigg[\sum_{(p,s<P_F;\,l,|\mathbf{m}+\mathbf{q}|<P_F)}\frac{(2\mathcal{V}^2+\mathcal{V}'^2)_{\mathrm{l},\mathbf{m}+\mathbf{q};\,\mathrm{ps}}}{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}+\mathbf{q}}}-\sum_{(p,s>P_F;\,l,m<P_F)}\frac{(2\mathcal{V}^2+\mathcal{V}'^2)_{\mathrm{lm};\,\mathrm{ps}}}{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}}}\Bigg]\\
&-\tfrac{1}{2}\sum_{\mathbf{m}\sigma} g_{\mathbf{m},-\mathbf{q}}{}^{\sigma}\frac{\xi^2}{\Omega^2}\Bigg[\sum_{(p,s>P_F;\,l,m<P_F)}\frac{(2\mathcal{V}^2+\mathcal{V}'^2)_{\mathrm{lm};\,\mathrm{ps}}}{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}}}-\sum_{(p,s>P_F;\,l,|\mathbf{m}-\mathbf{q}|<P_F)}\frac{(2\mathcal{V}^2+\mathcal{V}'^2)_{\mathrm{l},\mathbf{m}-\mathbf{q};\,\mathrm{ps}}}{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}-\mathbf{q}}}\Bigg]\\
&+\tfrac{1}{2}\sum_{\mathbf{mp}\tau}\Bigg[-2(f_{-\mathbf{m},-\mathbf{q}}{}^{\sigma}f_{\mathbf{p},\mathbf{q}}{}^{\sigma}+f_{-\mathbf{m},\mathbf{q}}{}^{\sigma}f_{-\mathbf{p},-\mathbf{q}}{}^{\sigma}+f_{-\mathbf{m},\mathbf{q}}{}^{\sigma}f_{\mathbf{p},-\mathbf{q}}{}^{\sigma}+f_{-\mathbf{m},-\mathbf{q}}{}^{\sigma}f_{-\mathbf{p},\mathbf{q}}{}^{\sigma})\frac{\xi^2}{\Omega^2}\sum_{(s>P_F;\,l<P_F)}\frac{(2\mathcal{V}^2-\mathcal{V}'^2)_{\mathrm{ml};\,\mathrm{ps}}}{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}}}\\
&+\tfrac{1}{2}(f_{-\mathbf{p},-\mathbf{q}}{}^{\sigma}f_{-\mathbf{s},\mathbf{q}}{}^{\sigma}+f_{-\mathbf{p},\mathbf{q}}{}^{\sigma}f_{-\mathbf{s},-\mathbf{q}}{}^{\sigma}+f_{\mathbf{p},\mathbf{q}}{}^{\sigma}f_{-\mathbf{s},-\mathbf{q}}{}^{\sigma}+f_{\mathbf{p},-\mathbf{q}}{}^{\sigma}f_{-\mathbf{s},\mathbf{q}}{}^{\sigma})\frac{\xi^2}{\Omega^2}\sum_{(l,m<P_F)}\frac{(2\mathcal{V}^2-\mathcal{V}'^2)_{\mathrm{ml};\,\mathrm{ps}}}{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}}}\\
&+\tfrac{1}{2}(f_{-\mathbf{l},\mathbf{q}}{}^{\sigma}f_{\mathbf{m},-\mathbf{q}}{}^{\sigma}+f_{-\mathbf{l},-\mathbf{q}}{}^{\sigma}f_{\mathbf{m},\mathbf{q}}{}^{\sigma}+f_{-\mathbf{l},-\mathbf{q}}{}^{\sigma}f_{-\mathbf{m},\mathbf{q}}{}^{\sigma}+f_{-\mathbf{l},\mathbf{q}}{}^{\sigma}f_{-\mathbf{m},-\mathbf{q}}{}^{\sigma})\frac{\xi^2}{\Omega^2}\sum_{(p,s>P_F)}\frac{(2\mathcal{V}^2-\mathcal{V}'^2)_{\mathrm{ml};\,\mathrm{ps}}}{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}}}\\
&-\tfrac{1}{2}(f_{-\mathbf{m},-\mathbf{q}}{}^{\sigma}f_{\mathbf{p},\mathbf{q}}{}^{\sigma}+f_{-\mathbf{m},\mathbf{q}}{}^{\sigma}f_{-\mathbf{p},-\mathbf{q}}{}^{\sigma}+f_{-\mathbf{m},\mathbf{q}}{}^{\sigma}f_{\mathbf{p},-\mathbf{q}}{}^{\sigma}+f_{-\mathbf{m},-\mathbf{q}}{}^{\sigma}f_{-\mathbf{p},\mathbf{q}}{}^{\sigma})\frac{\xi^2}{\Omega^2}\sum_{(s>P_F;\,l<P_F)}\frac{(\mathcal{V}'^2)_{\mathrm{ml};\,\mathrm{ps}}}{\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{s}}-\epsilon_{\mathbf{l}}-\epsilon_{\mathbf{m}}}\Bigg].
\end{aligned} \tag{26}$$

