

# Dispersion-Theoretical Fitting of Nucleon-Nucleon Scattering\*

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A general approach based on partial-wave dispersion relations is presented for obtaining a formula for phase shifts which depends on only a small number of parameters. This approach does not make any specific assumptions on, or approximations of, the unphysical discontinuity, and involves only approximations of functions that are already known from the general framework of the theory. The method approximately reproduces the  $^1S_0$  state in the 0- to 100-Mev range, with two parameters determined from the low-energy data and a third from a phase shift at a higher energy.

## I. INTRODUCTION

THE usual approach to the construction of effective-range formulas in dispersion theory consists of replacing unknown unphysical cuts by a small number of simple poles. Such a procedure is certainly justified for a physical energy range that is small compared with its distance from these cuts. This is often not the case in actual applications, however. The only way of justifying the pole approximation in such cases has been to solve the dispersion relations exactly for several simple assumed forms of the unphysical discontinuities and comparing these solutions with the pole formula. This also gives a rough estimate of the possible error associated with the approximation.

The main defects of the above approach are that it may involve a large amount of computation in solving dispersion relations exactly, and that one does not see in any natural way why the approximation works in the way it does. Indeed, it is comparable to the early work on low-energy nucleon-nucleon scattering, in which the problem was solved accurately for a number of standard potentials to establish the fact that the low-energy region could be characterized in terms of only two parameters per angular-momentum state. As in that problem, so here also, it would be desirable to find an approach in which the approximate formula could be justified in a natural way, without the consideration of specific examples.

In the present method this is attempted by making certain approximations only for functions that are already definitely known from the general theory. These are made in such a way that we are left with an expression depending on only a small number of unknown constants, which may then be determined experimentally. No approximations are made for unknown functions as in the pole approach. Thus our results are independent of the specific shape of the unphysical discontinuity. Moreover, the accuracy of our approximate formula in any given energy range, or, equivalently, its range of validity, can be easily estimated in an *a priori* manner, even before we have any knowledge of the orders of magnitude of our phenomenological parameters. Finally, some (or all) of these constants can

be calculated in a fairly simple manner from part (or all) of the unphysical cut if it is known, provided that we make our approximations suitably.

## II. SHAPE-INDEPENDENT APPROXIMATION

If the partial-wave dispersion relation in any particular angular-momentum state is solved by the  $N/D$  method, using nonrelativistic kinematics for simplicity, the phase shift  $\delta$  will be given by<sup>1</sup>

$$q \cot \delta = \frac{1 + \int_0^1 dy G(q^2, y) R(y) E(y)}{\int_0^1 dy H(q^2, y) R(y) E(y)}. \quad (1)$$

Here  $E(y)$  is determined by solving the integral equation

$$E(z) = 1 + \int_0^1 dy K(z, y) R(y) E(y), \quad (2)$$

where

$$R(y) = -\text{Im} h(q^2)/\pi y, \text{ with } 0 \leq y < 1 \text{ or } -\frac{1}{4} > q^2 > -\infty.$$

$$h(q^2) = e^{i\delta} \sin \delta / q,$$

with  $q$  = momentum of the nucleon,  $y = -\frac{1}{4}q^{-2}$ , if the pion mass is unity,

$$G(q^2, y) = 2q^2 y^{\frac{1}{2}} / (1 + 4q^2 y),$$

$$H(q^2, y) = 1 / (1 + 4q^2 y),$$

$$K(z, y) = \frac{1}{2}(z^{\frac{1}{2}} + y^{\frac{1}{2}}).$$

In principle, the interaction function  $R(y)$  can be determined from field theory, but we assume it to be unknown. The above results also hold for scattering by a superposition of Yukawa potentials.<sup>2</sup> The generalization using relativistic kinematics<sup>3</sup> does not affect our general approach.

To obtain an approximate formula for  $q \cot \delta$  without making any specific assumptions about the unknown

<sup>1</sup> H. P. Noyes and D. Y. Wong, Phys. Rev. Letters **3**, 191 (1959).

