# Selection Rules in the Odd-Even Shift of Certain Nuclear Rotational Bands

NEAL D. NEWBY, JR.*
*Lawrence Radiation Laboratory, University of California, Berkeley, California*
(Received August 14, 1961)

A displacement of energy levels has been observed in the $K=0$ band of odd-odd deformed nuclei. It is shown that this shift is due to a particular type of scattering in which the final state is obtained from the initial state by a rotation of 180°. This circumstance allows one to state certain selection rules on the parts of the $n$-$p$ residual interaction responsible for the shift. The Wigner component of the force cannot contribute to the shift. In certain cases the contribution of all *central* forces will be strongly damped allowing observation of the tensor-force scattering contribution. Numerical results are presented.

## I. INTRODUCTION

ONE of the striking successes of the nuclear collective model has been its prediction of rotational states in deformed nuclei. At the present time many examples of rotational bands are known and their energy level structure is found to agree excellently with the theoretical prediction. If we neglect for the moment the cases with $K=\frac{1}{2}$ ($K$ is the projection of the total spin, $I$, on the nuclear symmetry axis), then the prediction is

$$E(I)=\frac{\hbar^2}{2\mathcal{I}_{\text{eff}}}I(I+1), \quad (K\neq\tfrac{1}{2}), \tag{1}$$

where $\mathcal{I}_{\text{eff}}$ can be termed the "effective moment of inertia" and is defined by the above expression. This formula is valid for even-even, odd $A$, and odd-odd deformed nuclei. Deviations from Eq. (1) due to higher-order effects (such as rotation-vibration interaction, for example), amount, at most, to a few percent.

Recently evidence has begun to accumulate which indicates that in odd-odd nuclei Eq. (1) is not always correct and may require important modification. In these cases we find levels which are members of a rotational band but have a level structure completely at variance with the prediction of Eq. (1). In the cases under study the deviation from the expected level structure is large enough to change the level *ordering* from that implied by Eq. (1). As an illustration of this effect we refer to Fig. 1 which shows the experimental situation in $Ho^{166}$ as determined by Geiger and coworkers at Chalk River.[1] The three lowest levels are determined to be members of a $K=0$ rotational band. From the energy difference between the $I=0$ and $I=2$ states one uses Eq. (1) to find $\hbar^2/2\mathcal{I}_{\text{eff}}=9$ kev. Thus one would expect the $I=1$ state to lie at 18 kev. Instead one finds $I=1$ at 82 kev. It appears that the $I=1$ state has been shifted upward from its expected position *relative* to $I=0$ and $I=2$ by 64 kev.

In other cases, to be discussed later, one finds shifts in the opposite direction. In $Am^{242}$, for example, we apparently have a $K=0$ band in which the $I=1$ state has been shifted *downward* so that it lies below the $I=0$ state.

We may briefly summarize the experimental data in the following way: states belonging to a $K=0$ rotational band have been observed in several odd-odd deformed nuclei. These states do not have the expected level spacings. The observed level spacings may be accounted for by superposing an energy shift on the expected $I(I+1)$ level spacing. This energy shift may be described provisionally as a displacement of levels of odd $I$ relative to those of even $I$ within the rotational band. How can we understand this "odd-even shift?"

## II. THEORY

A recent paper[2] has suggested an explanation. This explanation, in effect, places the "blame" for the odd-even shift on the residual interaction between the odd neutron and odd proton. It is our purpose in what follows to examine this proposal in some detail.

We may write the wave function of an odd-odd deformed nucleus as

$$\psi_{MK}{}^{I}=[(2I+1)/16\pi^2]^{\frac{1}{2}}\sum_{j_1}\sum_{j_2}C_{j_1}C_{j_2}\times[\chi_{K_1}{}^{j_1}(1)\chi_{K_2}{}^{j_2}(2)D_{MK}{}^{I}+(-1)^{I-j_1-j_2}\times\chi_{-K_1}{}^{j_1}(1)\chi_{-K_2}{}^{j_2}(2)D_{M-K}{}^{I}], \quad (K=K_1+K_2). \tag{2}$$

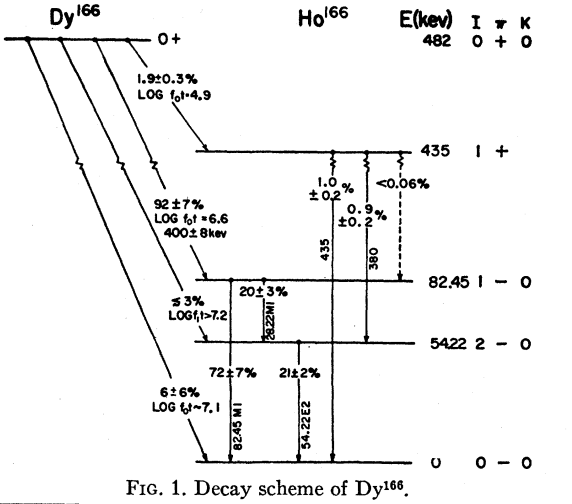


FIG. 1. Decay scheme of $Dy^{166}$.

* Present address: 1039–2 S. Westmoreland Ave., Los Angeles 6, California.

[1] I am grateful to J. S. Geiger, R. L. Graham, and G. T. Ewan for permission to use this illustration. Several conversations with Dr. Geiger were most helpful.

[2] F. Asaro, I. Perlman, J. O. Rasmussen, and S. G. Thompson, Phys. Rev. **120**, 934 (1960).