This rather lengthy expression is easily reduced with the help of Eqs. (15a) and (15b) to the sum of three terms,[26]

$$\begin{aligned}
\lim_{q\to 0} W(\mathbf{q})=&|B(\mathbf{q})|^2P_F\frac{4\pi\Omega}{(2\pi)^3}\Bigg[\frac{1}{M}+\left(\frac{1}{M^*}-\frac{1}{M}\right)^{(1)}+\frac{P_F}{(2\pi)^3}\int_{(p,p'=P_F)}d\Omega_{p'}\,\mathcal{K}_{\mathrm{ex}}{}^{(1)}(\mathbf{p},\mathbf{p}')\Bigg]^{-1}\\
&-|B(\mathbf{q})|^2P_F\frac{4\pi\Omega}{(2\pi)^3}\Bigg[\frac{1}{M}+\left(\frac{1}{M^*}-\frac{1}{M}\right)^{(1)}+\frac{P_F}{(2\pi)^3}\int_{(p,p'=P_F)}d\Omega_{p'}\,\mathcal{K}_{\mathrm{ex}}{}^{(1)}(\mathbf{p},\mathbf{p}')\Bigg]^{-2}\\
&\times\left\{\frac{1}{P_F}\frac{d}{dP_F}[\delta^{(2)}\epsilon(P_F)]+\frac{P_F}{(2\pi)^3}\int_{(p'',p'''=P_F)}d\Omega_{p''}\,\mathcal{K}_{\mathrm{ex}}{}^{(2)}(\mathbf{p}''',\mathbf{p}'')\right\},
\end{aligned} \tag{27}$$

where the superscript (1) [(2)] indicates the interaction correction of first [second] order. $\mathcal{K}_{\mathrm{ex}}(p,p')$ denotes the forward (exchange) K-matrix element.

The expression (27), which exhibits the first non-random-phase corrections to the R.P.A. (which is the first

[26] The identity,

$$f_{\mathbf{a},\mathbf{q}}{}^{\sigma}f_{\mathbf{b},-\mathbf{q}}{}^{\sigma}+f_{\mathbf{a},-\mathbf{q}}{}^{\sigma}f_{\mathbf{b},\mathbf{q}}{}^{\sigma}+f_{-\mathbf{a},\mathbf{q}}{}^{\sigma}f_{\mathbf{b},-\mathbf{q}}{}^{\sigma}+f_{-\mathbf{a},-\mathbf{q}}{}^{\sigma}f_{\mathbf{b},\mathbf{q}}{}^{\sigma}=\frac{|B(\mathbf{q})|^2}{P_F{}^2}\Bigg[\frac{1}{M^*}-\frac{\xi P_F}{(2\pi)^3}\int_{(p,p'=P_F)}d\Omega_{p'}v(\mathbf{p}-\mathbf{p}')\Bigg]^{-2}\delta(a-P_F)\delta(b-P_F),$$

is most useful here.