<sup>2</sup> R. Blankenbecler, N. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. **10**, 62 (1960).

<sup>3</sup> H. P. Noyes, Phys. Rev. **119**, 1736 (1960).

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quantities that come into expressions (1) and (2), we shall make approximations only for the kernels  $G(q^2, y)$ ,  $H(q^2, y)$ , and  $K(z, y)$ , which we already know. Suppose we have a finite set of functions  $F_i(y)$  such that, to a good approximation in the energy range of interest, we may write

$$G(q^2, y) \simeq \sum_{i=1}^N A_i(q^2) F_i(y), \quad (3a)$$

and

$$H(q^2, y) \simeq \sum_{i=1}^N B_i(q^2) F_i(y), \quad (3b)$$

where the  $A_i(q^2)$  and  $B_i(q^2)$  are functions such that, at each value of  $q^2$  in our range, the approximations are as good as possible. The reason for making these approximations will be obvious presently. The simplest accurate way of making them would probably be to use some kind of polynomial interpolation through  $N$  carefully chosen points (whose positions may vary with  $q^2$ ) at each value of  $q^2$ . The resulting approximations would then have the forms of Eq. (3), even if we use different polynomials in different regions rather than a single polynomial. In general, neither the  $F_i(y)$  nor the  $A_i(q^2)$  and  $B_i(q^2)$  have to be continuous. However, one should choose the  $F_i(y)$  in such a way that  $N$  is as small as possible for the desired degree of accuracy. Since  $G(q^2, y)$  and  $H(q^2, y)$  are known and are quite smooth functions of  $y$ , there should be no difficulty in doing this.

Substituting the approximations (3) into Eq. (1), we have

$$q \cot \delta = \left[ 1 + \sum_{i=1}^N A_i(q^2) \alpha_i \right] / \sum_{i=1}^N B_i(q^2) \alpha_i, \quad (4)$$

where

$$\alpha_i = \int_0^1 dy F_i(y) R(y) E(y). \quad (5)$$

The reason for making the approximations (3) is now obvious, since we have a formula [namely, Eq. (4)] depending on only a small number  $N$  of unknown parameters  $\alpha_i$ , which could be determined from experiment. A bound on the accuracy of this formula can be easily estimated, being of the same order as the accuracy of the approximations (3), which we can obtain at once since we know what  $G(q^2, y)$  and  $H(q^2, y)$  actually are.<sup>4</sup> Equivalently, the range of validity of Eq. (4) is of the same order as the corresponding range of validity of the approximations (3). An additional advantage of this approach is that it is always much easier to make approximations of known functions than of unknown ones (as in the usual pole approach), since no guesswork is involved. Finally, we do not have to make any very

special assumptions about  $R(y)$ , although a knowledge of some of its more detailed features could certainly be used to improve the accuracy of our results. For instance, if we know that  $R(y)E(y)$  is comparatively large in some region, it would certainly be desirable to make the approximations (3) particularly accurate there.

In the states with orbital angular momentum  $l > 0$ ,  $l$  of the  $\alpha_i$  can be determined from the condition that the phase shift goes to zero as  $q^{2l+1}$  for small  $q^2$ .

A particular way of obtaining the approximations (3) would be to interpolate through  $N$  fixed points  $y_i$  at each value of  $q^2$ . Then, no matter what type of interpolation we use, we always obtain expressions having the forms

$$G(q^2, y) \simeq \sum_{i=1}^N G(q^2, y_i) F_i(y), \quad (6a)$$

and

$$H(q^2, y) \simeq \sum_{i=1}^N H(q^2, y_i) F_i(y). \quad (6b)$$

This means that Eq. (4) becomes

$$q \cot \delta = \left[ 1 + \sum_{i=1}^N G(q^2, y_i) \alpha_i \right] / \sum_{i=1}^N H(q^2, y_i) \alpha_i, \quad (7)$$

which is identical with the result of applying the pole approximation. This time, of course, the positions of the poles are not free parameters.