In this expression the single-particle states of the odd nucleons (particles No. 1 and No. 2) are written in the body-fixed coordinate frame. The $D$ functions describe the orientation of the deformed nuclear potential well in the laboratory frame and are functions of the Euler angles which specify that orientation. The total wave function has the required symmetries—(a) invariance under rotations around the nuclear symmetry axis and (b) invariance under a rotation of $\pi$ about any axis perpendicular to the nuclear symmetry axis.[3] If we now turn on an interaction, $V$ between particles 1 and 2 and evaluate the diagonal matrix element, we find

$$\langle\psi_{MK}{}^{I}|\,V\,|\psi_{MK}{}^{I}\rangle=A+(-1)^{I}B\delta_{K0}, \tag{3}$$

with

$$B=\sum_{j_1}\sum_{j_2}\sum_{j_1'}\sum_{j_2'}C_{j_1}{}^{*}(1)C_{j_2}{}^{*}(2) \times C_{j_1'}(1)C_{j_2'}(2)(-1)^{-j_1-j_2} \times\langle\chi_{-K_1}{}^{j_1}(1)\chi_{+K_1}{}^{j_2}(2)\,|\,V\,|\,\chi_{K_1}{}^{j_1'}(1)\chi_{-K_1}{}^{j_2'}(2)\rangle. \tag{4}$$

The matrix element [Eq. (3)] has been written so that its dependence on $I$ is explicitly displayed. In the expression for $B$ we have made use of the fact that this term contributes to the matrix element only if $K=K_1+K_2=0$ and so $K_2=-K_1$. The four radial wave functions are omitted for the present.

Already we see that in an odd-odd deformed nucleus a residual *n-p* interaction may cause a shift in energy within the members of a rotational band. This displacement is to be expected if and only if $K=0$. The shift takes the form of a relative displacement of the odd-$I$ and even-$I$ members of the band and its magnitude is $2B$. The physical model under scrutiny here says that an experimental determination of the odd-even shift is a measurement of the sign and magnitude of $B$. Thus it is worthwhile to examine Eq. (4) more closely.

$B$ is an *off-diagonal* matrix element. The initial and final states may be very different. In cases of interest $K_1$ is often large ($K_1=\frac{7}{2}$ for $Ho^{166}$). Because $K_1$ is the projection of the total angular momentum of one of the particles on the nuclear symmetry axis this implies large total angular momentum for each of the particles 1 and 2. The large angular momentum and angular momentum projection means that one can conveniently think of the particles as classical particles traveling in orbits around the periphery of the nucleus in the region of the nuclear equator. $B$ is a sum of contributions from various scattering processes which connect initial and final state. If we ignore, for the moment, the fact that the interaction $V$ may contain some space-exchange components we may visualize the scattering processes which contribute to $B$ in the following way. In the initial state before the scattering we have particle 1 traveling around the nuclear surface in the region of the equator in a "counter-clockwise" direction. Particle 2 is traveling in the same plane but in a "clockwise" direction. The particles meet head-on in a catastrophic collision. The final state which they must scatter into is one in which each particle is in the same orbit as before the collision but is now traveling in the opposite direction. Thus we have a backward scattering in the center-of-mass system of particles 1 and 2. Although particles 1 and 2 will undergo a wide variety of scatterings only those of the above type involving angular momentum transfers greater than $2K_1$ can contribute to $B$. The above picture requires some modification when $V$ contains a space-exchange component. Our purpose here, however, is simply to emphasize the specialized nature of the scatterings which are contributing to $B$.

The interaction $V$ will in general be a mixture of a number of components—central and noncentral forces with various exchange properties. Due to the particularity of the scatterings measured by $B$ it is clear that the components of $V$ will differ in their ability to produce the type of scattering needed. We suspect, in fact, that the contributions of certain components of $V$ may be highly damped out thereby allowing other components to manifest themselves more clearly. Hopefully this selective effect might prove to be a rather sensitive probe of the *n-p* residual interaction between peripheral nucleons.

The preceding discussion becomes more meaningful if we recall a fact not mentioned thus far. Some of the nuclei of interest will be those having large equilibrium deformations. The intrinsic single-particle states in such nuclei are often close to their asymptotic forms. In the asymptotic limit, in addition to $K_1$, we have two more good quantum numbers, $\Lambda_1$ and $\Sigma_1$. $\Lambda_1$ and $\Sigma_1$ are the projections on the nuclear symmetry axis of the orbital angular momentum and intrinsic spin, respectively. That is, $K_1=\Lambda_1+\Sigma_1$. In the asymptotic limit, then, the types of scatterings contributing to $B$ are even more restricted than before. In this limit the four single-particle states in $B$ are states of 100% polarization of intrinsic spin. Only those components of $V$ which have the capability of linking such polarized states can contribute.

In order to investigate this matter further let us rewrite Eq. (4) in the asymptotic limit of large deformation—this time expanding the single-particle states in their orbital and intrinsic spin components. There are now two cases to consider. Either the intrinsic spin projections of particles 1 and 2 are antiparallel or parallel—that is, either $\Sigma_1+\Sigma_2=0$ or $\Sigma_1+\Sigma_2=1$.

Case I: $\Sigma_1+\Sigma_2=0$.

$$B=(-1)^{l_1+l_2+1}\langle Y_{l_1-\Lambda_1}(1)\chi_{-\Sigma_1}(1)Y_{l_2+\Lambda_1}(2)\chi_{+\Sigma_1}(2) \times|\,V\,|\,Y_{l_1'\Lambda_1}(1)\chi_{+\Sigma_1}(1)Y_{l_2'-\Lambda_1}(2)\chi_{-\Sigma_1}(2)\rangle, \quad K_1=\Lambda_1+\Sigma_1. \tag{5}$$

Case II: $\Sigma_1+\Sigma_2=1$.

$$B=(-1)^{l_1+l_2+1}\langle Y_{l_1-\Lambda_1}(1)\chi_{-\Sigma_1}(1)Y_{l_2(+\Lambda_1+2\Sigma_1)}(2)\chi_{-\Sigma_1}(2) \times|\,V\,|\,Y_{l_1'\Lambda_1}(1)\chi_{+\Sigma_1}(1)Y_{l_2'(-\Lambda_1-2\Sigma_1)}(2)\chi_{+\Sigma_1}(2)\rangle, \quad K_1=\Lambda_1+\Sigma_1. \tag{6}$$

[3] See, for example, S. A. Moszkowski, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 482.

