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Spin and Exchange Corrections to Plasma Dispersion Relations

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Longitudinal and transverse dispersion relations including spin and exchange contributions are derived for electromagnetic waves in a fully ionized plasma. The nonrelativistic Pauli description of the electrons is used, and the electron density matrix is assumed to obey the quantum analog of the Vlasov equation. The electromagnetic fields satisfy Maxwell's equations with self-consistent source terms.

The dispersion relation for longitudinal modes is unaffected by spin, and the exchange correction derived agrees with that obtained by previous investigators. The transverse dispersion relation is evaluated in the long-wavelength limit for the completely degenerate plasma at 0°K. If the transverse frequency be written $\omega=\omega_0+\omega_{\mathrm{sp}}+\omega_{\mathrm{ex}}$, where ω_0 is the frequency with spin and exchange effects omitted, our results for the corrections are $\omega_{\mathrm{sp}}=(\omega_p^2/8\omega_0^3)(\hbar^2/m^2)$, $\omega_{\mathrm{ex}}=-3\omega_p^4q^2/80\omega_0^3K_F^2$, where $\mathbf{q}$ is the wave vector of a disturbance and $\hbar K_F$ is the momentum limit of the Fermi distribution.

I. INTRODUCTION

THE study of plasmas in which quantum mechanical effects are significant has been facilitated through the use of the quantum analog of the classical Vlasov equation for the density matrix or quantum mechanical distribution function of the system.[1–3] In particular, dispersion relations for longitudinal and transverse electromagnetic waves have been derived by several authors.[4–7] The effect of electron exchange of the longitudinal dispersion relation has been calculated recently by Silin[8] and von Roos and Zmuidzinas[9] using different forms of quantum mechanical distribution functions.

In view of the fact that for real systems the particles comprising the plasma have spin, it is of interest to calculate the effects of spin on the dispersion relations. Since spin couples only to the transverse field, it is desirable also to determine the contribution of particle exchange to the transverse dispersion relation in order to compare the two. Although a proper theory of particle spin and transverse electromagnetic waves should be relativistic, we shall use the nonrelativistic Pauli description of the particles. Thus, the results will be strictly valid only for low-temperature plasmas and weak fields. In the course of calculating the exchange contribution to the dispersion relations, one is faced with solving an intractable integral equation. Here we have used a perturbation treatment and since, in fact, both spin and exchange contributions are expected to be small, they have been calculated entirely separately. In Sec. II we consider the spin correction and in Sec. III that due to exchange.

Since the derivation of the quantum Vlasov equation has been examined elsewhere,[2,7,10–11] only a brief summary will be given here. The system of particles is described in terms of a distribution function of the type defined by Wigner[12] or von Roos[2] or by means of

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the density matrix.[13–14] The particle interactions take place through self-consistent electromagnetic fields, the sources of which are derived from the one-particle distributions. Collisions are neglected. Finally, exchange effects are included by adding to the Hamiltonian a Coulomb term of the form[7,9]

$$\phi_{\mathrm{ex}}=4\pi e\left[\mathrm{Tr}^{(2)}E\left(\frac{R^{(2)}(x_1,x_2;x_1',x_2')}{|\mathbf{r}_1-\mathbf{r}_2|}\right)\right], \tag{1}$$

where $R^{(2)}$ is the two-particle density matrix and the trace is taken over all coordinates, space and spin, of particle 2. The operator E interchanges one set of coordinates of particles 1 and 2; e.g.,

$$ER^{(2)}(x_1,x_2;x_1',x_2')=\pm R^{(2)}(x_1,x_2;x_2',x_1'), \tag{2}$$

where the positive sign holds for bosons and the negative for fermions.

The system to be treated is comprised of N electrons per unit volume and an equal number of stationary positive ions to provide net electrical neutrality. We assume that the one-particle density matrix for electrons in wave-vector space may be written

$$R^{(1)}(\mathbf{k}',\mathbf{k};s',s;t)=F(|\mathbf{k}|)+f(\mathbf{k}',\mathbf{k};s',s;t), \tag{3}$$

where F is the equilibrium Fermi-Dirac distribution, diagonal in both momentum and spin space, and f is a small perturbation. Finally, the sources of the self-consistent electromagnetic field are determined by the perturbation f alone, since the positive-ion background neutralizes the equilibrium part.