term on the right-hand side of (27)), has the appearance of the beginning of an expansion of

$$\lim_{q\to 0} W_{\text{exact}}(\mathbf{q}) = |B(\mathbf{q})|^2 P_F \frac{4\pi\Omega}{(2\pi)^3}\left[\frac{1}{(M^*)_{\text{exact}}} + \frac{P_F}{(2\pi)^3}\int_{(p,p'=P_F)} d\Omega_{p'}\, \mathcal{K}_{\text{ex}}{}^{\text{exact}}(\mathbf{p},\mathbf{p}')\right]^{-1} \quad (28)$$

in irreducible self-energy and (exchange) scattering diagrams. We shall prove the truth of this statement in the next section.

IV. PROOF TO ALL ORDERS OF PARTICLE-PARTICLE COUPLING

The general proof of (28) begins with the *exact* statement,[27]

$$W(\mathbf{q}) = -\frac{1}{4}\frac{\partial^2}{\partial\lambda^2}\langle\Phi_0|(H_v+\lambda H_{I,\mathbf{q}})\frac{1}{-(H_T+\lambda H_{I,\mathbf{q}})+i\epsilon}(H_v+\lambda H_{I,\mathbf{q}})|\Phi_0\rangle_L\Big|_{(\lambda=0)} \quad (29)$$

$$= -\tfrac{1}{2}\langle\Phi_0|H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}|\Phi_0\rangle_L - \tfrac{1}{2}\langle\Phi_0|H_v\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}|\Phi_0\rangle_L$$

$$-\tfrac{1}{2}\langle\Phi_0|H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_v|\Phi_0\rangle_L - \tfrac{1}{2}\langle\Phi_0|H_v\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_v|\Phi_0\rangle_L. \quad (30)$$

The last of the terms in Eq. (30) above vanishes in the limit $q\to 0$, since one has

$$\lim_{q\to 0}\langle\Phi_0|H_v\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_v|\Phi_0\rangle_L \to$$

$$2|B(\mathbf{q})|^2\langle\Phi_0|H_v\frac{1}{-H_T+i\epsilon}2S_z\frac{1}{-H_T+i\epsilon}2S_z\frac{1}{-H_T+i\epsilon}H_v|\Phi_0\rangle_L = 0, \quad (31)$$

where

$$S_z = \tfrac{1}{2}\sum_{\mathbf{k}\sigma} p(\sigma)(n_{\mathbf{k}\sigma}{}^a - n_{\mathbf{k}\sigma}{}^b),$$

since

$$[S_z, H_T] = [S_z, H_v] = 0,$$

and

$$S_z|\Phi_0\rangle = 0.$$

We consider next the propagator corrections in the second and third terms of (30). We need only discuss the second term where one finds

$$\langle\Phi_0|H_v\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}|\Phi_0\rangle_L\Big|_{\text{(propagator corrections)}} \to 2|B(\mathbf{q})|^2 \lim_{q\to 0}\sum_{\sigma\sigma'} p(\sigma)p(\sigma')\sum_{\mathbf{k}}\langle\Phi_0|H_v\frac{1}{-H_T+i\epsilon}$$

$$\times\Big\{\sum_{\mathbf{k}'\neq\mathbf{k}}(a_{\mathbf{k}'\sigma'}{}^\dagger a_{\mathbf{k}'+\mathbf{q}\sigma'} - b_{\mathbf{k}'+\mathbf{q}\sigma'}{}^\dagger b_{\mathbf{k}'\sigma'}) + b_{\mathbf{k}\sigma}a_{\mathbf{k}+\mathbf{q}\sigma}\Big\}\frac{1}{-H_T+i\epsilon}a_{\mathbf{k}+\mathbf{q}\sigma}{}^\dagger b_{\mathbf{k}\sigma}{}^\dagger|\Phi_0\rangle_L = 0. \quad (32)$$

The principle we invoke in this instance is that of "particle number conservation," by which we may pair all the diagrams of (32) into null[3] dyads.