Here the $Y_{lm}$ are the spherical harmonics and the $\chi_\sigma$ are the intrinsic spin functions. It is understood that the above expressions are to be summed over $l_1l_2l_1'l_2'$ with the expansion coefficients appropriate to the asymptotic single-particle states. For the present we omit these sums—as well as the four radial wave functions—as immaterial to our discussion. In each of the two cases we verify that both initial and final states have $K=K_1+K_2=0$.

In both cases I and II we observe that particles 1 and 2 must flip their intrinsic spin in going from initial to final state. It is clear, then, that only spin-dependent forces can contribute to $B$ in the asymptotic limit. However, examination of case II reveals a much more drastic restriction. Here the spin space of the two particles has experienced a change of two units of angular momentum projection. (This change has, of course, been balanced by a corresponding change in the angular momentum projection in the orbital space.) A moment's reflection reveals that *no* central force regardless of its exchange character can produce such a scattering. Central forces are scalars in the orbital, spin, and isotopic spin spaces separately. Thus they conserve angular momentum independently in all three spaces. Only a tensor force is capable of transferring angular momentum between spin and orbital spaces as required in case II.[4]

The above discussion can be summarized in the following selection rules.

(1) (Asymptotic) Only spin dependent forces can contribute to the odd-even shift.

(2) (Asymptotic) If $\Sigma_1+\Sigma_2=1$ only a tensor interaction can produce an odd-even shift.

(3) A Wigner force cannot contribute to the odd-even shift-independent of nuclear deformation.

The third rule (proved in Appendix I) is true for all nonzero deformations. The effect of rule (3) is to remove a degree of freedom from the problem thus simplifying considerably the task of arriving at an unambiguous $n$-$p$ residual force.

The first two selection rules are rigorously true only in the asymptotic limit of large nuclear deformation. Their approximate validity in any given nucleus will depend on how close the single-particle wave functions are to their asymptotic forms. Or more precisely—how effective the deformation has been in polarizing the intrinsic spin. In a particular case preliminary examination of the single-particle states may reveal that they contain only small admixtures of nonasymptotic amplitudes. In such a case selection rules (1) and (2) may be employed as useful guides in understanding which parts of the $n$-$p$ interaction are most effective in producing the shift.

In addition to the shift due to a residual $n$-$p$ interaction, the members of a $K=0$ rotational band will also be perturbed by the Coriolis coupling. It is necessary to understand the effect of this latter perturbation in order to disentangle the shift due to the residual force. In the special case that $K=K_1+K_2=0$ with $K_1=\frac{1}{2}$ the Coriolis interaction makes a first order contribution to the energy. An expression for this contribution is given in Appendix II.

In all other cases ($K_1\neq\frac{1}{2}$) the Coriolis interaction will enter only in second order—that is, mixing between rotational bands. The Coriolis interaction only mixes bands which differ by one unit in $K$ value. Therefore if there is a $K=1$ band near the $K=0$ band then the energy levels of the $K=0$ band will be perturbed. However, if the effect is weak so that we may use second order perturbation theory, then it can be shown[5] that the energy shifts in the $K=0$ band are proportional to $I(I+1)$. Therefore these shifts can be taken into account by simply renormalizing the moment of inertia.

Kerman's study of the odd-$A$ nucleus $W^{183}$ provides an example.[6] Detailed analysis showed that although there was a substantial amount of interband mixing the effect on the *energy levels* could be essentially completely accounted for by renormalizing the moment of inertia. Thus, even in the presence of mixing, the energy levels within the band were given by $E(I)=(\hbar^2/2\mathcal{I}_{\text{eff}})I(I+1)$.

The procedure for measuring the odd-even shift is just as before. One determines the renormalized moment of inertia from the $(I=0)$–$(I=2)$ energy spacing, for example. This contains the effect of interband mixing on the energy levels. This value is sued to predict the location of $I=1$. The deviation of $I=1$ from this predicted position is then interpreted as the odd-even shift due to the $n$-$p$ residual interaction. (An exceptional case is discussed in Appendix III.)

## III. CALCULATIONS

In carrying out the numerical work the single-particle states for finite nuclear deformation were taken from the tables of Nilsson[7] and Mottelson and Nilsson.[8]

These wave functions are[9]

$$\phi_{+K}(1)=\sum_{l_1m_1} a_{l_1m_1}R_{n_1l_1}(1)Y_{l_1m_1}(1)\chi_{K-m_1}{}^{\frac{1}{2}}(1) \tag{7}$$

$$\phi_{-K}(2)=(-1)^K\sum_{l_2m_2} a_{l_2m_2}R_{n_2l_2}(2) \times Y_{l_2-m_2}(2)\chi_{-K+m_2}{}^{\frac{1}{2}}(2). \tag{8}$$

[4] We note that a two-body spin-orbit force is also incapable of producing the required scattering.

[5] This follows from a simple generalization of the derivation for the odd-$A$ case. See, for example, reference 3.

[6] A. K. Kerman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 15 (1956).

[7] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 16 (1955). It is important to note that Nilsson's phase convention (see p. 35 of this reference) differs from the usual one. The author, who had overlooked this point, thanks Dr. Nilsson for calling it to his attention.

[8] B. R. Mottelson and S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Skrifter **1**, No. 8 (1959). See pages 21, 22, 52, and 53 of this reference for guidance on use of the tables.