If one Fourier-analyzes the time dependence of the distributions and fields, the governing equations written in wave vector space are

$$\begin{aligned}[-\omega+\hbar\mathbf{k}\cdot\mathbf{q}/m+\hbar q^2/2m]f(\mathbf{k}+\mathbf{q},\,\mathbf{k};\,s',\,s;\,\omega)\\ =\frac{e}{\hbar}\bigg\{\phi(\mathbf{q},\omega)+\phi_{\mathrm{ex}}(\mathbf{q},\omega)-\frac{\hbar}{mc}\mathbf{k}\cdot\mathbf{A}(\mathbf{q},\omega)\\ +\frac{i\hbar}{2mc}[\mathbf{q}\times\mathbf{A}(\mathbf{q},\omega)\cdot\boldsymbol{\sigma}]\bigg\}\Delta F,\end{aligned} \tag{4}$$

$$q^2\phi(\mathbf{q},\omega)=4\pi e\int d^3k\,\mathrm{Tr}^{(s)}\{f(\mathbf{k}+\mathbf{q},\,\mathbf{k};\,s',\,s;\,\omega)\}, \tag{5}$$

$$\begin{aligned}\left[q^2-\frac{\omega^2-\omega_p{}^2}{c^2}\right]\mathbf{A}(\mathbf{q},\omega)\\ =\frac{4\pi e\hbar}{mc}\int d^3k[\mathbf{k}-\mathbf{q}(\mathbf{k}\cdot\mathbf{q})/q^2]\,\mathrm{Tr}^{(s)}f\\ -\frac{i4\pi e\hbar}{2mc}\int d^3k\mathbf{q}\times\mathrm{Tr}^{(s)}\{f\boldsymbol{\sigma}\},\end{aligned} \tag{6}$$

where ϕ and $\mathbf{A}$ are the electromagnetic potentials in Coulomb gauge, $\mathbf{q}$ is the wave vector and ω the angular frequency of the disturbance, $\boldsymbol{\sigma}$ is the Pauli spin vector, $\mathrm{Tr}^{(s)}$ means trace over spin coordinates only, and

$$\Delta F=F(|\mathbf{k}+\mathbf{q}|)-F(|\mathbf{k}|). \tag{7}$$

Exchange enters the equations by the additional exchange potential appearing in Eq. (4), the linearized quantum Vlasov equation. Spin contributes two terms; the last in Eq. (4) which is, of course, the interaction of the electron's intrinsic magnetic moment with the magnetic field, and the second source term of Eq. (6) which is the so-called "spin current."

II. SPIN CORRECTION

The dispersion relations including spin contributions may be calculated in a straightforward manner using Eqs. (4)–(6) when the exchange term is omitted. The sources in Eqs. (5) and (6) can be evaluated by solving Eq. (4) for f. One then obtains a set of homogeneous equations for the amplitudes $\phi(\mathbf{q},\omega)$ and $\mathbf{A}(\mathbf{q},\omega)$. The condition for these to have a solution is that the determinant of the coefficients vanish. This leads to the dispersion relations.

Inserting f from Eq. (4) into Eq. (5) and (6) and performing the indicated operations gives

$$[1-\omega_p{}^2I_1(q^2,\omega)]\phi(\mathbf{q},\omega)=0, \tag{8}$$

and

$$\left[-\omega^2+c^2q^2+\omega_p{}^2+\frac{\hbar^2q^2\omega_p{}^2}{2m^2}I_2(q^2,\omega)+\frac{\hbar^2q^4\omega_p{}^2}{4m^2}I_1(q^2,\omega)\right]\times\mathbf{A}(\mathbf{q},\omega)=0, \tag{9}$$

with

$$\omega_p{}^2=4\pi Ne^2/m, \tag{10}$$

$$I_1(q^2,\omega)=\frac{2}{N}\int\frac{d^3k\,F(|\mathbf{k}|)}{(\omega-\hbar\mathbf{k}\cdot\mathbf{q}/m)^2-(\hbar q^2/2m)^2}, \tag{11}$$

$$I_2(q^2,\omega)=\frac{2}{N}\int\frac{d^3k\,[k^2-(\mathbf{k}\cdot\mathbf{q})^2/q^2]F(|\mathbf{k}|)}{(\omega-\hbar\mathbf{k}\cdot\mathbf{q}/m)^2-(\hbar q^2/2m)^2}. \tag{12}$$