Thus we are left to discuss

$$W(\mathbf{q}) = -\tfrac{1}{2}\langle\Phi_0|H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}|\Phi_0\rangle_L$$

$$-\sum_{\sigma\sigma'} p(\sigma)p(\sigma')|B(\mathbf{q})|^2\sum_{\mathbf{k}\mathbf{k}';(\mathbf{k}'\neq\mathbf{k})}\langle\Phi_0|H_v\frac{1}{-H_T+i\epsilon}(b_{\mathbf{k}'\sigma'}a_{\mathbf{k}'+\mathbf{q}\sigma'} + a_{-\mathbf{k}'-\mathbf{q}\sigma'}{}^\dagger b_{-\mathbf{k}'\sigma'}{}^\dagger)\frac{1}{-H_T+i\epsilon}a_{\mathbf{k}+\mathbf{q}\sigma}{}^\dagger b_{\mathbf{k}\sigma}{}^\dagger|\Phi_0\rangle_L$$

$$-\sum_{\sigma\sigma'} p(\sigma)p(\sigma')|B(\mathbf{q})|^2\sum_{\mathbf{k}\mathbf{k}';(\mathbf{k}'\neq\mathbf{k})}\langle\Phi_0|b_{\mathbf{k}\sigma}a_{\mathbf{k}+\mathbf{q}\sigma}\frac{1}{-H_T+i\epsilon}(a_{\mathbf{k}'+\mathbf{q}\sigma'}{}^\dagger b_{\mathbf{k}'\sigma'}{}^\dagger + b_{-\mathbf{k}'\sigma}a_{\mathbf{k}'-\mathbf{q}\sigma'})\frac{1}{-H_T+i\epsilon}H_v|\Phi_0\rangle_L. \quad (33)$$

[27] By H_I, $\mathbf{q}$, we mean the *sum* of the Fourier components of H_I for $\mathbf{q}$ and $-\mathbf{q}$; otherwise, our notation is that of II.

At this point the discussion given in II *may be taken over bodily;* the complication of spin-dependence offers no problem. As in II, one finds for the contribution from forward pair scattering,

$$-\tfrac{1}{2}\langle\Phi_0|H_{I,\mathbf{q}}\frac{1}{-H_T+i\epsilon}H_{I,\mathbf{q}}|\Phi_0\rangle_L=-|B(\mathbf{q})|^2\sum_{\mathbf{k}\sigma}(p(\sigma))^2\langle\Psi_0|\rho_{\mathbf{k},\mathbf{q}}{}^{\sigma}(E_0-H_T+i\epsilon)^{-1}\rho_{\mathbf{k},\mathbf{q}}{}^{\sigma\dagger}|\Psi_0\rangle$$
$$=|B(\mathbf{q})|^2 2\sum_{\mathbf{p}}\eta(\hat{p}\cdot\mathbf{q})\delta(p-P_F)(M^*{}_{\text{exact}}/P_F)$$
$$=|B(\mathbf{q})|^2\frac{4\pi}{(2\pi)^3}P_F\Omega M^*{}_{\text{exact}}. \tag{34}$$

Further, the contribution from nonforward pair scattering and pair annihilation (and pair creation) may be expressed schematically[3] as the matrix[28]

$$G\Delta\tilde{G} \tag{35}$$

where $\tilde{G}$ satisfies the matrix integral equation

$$\tilde{G}=G+G\Delta\tilde{G}, \tag{36}$$

with

$$[\Delta]_{\mathbf{k}',\mathbf{q}}{}^{\sigma'}{}_{;\mathbf{k},\mathbf{q}}{}^{\sigma}=(\mathbf{k}'+\mathbf{q}_>,\mathbf{k}'_<;\sigma'|R|\mathbf{k}+\mathbf{q}_>,\mathbf{k}_<;\sigma)+(0|R|-\mathbf{k}'-\mathbf{q}_>,-\mathbf{k}'_<,\sigma';\mathbf{k}+\mathbf{q}_>,\mathbf{k}_<,\sigma). \tag{37}$$