[9] $R_{nl}$ is a harmonic oscillator radial wave function. We have chosen to define $\phi_{-K}(2)=T\phi_{+K}(2)$ where $T=\sigma_yK$ is the time-reversal operator. $\sigma_y$ is the Pauli spin operator and $K$ indicates complex conjugation. This choice leads to the phase factor included in Eq. (8). The final results are, of course, independent of this phase.

The expansion coefficients $a_{lm}$ are tabulated in references 7 and 8 for various nuclear deformations. Using this representation for the single-particle states, we find

$$B=(-1)^{l_1+l_2+1}\langle\phi_{-K}(1)\phi_{+K}(2)|V|\phi_{+K}(1)\phi_{-K}(2)\rangle. \quad (9)$$

We note that the factor preceding the matrix element is just $(-1)$ times the nuclear parity. The central-force part of the $n$-$p$ interaction is written as

$$V=U(r)[U_1P_M+U_2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+U_3(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2P_M)+U_4]. \quad (10)$$

$P_M$ is the space exchange operator; $U(r)$ is the well shape factor; and $U_1$, $U_2$, $U_3$, and $U_4$ are the well depths for the various components of the force as indicated. Having shown that the Wigner interaction does not contribute, $B$ is written

$$B=U_1\langle P_M\rangle+U_2\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle+U_3\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2P_M\rangle, \quad (11)$$

where

$$\langle P_M\rangle=P\sum_{l_1l_2}\sum_{l_1'l_2'}\sum_k\bar{A}(l_1l_2l_1'l_2'k)B1(l_1l_2l_1'l_2'k), \quad (12)$$

$$\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle=(-1)P\sum_{l_1l_2}\sum_{l_1'l_2'}\sum_k A(l_1l_2l_1'l_2'k)\times B2(l_1l_2l_1'l_2'k), \quad (13)$$

$$\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2P_M\rangle=(-1)P\sum_{l_1l_2}\sum_{l_1'l_2'}\sum_k\bar{A}(l_1l_2l_1'l_2'k)\times B3(l_1l_2l_1'l_2'k), \quad (14)$$

with

$$A(l_1l_2l_1'l_2'k)=[(2l_1+1)(2l_2+1)(2l_1'+1)(2l_2'+1)]^{\frac{1}{2}}\times F_k(n_1l_1n_2l_2n_1'l_1'n_2'l_2'). \quad (15a)$$

$$\bar{A}(l_1l_2l_1'l_2'k)=[(2l_1+1)(2l_2+1)(2l_1'+1)(2l_2'+1)]^{\frac{1}{2}}\times\bar{F}_k(n_1l_1n_2l_2n_1'l_1'n_2'l_2'), \quad (15b)$$

$$B1(l_1l_2l_1'l_2'k)=\left(\frac{1}{2k+1}\right)^2\sum_{\mu\nu}a_{l_1,K-\mu}a_{l_2,K+\nu}a_{l_1',K+\mu}a_{l_2',K-\nu}\times\langle l_1,l_2',-K+\mu,K-\nu|k,\mu-\nu\rangle\times\langle l_2,l_1',K+\nu,-K-\mu|k,-\mu+\nu\rangle\times\langle l_1,l_2',0,0|k,0\rangle\langle l_2,l_1',0,0|k,0\rangle, \quad (16)$$

$$B2(l_1l_2l_1'l_2'k)=2\left(\frac{1}{2k+1}\right)^2\sum_{\mu\nu\mu'}a_{l_1,K-\mu}a_{l_2,K+\nu}a_{l_1',K-\mu'}a_{l_2',K-\nu-\mu-\mu'}\times\langle l_1,l_1',K-\mu,K-\mu'|k,2K-\mu-\mu'\rangle\times\langle l_2,l_2',-K-\nu,-K+\nu+\mu+\mu'|k,-2K+\mu+\mu'\rangle\times\langle l_1,l_1',0,0|k,0\rangle\langle l_2,l_2',0,0|k,0\rangle\langle\tfrac{1}{2},\tfrac{1}{2},\mu,\mu'|1,\mu+\mu'\rangle\times\langle\tfrac{1}{2},\tfrac{1}{2},\nu,-\nu-\mu-\mu'|1,-\mu-\mu'\rangle, \quad (17)$$

$$B3(l_1l_2l_1'l_2'k)=2\left(\frac{1}{2k+1}\right)^2\sum_{\mu\nu\mu'}a_{l_1,K-\mu}a_{l_2,K+\nu}a_{l_1',K-\mu'}a_{l_2',K-\nu-\mu-\mu'}\times(-1)^{\mu+\nu}\langle l_1,l_2',K-\mu,-K+\nu+\mu+\mu'|k,\mu'+\nu\rangle\times\langle l_2,l_1',-K-\nu,K-\mu'|k,-\mu'-\nu\rangle\times\langle l_1,l_2',0,0|k,0\rangle\langle l_2,l_1',0,0|k,0\rangle\langle\tfrac{1}{2},\tfrac{1}{2},\mu,\mu'|1,\mu+\mu'\rangle\times\langle\tfrac{1}{2},\tfrac{1}{2},\nu,-\nu-\mu-\mu'|1,-\mu-\mu'\rangle. \quad (18)$$

$P$=nuclear parity$=\pm1$. The $C$'s are the usual Clebsch-Gordan coefficients and $F_k$ and $\bar{F}_k$ are the radial integrals

$$F_k=(2k+1)\int_0^\infty\int_0^\infty r_1^2dr_1\,r_2^2dr_2\times R_{n_1l_1}(r_1)R_{n_2l_2}(r_2)R_{n_1'l_1'}(r_1)R_{n_2'l_2'}(r_2)\times\int_{-1}^{+1}\frac{d\mu}{2}U(r)P_k(\mu), \quad (19a)$$

$$\bar{F}_k=(2k+1)\int_0^\infty\int_0^\infty r_1^2dr_1\,r_2^2dr_2\times R_{n_1l_1}(r_1)R_{n_2l_2}(r_2)R_{n_2'l_2'}(r_1)R_{n_1'l_1'}(r_2)\times\int_{-1}^{+1}\frac{d\mu}{2}U(r)P_k(\mu), \quad (19b)$$

where $P_k(\mu)$ is a Legendre polynomial.[10,10a] One can show, as expected, that $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2P_M\rangle\equiv\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle$ and $\langle P_M\rangle\equiv0$ in the limit of a zero-range force.