Setting the terms in brackets in Eqs. (8) and (9) equal to zero gives the dispersion relations. From Eq. (9) one obtains the usual relation for longitudinal modes[1–2,4,5] which is uninfluenced by spin as expected. The last term in the bracket of Eq. (9) is the spin correction to the transverse dispersion relation. If one expands the integrals I_1 and I_2 in powers of q^2, the transverse dispersion relation for long wavelengths becomes

$$\begin{aligned}\omega^2=c^2q^2+\omega_p{}^2+\frac{\hbar^2\omega_p{}^2}{m^2\omega^2}\left[\frac{\langle k^2\rangle}{3}+\frac{\hbar^2q^2}{5m^2\omega^2}\langle k^4\rangle\right]\\ +\left(\frac{\hbar^2\omega_p{}^2q^4}{4m^2\omega^2}\right)\left[1+\frac{\hbar^2q^2}{m^2\omega^2}\langle k^2\rangle\right],\end{aligned} \tag{13}$$

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where

$$\langle k^n\rangle = \int d^3k\, \mathbf{k}^n F(|\mathbf{k}|). \tag{14}$$

Finally, evaluating the averages at zero temperature gives

$$\omega^2 = c^2q^2 + \omega_p^2 + \frac{\hbar^2 q^2 \omega_p^2}{m^2}\frac{(K_F^2/5 + q^2/4)}{c^2q^2+\omega_p^2}, \tag{15}$$

where $\hbar K_F$ is the momentum at the top of the Fermi distribution, $\frac{1}{5}K_F^2$ is the leading quantum term from the ordinary transverse current and $\frac{1}{4}q^2$ that due to the spin interactions. While the expansion used to obtain Eq. (15) is mathematically valid for q comparable to K_F, the theory itself is probably inaccurate for such short-wavelength disturbances. In this region correlation effects are likely to be significant so the self-consistent field approximation breaks down. Further, in a real metal interactions with the ionic lattice, which are here ignored, will become important. Eq. (15) should, therefore, for physical reasons be restricted to $q \ll K_F$, and the spin correction consequently will be smaller than the usual quantum mechanical correction term.

III. EXCHANGE CORRECTION

Since the spin correction is small for $q \ll K_F$, the exchange contribution to the dispersion relation will be calculated with the spin interaction omitted. In keeping with the self-consistent field approximation and the first-order perturbation of Eq. (3), the exchange potential may be written

$$\phi_{\text{ex}} = \frac{4\pi e}{2}\left(f\int \frac{\Delta F(\mathbf{k}',\mathbf{q})d^3k'}{|\mathbf{k}-\mathbf{k}'|^2} - \Delta F\int \frac{f(\mathbf{k}'+\mathbf{q},\mathbf{k}';\omega)}{|\mathbf{k}-\mathbf{k}'|^2}d^3k'\right), \tag{16}$$

where ΔF is given by Eq. (7), the factor of $\frac{1}{2}$ comes from averaging over spins[9] and

$$f(\mathbf{k}+\mathbf{q},\mathbf{k};\omega) = \mathrm{Tr}^{(s)}[f(\mathbf{k}+\mathbf{q},\mathbf{k};s',s;\omega)].$$