In the limit of $q\rightarrow 0$, we make the customary identification,[3]

$$\lim_{q\to 0}[\Delta]_{\mathbf{k}',\mathbf{q}}{}^{\sigma'}{}_{;\mathbf{k},\mathbf{q}}{}^{\sigma}\rightarrow\eta(\hat{k}'\cdot\mathbf{q})[\mathcal{K}(\mathbf{k}'\sigma',\mathbf{k}\sigma)+\mathcal{K}(-\mathbf{k}'\sigma',\mathbf{k}\sigma)]_{(k,k'=P_F)}. \tag{38}$$

Following Landau,[6,7] the K matrix may be written as[29]

$$\mathcal{K}_0+\tfrac{1}{2}\mathcal{K}_{\text{ex}}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}', \tag{39}$$

so that, for example,

$$\mathcal{K}(\mathbf{k}'\pm,\mathbf{k}\pm)-\mathcal{K}(\mathbf{k}'\pm,\mathbf{k}\mp)=\mathcal{K}_{\text{ex}}(\mathbf{k}',\mathbf{k}). \tag{40}$$

On the other hand, we will find the usual matrix representation of $\mathcal{K}$ convenient for the solution of (36). [That integral equation we shall consider in its premultiplied form,

$$\tilde{\Gamma}=\Delta G+\Delta G\tilde{\Gamma}, \tag{41}$$

where

$$\tilde{\Gamma}=\Delta\tilde{G}.] \tag{42}$$

Namely,[30]

$$\mathcal{K}=\begin{bmatrix}\mathcal{K}_{++} & \mathcal{K}_{+-}\\ \mathcal{K}_{-+} & \mathcal{K}_{--}\end{bmatrix}. \tag{43}$$

Then we have,

$$W(\mathbf{q})=|B(\mathbf{q})|^2\frac{4\pi}{(2\pi)^3}P_F\Omega M^*{}_{\text{exact}}-|B(\mathbf{q})|^2\sum_{\sigma\sigma'}p(\sigma)p(\sigma')\lim_{q\to 0}\sum_{\mathbf{k}\mathbf{k}'}[G\tilde{\Gamma}]_{\mathbf{k}',\mathbf{q}}{}^{\sigma'}{}_{;\mathbf{k},\mathbf{q}}{}^{\sigma}. \tag{44}$$

If we take

$$u=\begin{bmatrix}1\\-1\end{bmatrix}, \tag{45}$$

we arrive at the scalar integral equation,

$$u^\dagger\tilde{\Gamma}u=u^\dagger\Delta u\cdot G+u^\dagger\Delta G\tilde{\Gamma}u, \tag{46}$$

with

$$W(\mathbf{q})=|B(\mathbf{q})|^2\frac{4\pi}{(2\pi)^3}P_F\Omega M^*{}_{\text{exact}}-|B(\mathbf{q})|^2\lim_{q\to 0}\sum_{\mathbf{k}\mathbf{k}'}[Gu^\dagger\tilde{\Gamma}u]_{\mathbf{k}',\mathbf{q};\mathbf{k},\mathbf{q}}. \tag{47}$$

Equation (46) is easily reduced to the algebraic equation,

$$(\tilde{\Gamma}_{++}-\tilde{\Gamma}_{+-})_{\mathbf{k}'\mathbf{k}}=(\Delta_{++}-\Delta_{+-})G_{\mathbf{k},q}{}^{(0)}-\frac{4\pi P_F}{(2\pi)^3}M^*{}_{\text{exact}}\langle\Delta_{++}-\Delta_{+-}\rangle_{\text{av}}(\tilde{\Gamma}_{++}-\tilde{\Gamma}_{+-}), \tag{48}$$

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[28] We refer here to spin space. These indices were generally suppressed in II.

[29] Our notation differs slightly from his.