In Sec. II it was pointed out that a space exchange type interaction would destroy the picture of "catastrophic collisions." This is observed above in that both $\langle P_M\rangle$ and $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2P_M\rangle$ involve sums over all angular momentum transfers $k$, whereas $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle$ contains only those $k$ for which $k\geqq2K-1$ as is seen in the Clebsch-Gordan coefficients. It is clear the results will be sensitive to the amount of Majorana exchange in the force.

In carrying out the numerical work we have used for $U(r)$ the Gaussian

$$U(r)=\exp(-\beta r^2), \quad (20)$$

with $\beta^{-\frac{1}{2}}=1.6\times10^{-13}$ cm.

The nuclear radius enters the problem through the parameter $\nu$ in the radial wave function. This parameter is fixed as follows. One examines the single particle states to see what is the highest orbital angular momentum involved. Call this number $l$. (In the cases of

[10] K. W. Ford and E. J. Konopinski, Nuclear Phys. **9**, 218 (1959).

[10a] Thanks go to Dr. N. K. Glendenning for the use of his computer program in evaluating the radial integrals.

interest here $l=4$ or 5 or 6.) This component of the wave function will have a radial dependence,

$$R_{1l}\sim r^l \exp[-(1/2)\nu r^2]. \quad (21)$$

The classical turning point of such a component will be at a radius $R$ where

$$R^2=(2l+3)/\nu. \quad (22)$$

$\nu$ is now fixed by setting $R=1.2\times A^{\frac{1}{3}}\times 10^{-13}$ cm. Using these values of the parameters the dimensionless quantities $\langle P_M\rangle$, $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle$, and $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 P_M\rangle$ will be presented in each case. The reader may then use Eq. (11) to examine the results arising from various force mixtures.

In order to have a definite number to compare with experiment a particular choice was made for $V$. As will be apparent later, this force is in no sense a "best fit' but merely serves as a convenient standard of comparison. The experimental data is, as yet, too limited to make adjustment of force parameters very meaningful. Our choice is

$$V=U_0\exp(-\beta r^2)[\tfrac{15}{16}+\tfrac{1}{16}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2][\tfrac{1}{2}[1+P_M], \quad (23)$$

with $U_0=-60$ Mev and $\beta^{-\frac{1}{2}}=1.6\times 10^{-13}$ cm. This interaction is similar to those used in other calculations.[11,12] It yields a singlet and triplet effective range and scattering length comparable with those obtained from low-energy nucleon-nucleon data. The use of a Serber mixture seems a reasonable first choice for the space exchange character.

$_{67}Ho_{99}^{166}$: The single-particle states seem quite certain.[13] They are neutron: $\frac{7}{2}+$ [633], $\eta=+6$ and proton: $\frac{7}{2}-$ [523], $\eta=+6$. The asymptotic quantum numbers (enclosed in brackets) and $\eta$, which is related to the deformation, are defined in references 7 and 8. Here we have used Eq. (22) with $l=6$ to specify $\nu$. We find $\langle P_M\rangle=+0.00215$, $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle=+0.00479$, and $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 P_M\rangle=+0.02222$. The approximate validity of selection rule 1 can be seen by comparing $\langle P_M\rangle$ and $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 P_M\rangle$. The introduction of $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ causes the matrix element to increase by an order of magnitude.

Using $V$ defined by Eq. (23), we find that $B=-111$ kev. Referring to Eq. (3) shows that the $I=1$ state should be moved up in energy relative to $I=0$ and $I=2$ by 222 kev. This is to be compared with a shift of 64 kev indicated by experiment (Fig. 1). It is possible that the choice of $l=4$ rather than $l=6$ is more reasonable in this region of the periodic table. If we use $l=4$ in Eq. (22) we can estimate the effect on the energy shift. With $l=4$ the $I=1$ state is moved up relative to $I=0$ and 2 by $\sim$150 kev. Therefore, using $V$ as specified by Eq. (23), we find a shift which is of the correct sign but too large.

By decreasing the percentage of Majorana exchange we can, of course, fit the experimental shift.

$_{95}Am_{147}^{242}$: The experimental evidence is reviewed in reference 2. The ground state seems to have $I=1$, $K=0$. This is surprising. If the gound state is a member of a $K=0$ band one would expect it to have $I=0$. A possible interpretation is that there is an odd-even shift in the $K=0$ band which has pushed the $I=1$ state below the $I=0$ state. Although the magnitude of the shift is unknown at present, it is of interest to see if we can account for the sign of the shift. The most likely single particle states are neutron: $\frac{5}{2}+$ [622], $\eta=+6$ and proton: $\frac{5}{2}-$ [523], $\eta=+6$. $\nu$ is fixed by setting $l=6$ in Eq. (22). A preliminary examination of the wave functions shows that the intrinsic spins of both neutron and proton are highly polarized. This, together with the fact that the situation is in the category covered by selection rule 2, leads us to expect that central-force effects will be highly damped. Calculation yields $\langle P_M\rangle=-0.00022$, $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle=+0.00111$, $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 P_M\rangle=+0.00320$. Using $V$ as specified by Eq. (23), we find that $B=-2$ kev. (*Each* of the three contributions is 6 kev or less in magnitude.) This is the central force damping which was anticipated. The effect of a tensor interaction is discussed below.