Eliminating ϕ and $\mathbf{A}$ through use of the field equations leads to an integral equation for which exact solutions have not been obtained. Consequently, we shall apply a simple perturbation treatment similar to that used in the longitudinal case by von Roos and Zmuidzinas.[9] That is, assuming the effect of exchange to be small, we write

$$\begin{aligned} \omega &= \omega_0 + \omega_1, \\ f &= f_0(\omega_0) + f_1(\omega_1), \\ \phi &= \phi_0(\omega_0) + \phi_1(\omega_1), \\ \mathbf{A} &= \mathbf{A}_0(\omega_0) + \mathbf{A}_1(\omega_1), \end{aligned} \tag{17}$$

where ω_0 is the frequency without exchange. Equating terms of like order in the perturbation expansion of Eqs. (4)–(6) gives the following set of equations for the distributions and fields:

$$f_0 = \left[\frac{e}{\hbar}\phi_0 - \frac{e}{mc}\mathbf{k}\cdot\mathbf{A}_0\right]\Delta F/D, \tag{18}$$

$$q^2\phi_0 = 4\pi Ne\int d^3k\, f_0(\mathbf{k}+\mathbf{q},\mathbf{k};\omega_0), \tag{19}$$

$$\left(q^2 - \frac{(\omega_0^2-\omega_p^2)}{c^2}\right)\mathbf{A}_0 = \frac{4\pi Ne\hbar}{mc}\int d^3k\, (\mathbf{k}-\mathbf{k}\cdot\mathbf{q}\mathbf{q}/q^2)f_0, \tag{20}$$

and

$$f_1 = D^{-1}[\omega_1 f_0 + (e\phi_1/\hbar - e\mathbf{k}\cdot\mathbf{A}_1/mc)]\Delta F + \frac{m\omega_p^2}{2\hbar}\left(f_0\int\frac{d^3k'\,\Delta F}{|\mathbf{k}-\mathbf{k}'|^2} - \Delta F\int\frac{d^3k'\,f_0}{|\mathbf{k}-\mathbf{k}'|^2}\right), \tag{21}$$

$$q^2\phi_1 = 4\pi Ne\int d^3k\, f_1(\mathbf{k}+\mathbf{q},\mathbf{k};\omega_1), \tag{22}$$

$$\left(q^2 - \frac{(\omega_0^2-\omega_p^2)}{c^2}\right)\mathbf{A}_1 - \frac{2\omega_0\omega_1}{c^2}\mathbf{A}_0 = \frac{4\pi Ne\hbar}{mc}\int d^3k\, (\mathbf{k}-\mathbf{k}\cdot\mathbf{q}\mathbf{q}/q^2)f_1, \tag{23}$$

where

$$D = -\omega_0 + \hbar\mathbf{k}\cdot\mathbf{q}/m + \hbar q^2/2m. \tag{24}$$

Solution of Eqs. (18)–(20) gives the usual dispersion relations. Making use of these relations in the solution of Eqs. (21)–(23) leads to separate dispersion relations for longitudinal and transverse waves with exchange effects included. The longitudinal dispersion relation agrees with that found previously.[8,9] The transverse dispersion relation is obtained from the following equation:

$$\omega_1\left[2\omega_0\mathbf{I} - \frac{\hbar\omega_p^2}{m}\int d^3k\, \mathbf{k}_\perp\mathbf{k}_\perp\Delta F D^{-2}\right] = \tfrac{1}{2}\omega_p^4\int d^3k d^3k'\, \frac{\Delta F(k)\Delta F(k')}{|\mathbf{k}-\mathbf{k}'|^2}\left(\frac{\mathbf{k}_\perp\mathbf{k}_\perp'}{D'D} - \frac{\mathbf{k}_\perp\mathbf{k}_\perp}{D^2}\right), \tag{25}$$

where $\mathbf{I}$ is the unit dyadic and

$$\mathbf{k}_\perp = \mathbf{k} - \mathbf{k}\cdot\mathbf{q}\mathbf{q}/q^2. \tag{26}$$

Inspection of the integrals in Eq. (25), taking into account the parity of the integrands, reveals that this dyadic is diagonal. The dispersion relation for transverse waves may be obtained by equating any one of the diagonal elements to zero.