[30] There are, as a result of (39), only two independent elements in this matrix. In particular,[7] $\mathcal{K}_{++}=\mathcal{K}_{--}=\mathcal{K}_0+\frac{1}{2}\mathcal{K}_{\text{ex}}$, $\mathcal{K}_{+-}=\mathcal{K}_{-+}=\mathcal{K}_0-\frac{1}{2}\mathcal{K}_{\text{ex}}$.

with the solution,

$$u^\dagger\tilde{\Gamma}_{\mathbf{k}'\mathbf{k}}u=2(\tilde{\Gamma}_{++}-\tilde{\Gamma}_{+-})_{\mathbf{k}'\mathbf{k}}$$

$$=2(\Delta_{++}-\Delta_{+-})_{\mathbf{k}'\mathbf{k}}G_{\mathbf{k},\mathbf{q}}{}^{(0)}\left[1+\frac{4\pi P_F}{(2\pi)^3}M^*{}_{\text{exact}}\langle\Delta_{++}-\Delta_{+-}\rangle_{\text{av}}\right]^{-1}. \tag{49}$$

The additional relations,

$$4\pi\langle\Delta_{++}-\Delta_{+-}\rangle_{\text{av}}=\int_{(k,k'=P_F)}d\Omega_{k'}\;\mathcal{K}_{\text{ex}}{}^{(\text{exact})}(\mathbf{k},\mathbf{k}'),$$

and

$$\lim_{q\to0}2\sum_{\mathbf{k}\mathbf{k}'}G_{\mathbf{k}',\mathbf{q}}{}^{(0)}(\Delta_{++}-\Delta_{+-})_{\mathbf{k}'\mathbf{k}}G_{\mathbf{k},\mathbf{q}}{}^{(0)}=(M^*{}_{\text{exact}})^2\frac{\Omega 4\pi P_F{}^2}{(2\pi)^6}\int_{(k'',k'''=P_F)}d\Omega_{k'''}\;\mathcal{K}_{\text{ex}}{}^{(\text{exact})}(\mathbf{k}'',\mathbf{k}'''),$$

then yield the anticipated result.

$$\lim_{q\to0}W(\mathbf{q})=|B(\mathbf{q})|^2\frac{4\pi P_F\Omega M^*{}_{\text{exact}}}{(2\pi)^3}\left[1+\frac{P_FM^*{}_{\text{exact}}}{(2\pi)^3}\int_{(k,k'=P_F)}d\Omega_{k'}\;\mathcal{K}_{\text{ex}}{}^{(\text{exact})}(\mathbf{k},\mathbf{k}')\right]^{-1}. \tag{28}$$

which is the proper generalization (to all orders) of that of reference 4. If one passes to the limit $q=0$ and replaces $B(0)$ by B in (28), one obtains for ths susceptibility per unit volume, the expression

$$\chi=\frac{8\pi\lambda^2P_FM^*{}_{\text{exact}}}{(2\pi)^3}\left[1+\frac{P_FM^*{}_{\text{exact}}}{(2\pi)^3}\int_{(k,k'=P_F)}d\Omega_{k'}\;\mathcal{K}_{\text{ex}}{}^{(\text{exact})}(\mathbf{k},\mathbf{k}')\right]^{-1}, \tag{50}$$

which has been previously derived by Landau[5,6] from a somewhat intuitive basis; however in the present instance it is found as a direct consequence of the many-body perturbation theory.[3,31] We shall postpone to a subsequent communication a more detailed discussion of the static responses of a normal fermion system. There we will find that these have a precise formulation in terms of $M^*{}_{\text{exact}}$ and $\mathcal{K}(\mathbf{k}\sigma,\mathbf{k}'\sigma')|_{(k,k'=P_F)}$ so long as they depend strongly on contributions from the region of small wave numbers.

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[31] Indeed we have provided the connecting link between Landau's phenomenological result[6,7] and Luttinger's microscopic result [J. M. Luttinger, Phys. Rev. **19**, 1153 (1960)]. Note that we do assume a spherical Fermi surface.

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