$_{71}Lu_{101}^{172}$: There is a group of three levels which may be members of a $K=0$ excited rotational band. The levels at 65.6, 109.7, and 191.4 kev are assigned spins of 0, 2, 1, respectively.[14] (The interpretation as a $K=0$ band must be considered as tentative. The 0–2 spacing is anomalously small.) Applying Eq. (1) we expect to find the $I=1$ state at 15 kev above the $I=0$ state. Instead it is 126 kev above $I=0$. Therefore it seems that the $I=1$ state has been pushed up relative to $I=0$ and 2 by 111 kev.

The most reasonable choice for single particle states seems to be neutron: $\frac{7}{2}+$ [633], $\eta=+6$ and proton $\frac{7}{2}+$ [404], $\eta=+4$.[8,15] Examination of these wave functions shows that the states are strongly polarized. Since the configuration is of the type discussed in selection rule 2 we expect central forces to be damped and the tensor force to make a crucial contribution. The parameter $\nu$ is fixed by using $l=5$ in Eq. (22). Calculation yields $\langle P_M\rangle=+0.00096$, $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle=-0.00050$, and $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 P_M\rangle=-0.00355$. Using $V$ from Eq. (23), $B=-19$ kev. (Comparing this with the result obtained for $Ho^{166}$—a nucleus of the same size—one sees that the central-force contribution has dropped by a factor of about 5.) Our central-force result is, then, that states of odd $I$ will be pushed up by 38 kev relative to states of even $I$. This is in the same direction as the experimental shift but only about $\frac{1}{3}$ as large. Next we turn our attention to the tensor force.[16]

[11] N. Newby, Jr. and E. J. Konopinski, Phys. Rev. **115**, 434 (1959).

[12] W. True and K. Ford, Phys. Rev. **109**, 1675 (1958).

[13] See reference 2 for further details on the single-particle states involved in $Ho^{166}$ and $Am^{242}$.

[14] J. Valentin, D. J. Horen, and J. M. Hollander, Nuclear Phys. (to be published).

[15] R. G. Wilson and M. I. Pool, Phys. Rev. **120**, 1843 (1960).

[16] It was suggested (S. G. Nilsson, private communication) that in $Lu^{172}$ we might see the proton state $\frac{1}{2}-$ [541] coming down from the fifth oscillator shell. A $K=0$ band was constructed by coupling this state with the neutron state: $\frac{1}{2}-$ [521]. A calculation using the standard $V$ defined by Eq. (23) revealed that even-$I$ states would be shifted upwards relative to odd-$I$ states by 74 kev. Further, each central-force term contributed with the same sign. Next, the term discussed in Appendix II was evaluated (deter-

In $Am^{242}$ and to a lesser extent in $Lu^{172}$ it was found that the central-force contribution to the odd-even shift was damped. This was expected since the single-particle states in both cases made selection rule 2 applicable and inspection of the single-particle states showed that the intrinsic spins were highly polarized. It was felt, therefore, that in these cases a calculation of tensor-force effects was in order.

The calculation of tensor-force effects in spherical shell model states is rather laborious.[11] Here the labor is compounded since we have a fourfold sum over spherical states. It was found in calculating the central-force matrix elements, however, that the dominant contribution to the sum came, in each case, from the term where both particles, initially and finally, were in their component of maximum orbital angular momentum. This was expected since the highest orbital momentum component in the single-particle wave functions was either the largest or comparable with the largest component. More importantly, these highest angular momentum components always have a nodeless radial wave function thus guaranteeing the best possible radial overlap. It was found in $Am^{242}$, for example, that the central-force term $l_1=l_1'=5$, $l_2=l_2'=6$ dominated the sums. Therefore the tensor force matrix element was calculated with $\Lambda$ and $\Sigma$ as good quantum numbers assuming that the above-mentioned term would give the dominant contribution. A method was developed which gives an exact answer in the limit of a short-range tensor force. The numerical value attained by this procedure is not expected to be significant beyond indicating the sign of the tensor force contribution to the odd-even shift.

In $Am^{242}$ such a calculation showed that an "attractive" tensor interaction (of such sign as to give the deuteron quadrupole moment) would raise states of even $I$ and lower those of odd $I$. This is in agreement with the limited experimental data and consistent with the implication of the selection rules—that the shift in this case should be decided by the tensor interaction.

A similar calculation for $Lu^{172}$ yielded the result that the tensor force would raise states of odd $I$ and lower those of even $I$. A shift of such a sign when added to the previously calculated central-force shift is such as to move the calculated value of the shift closer to the experimental value.

## IV. DISCUSSION

The point of view taken in the preceding section was to calculate only central-force effects when the configuration did not tend to emphasize the tensor-force contribution.[17] In such cases it was believed that the effect of the tensor interaction could be taken into account by simply making adjustments of the central-force parameters. Only in those cases where the configuration was such that central-force contributions were strongly damped was it deemed necessary to make an explicit tensor calculation. $Am^{242}$ is such a case.

The effects which we are calculating are small and depend sensitively on the overlap of the wave functions involved. However, the results are dominated by terms in which $l_1=l_1'$ and $l_2=l_2'$. Such terms involve the same radial wave functions in initial and final state. Thus the radial integrals are "diagonal" and the results should not be especially sensitive to the particular choice of radial functions (harminic oscillator functions here).

The matrix element $B$ is nondiagonal in a very special sense. The final state is simply the initial state rotated through 180°. This symmetry is supplemented by the symmetries imposed on the single-particle states due to the nuclear deformation. These symmetry conditions combine to impose grave restrictions on the type of scattering which can join initial and final states. The Wigner-force contribution is completely damped. As one approaches the asymptotic limit, the pure Majorana contribution goes to zero and only spin-dependent forces can contribute to the shift. For certain configurations the effect of all central forces becomes damped at large deformation and the shift is due to the tensor interaction alone.