Again, we examine this expression for the zero temperature Fermi distribution and for $q \ll K_F$. One then has

$$\omega_1 = -3\omega_p^4 q^2/80\omega_0^3 K_F^2, \tag{27}$$

and the corrected transverse dispersion relation will be

$$\omega^2=(\omega_0+\omega_1)^2\cong\omega_0^2+2\omega_0\omega_1=\omega_0^2-3\omega_p^4q^2/40\omega_0^2K_F^2, \quad (28)$$

with ω_0^2 given by the first three terms of Eq. (15). Since for physical reasons we must restrict our treatment to $q\ll K_F$ as discussed before and the exchange correction is then small compared to the zero-order frequency, the approach of calculating spin and exchange contributions separately is justified.

IV. CONCLUSION

Taking the ratio of spin to exchange corrections from Eqs. (15) and (28) gives

$$\frac{10}{3}\frac{\hbar^2q^2K_F^2}{m^2\omega_p^2}\ll\frac{10}{3}\frac{\hbar^2K_F^4}{m^2\omega_p^2}\cong10^{-2} \quad (29)$$

for a typical metal, indicating that for the region $q\ll K_F$, particle exchange is more significant than the spin interaction. This is hardly surprising since the typical metal at zero temperature is certainly nonrelativistic whereas particle statistics are important in this system. An interesting feature of the calculation is that when spin and exchange effects are included, the transverse and longitudinal modes are still decoupled in the sense of having separate dispersion relations, at least for first order. They are not entirely independent, however, since the exchange contribution to the transverse relation arises directly from the Coulomb interaction. Thus, even in the "zero interaction" limit to which the self-consistent field treatment corresponds, the indistinguishability of the particles gives rise to currents which couple the radiating and nonradiating modes.

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Quantum-Mechanical Sum-Rule for Infinite Sums Involving the Operator $\partial H/\partial\lambda$*†

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The sum-rule

$$\sum_n{}'\frac{\langle m|A|n\rangle\langle n|\partial\Omega/\partial\lambda|m\rangle+\langle m|\partial\Omega/\partial\lambda|n\rangle\langle n|A|m\rangle}{\epsilon_n-\epsilon_m}=\left\langle m\left|\frac{\partial A}{\partial\lambda}\right|m\right\rangle-\frac{\partial}{\partial\lambda}(\langle m|A|m\rangle)$$

is derived. In this relation $|m\rangle,\cdots,|n\rangle,\cdots$ form a complete set of orthonormal vectors, which are the eigenvectors of the Hermitian linear operator Ω, with eigenvalues $\epsilon_m,\cdots\epsilon_n,\cdots$; λ is a parameter which occurs in Ω, and A is an arbitrary linear operator. In many sums of this type, Ω is the Hamiltonian operator H. Particular examples are considered, and a differential equation, relating the mass dependence and coordinate dependence of the wave function ψ, is derived.

1. INTRODUCTION

INFINITE sums of the form

$$S\equiv\sum_n{}'\frac{\langle m|A|n\rangle\langle n|B|m\rangle}{\epsilon_n-\epsilon_m} \quad (1)$$

appear in many quantum-mechanical problems. In (1), A and B are linear operators, while $|m\rangle,\cdots|n\rangle,\cdots$ form a complete set of orthonormal vectors, which are the eigenvectors of some Hermitian linear operator Ω, with eigenvalues $\epsilon_m,\cdots\epsilon_n,\cdots$; the prime in the summation sign indicates that the summation is over all states except $|n\rangle\equiv|m\rangle$. A familiar case occurs when Ω is identical with the Hamiltonian operator H, so that the ϵ_n's become the energies E_n associated with the different stationary states $|n\rangle$. The summation in S then includes, of course, an integral over the continuum states.

The reduction of S to a simple expression which depends on the properties of $|m\rangle$ alone has long been a challenging problem. A very simple, though dangerously uncertain, expression can be obtained by using the closure approximation.[1] Clinton[2] seems to have been the first to discover a case where, although one of the two operators A and B remains completely arbitrary, S is exactly reducible. He treated those sums in which one operator is identical to $2T-\sum_i \mathbf{r}_i\cdot(\partial V/\partial\mathbf{r}_i)$ (T=kinetic energy operator, V=potential energy operator), while the other operator remains arbitrary,

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