It is felt that the present preliminary calculations substantiate the physical model under study here. More importantly, we have tried to show why the odd-even shift in the $K=0$ band is a rather sensitive probe of the residual $n$-$p$ interaction between surface nucleons.

## ACKNOWLEDGMENTS

Professor John O. Rasmussen suggested this investigation. His advice and enthusiasm are greatly appreciated.

My thanks go to Dr. Brian Judd and Dr. Sven G. Nilsson for helpful conversations.

I am grateful to Professor Isadore Perlman for the opportunity of working in the Chemistry Division.

## APPENDIX I

If we calculate $B$ (Eq. 9) using a pure Wigner force for $V$, we find

$$B=\sum_{l_1 l_2}\sum_{l_1' l_2' k}\sum_{\mu\nu} a_{l_1,K-\mu}a_{l_2,K+\nu}a_{l_1',K+\mu}a_{l_2',K-\nu}(-1)^{l_1+l_2}$$
$$\times[(2l_1+1)(2l_2+1)(2l_1'+1)(2l_2'+1)]^{\frac{1}{2}}$$
$$\times F_k(l_1 l_2 l_1' l_2')(-1)^{\mu+\nu}\left(\frac{1}{2k+1}\right)^2$$
$$\times\langle l_1, l_1', K-\mu, K+\mu | k, 2K\rangle$$
$$\times\langle l_2, l_2', -K-\nu, -K+\nu | k, -2K\rangle$$
$$\times\langle l_1,l_1',0,0 | k,0\rangle\langle l_2,l_2',0,0 | k,0\rangle.$$

mining $\hbar^2/2\mathcal{I}$ from the 0–2 energy spacing). It was found that this term moved even-$I$ states down relative to odd-$I$ states by 14 kev. Thus this rotational term is bucking the shift due to the residual $n$-$p$ interaction but is only about $\frac{1}{5}$ as large. The conclusion is that this choice of single-particle states cannot give agreement with the experimental level scheme.

[17] Neal D. Newby, Jr., Phys. Rev. **119**, 747 (1960).

Next we re-express $B$ as

$$B=\sum_{l_2l_2'}\sum_{k\nu}a_{l_2,K+\nu}a_{l_2',K-\nu}(-1)^{l_2+\nu}\left(\frac{1}{2k+1}\right)^2$$
$$\times\langle l_2, l_2', -K-\nu, -K+\nu\,|\,k, -2K\rangle\langle l_2,l_2',0,0\,|\,k,0\rangle$$
$$\times[(2l_2+1)(2l_2'+1)]^{\frac{1}{2}}\sum_{l_1l_1'}G(l_1l_2l_1'l_2')H(l_1l_2l_1'l_2'),$$

with

$$G(l_1l_2l_1'l_2')=(-1)^{l_1}[(2l_1+1)(2l_1'+1)]^{\frac{1}{2}}F_k(l_1l_2l_1'l_2'),$$
$$H(l_1l_2l_1'l_2')=\sum_\mu(-1)^\mu a_{l_1,K-\mu}a_{l_1',K+\mu}$$
$$\times\langle l_1, l_1', K-\mu, K+\mu\,|\,k, 2K\rangle\langle l_1,l_1',0,0\,|\,k,0\rangle$$

The indices $l_1$ and $l_1'$ range over the same set of values which are either all even or all odd integers. $\mu=\pm\frac{1}{2}$. Observing that

$$G(l_1l_2l_1'l_2')=+G(l_1'l_2l_1l_2'),$$
$$H(l_1l_2l_1'l_2')=-H(l_1'l_2l_1l_2'),$$

it is clear that

$$\sum_{l_1l_1'}G(l_1l_2l_1'l_2')H(l_1l_2l_1'l_2')=0,$$

and therefore

$$B\equiv 0.$$

Thus the Wigner interaction cannot contribute to the shift. The result is, of course, independent of force range and nuclear deformation—assuming the latter is non-zero. If one first expands the single-particle states in eigenstates of $j$ and then evaluates the matrix element, it becomes clear that the above result is due to the rotation properties of spinor particles.

## APPENDIX II

For an odd-odd nucleus with $K=K_1+K_2=0$ and $K_1=\frac{1}{2}$ there is a first order contribution to the energy level structure due to a term in the collective rotational part of the Hamiltonian. The Hamiltonian can be written[3]

$$H=H_{\text{intr}}(\mathbf{r}^1,\mathbf{r}^2)+T_{\text{rot}},$$

where $H_{\text{intr}}(\mathbf{r}^1,\mathbf{r}^2)$ is the Hamiltonian for the motion of particles 1 and 2 in the body-system and $T_{\text{rot}}$ is

$$T_{\text{rot}}=(\hbar^2/2\mathcal{I})\{[\mathbf{I}-\mathbf{j}(1)-\mathbf{j}(2)]^2-[I_3-j_3(1)-j_3(2)]^2\}$$
$$=(\hbar^2/2\mathcal{I})\{[I^2+j^2(1)+j^2(2)]-[I_3-j_3(1)-j_3(2)]^2\}$$
$$+(\hbar^2/2\mathcal{I})\{-2[\mathbf{I}\cdot\mathbf{j}(1)+\mathbf{I}\cdot\mathbf{j}(2)]+2\mathbf{j}(1)\cdot\mathbf{j}(2)\}.$$

Evaluating the diagonal matrix element of $(\hbar^2/\mathcal{I})\mathbf{j}(1)\cdot\mathbf{j}(2)$ we find

$$\langle\psi_{MK=0}{}^I|\,(\hbar^2/\mathcal{I})\mathbf{j}(1)\cdot\mathbf{j}(2)\,|\psi_{MK=0}{}^I\rangle$$
$$=\text{const}+(-1)^{I+1}(\hbar^2/2\mathcal{I})a_1a_2,$$

where $a_1$, $a_2$ are the decoupling parameters for particles 1 and 2 defined by

$$a=\sum_j(-1)^{j-\frac{1}{2}}(j+\tfrac{1}{2})\,|C_j|^2.$$

Here the $C_j$ are the expansion coefficients of the single-particle states in terms of eigenstates of $j$. The energy level formula now becomes

$$E(I)=(\hbar^2/2\mathcal{I})[I(I+1)+(-1)^{I+1}a_1a_2\delta_{K,0}\delta_{|K_1|,\frac{1}{2}}].$$

The situation here is, in a sense, simpler than the analogous situation in odd-$A$ nuclei with $K=\frac{1}{2}$. Because the $I$ dependence of the above expression enters only through $(-1)^I$, we can use the 0–2 energy spacing to find $\hbar^2/2\mathcal{I}$ immediately. Then the above term may be calculated to find the predicted position of $I=1$. The deviation of $I=1$ from this position is interpreted as due to the odd-even shift arising from the residual $n$-$p$ interaction.

## APPENDIX III

There is a special case where the effect on energy levels of interband mixing cannot be taken into account by renormalizing the moment of inertia—even though second-order perturbation theory may be valid. Consider rotational particle mixing between a $K=0$ band and a $K=1$ band. The $K=0$ band is to be constructed from the single particle states

$$\psi_{K_1}(1)=\sum_{j_1}C_{j_1}(1)\phi_{K_1}{}^{j_1}(1),$$
$$\psi_{K_1}(2)=\sum_{j_2}C_{j_2}(2)\phi_{K_1}{}^{j_2}(2),$$

and the $K=1$ band from

$$\bar{\psi}_{K_1+1}(1)=\sum_{j_1}\bar{C}_{j_1}(1)\phi_{K_1+1}{}^{j_1}(1),$$
$$\psi_{K_1}(2)=\sum_{j_2}C_{j_2}(2)\phi_{K_1}{}^{j_2}(2).$$

That is, in the $K=1$ band we have promoted particle 1 to a state with projection $K_1+1$ whereas particle 2 is still in the same state.

Examination of the Hamiltonian in Appendix II reveals that the matrix element connecting the two bands is

$$\left\langle\psi_{MK=0}{}^I\left|-\frac{\hbar^2}{\mathcal{I}}[\mathbf{I}\cdot\mathbf{j}(1)+\mathbf{I}\cdot\mathbf{j}(2)]\right|\psi_{MK=1}{}^I\right\rangle.$$

Calling this matrix element $\alpha$ we find

$$\alpha=-\left(\frac{\hbar^2}{2\mathcal{I}}\right)[I(I+1)]^{\frac{1}{2}}\sum_{j_1j_2}C_{j_1}{}^*C_{j_2}{}^*\bar{C}_{j_1}C_{j_2}$$
$$\times\{[(j_1+K_1+1)(j_1-K_1)]^{\frac{1}{2}}+(-1)^{I-j_1-j_2}$$
$$\times[(j_2+K_1+1)(j_2-K_1)]^{\frac{1}{2}}\delta_{+K_1,-K_1-1}\}.$$

In second-order perturbation theory the energy shift due to the mixing is proportional to $\alpha^2$. Thus, provided the second term inside the brackets does not contribute, the shifts are proportional to $I(I+1)$ and can be absorbed by renormalizing $\mathcal{I}$. However if $K_1=-\frac{1}{2}$ the second term will contribute.

It is clear that we can absorb the energy shifts by renormalizing $I$ *separately* for even-$I$ and odd-$I$ states.

We may illustrate the situation by the following case of practical interest. Suppose we have a neutron and proton state each having a projection of $\frac{1}{2}$. We can couple these single particle states to form a $K=0$ state and a $K=1$ state. The $K=0$ rotational band will be perturbed by the $K=1$ band. Both the 0–2 energy spacing and the 1–3 energy spacing of the $K=0$ band will be independent of the odd-even shift discussed in Appendix II and the residual $n$-$p$ force shift. The moment of inertia derived from the 0–2 spacing will, however, *differ* from that derived from the 1–3 spacing due to the fact that $\alpha^2$ above is no longer simply proportional to $I(I+1)$.

An extreme example of this $I$-dependent interband-mixing is found when the isotopic spin $T$ is a good quantum number. Suppose neutron and proton are in the *same* single particle state with projection $\frac{1}{2}$. In the $K=0$ band the states with $I=$even have $T=1$ and $I=$odd states have $T=0$. In the $K=1$ band all states have $T=0$. It is clear that only the $I=$odd states in the $K=0$ band will mix with the $K=1$ band. The $I=$even states will not mix. The matrix element between the two bands is now

$$\alpha=i(-1)^{l}(\hbar^2/2\mathcal{I})[I(I+1)]^{\frac{1}{2}} \times[\textstyle\sum_{j_1}(-1)^{j_1-\frac{1}{2}}(j_1+\frac{1}{2})|C_{j_1}|^2][1+(-1)^{I+1}] =i(-1)^{l}(\hbar^2/2\mathcal{I})[I(I+1)]^{\frac{1}{2}}a[1+(-1)^{I+1}].$$

[When $T$ is a good quantum number it is of course clear what the *sign* of the shift due to the residual $n$-$p$ force will be. The odd-$I$ states ($T=0$) symmetric in space and spin will feel the effect of $V$ more strongly than the even-$I$ states ($T=1$). Thus the odd-$I$ states will be shifted downward in energy relative to the even-$I$ states.]