Perhaps the most familiar interpolation formula is Lagrange's formula for polynomial interpolation. (One may, of course, use different polynomials in different regions.) In this case one has (in any region)

$$F_i(y) = \left[ \prod_{\substack{j=1 \\ j \neq i}}^N (y - y_j) \right] / \left[ \prod_{\substack{j=1 \\ j \neq i}}^N (y_i - y_j) \right]. \quad (8)$$

### III. DETERMINATION OF THE $\alpha_i$ FROM $R(y)$

Suppose now that the interaction function  $R(y)$  is known—perhaps in some approximate form that we may wish to check against experiment. Suppose, also, that we had chosen our  $F_i(y)$  in such a way that, to a good approximation, we may also put

$$K(z, y) \simeq \sum_{i=1}^N C_i(z) F_i(y), \quad (9)$$

where the  $C_i(z)$  are functions such that, at each value of  $z$ , the approximation (9) is optimum. [This approximation, once again, does not depend on  $R(y)$  and can thus be made before  $R(y)$  becomes known. However, if  $R(y)$  is in fact known, it may be desirable to choose the  $F_i(y)$  in such a way that the integrals in Eq. (11) will be simple to evaluate without further approximations.] We may then calculate the  $\alpha_i$  from  $R(y)$ .

To do this, we substitute the approximation (9) into

<sup>4</sup> We are referring here to the percentage accuracy, since the product of an approximate (known) function by an exact (unknown) function always has the same percentage accuracy as the approximate function at any point.

Eq. (2). This gives

$$E(z) = 1 + \sum_{i=1}^N C_i(z) \alpha_i. \quad (10)$$

Inserting this into expression (5), we obtain

$$\alpha_i = \left[ \int_0^1 dy F_i(y) R(y) \right] + \sum_{j=1}^N \left[ \int_0^1 dy F_i(y) R(y) C_j(y) \right] \alpha_j, \quad (11)$$

which is a set of  $N$  linear equations with known coefficients, and may be solved for the  $N$  parameters  $\alpha_i$ . These, in turn, give the phase shift through Eq. (4), as well as the solution of the integral equation through Eq. (10).

In practice, only part of  $R(y)$  can be calculated with present techniques.<sup>5,6</sup> Thus we may take  $R(y)$  to be known only for  $y > a$ , where  $a = \frac{1}{4}$  if only the one-pion contribution is assumed to be known, and  $a = \frac{1}{9}$  if the two-pion contribution is also assumed. Suppose one chooses the  $F_i(y)$  such that

$$F_i(y) = 0 \quad \text{for } y > a, \quad \text{with } i = 1, \dots, N', \quad (12a)$$

and

$$F_i(y) = 0 \quad \text{for } y < a, \quad \text{with } i = N' + 1, \dots, N. \quad (12b)$$

Then all the coefficients in the last  $(N - N')$  of Eq. (11) will be known, even if we make the approximation (9) only for  $z > a$ . We may thus use these equations to eliminate  $(N - N')$  of the  $\alpha_i$ , expressing them linearly in terms of the remaining  $N'$ . In a sense, we have thus eliminated the parameters describing the outer forces (nearby singularities) and are only left with those describing the inner forces (distant singularities).

#### IV. APPLICATION TO THE $^1S_0$ IN THE 0- TO 100-MEV RANGE

Consider the  $^1S_0$  state in the 0- to 100-Mev range. From Fig. 1(a) we see that, for  $y > y_2$  ( $= 0.35$ ), we may approximate  $[G(q^2, y)/2q^2]$  by a straight line passing through its values at  $y_2$  and  $y_1$  ( $= 0.85$ ), and for  $y < y_2$  by the sum of this same line and a parabola in such a way that this sum passes through the value of  $[G(q^2, y)/2q^2]$  at  $y_3$  ( $= 0.06$ ). In other words, we have

$$G(q^2, y) \approx \left[ \left( \frac{y - y_2}{y_1 - y_2} \right) G(q^2, y_1) + \left( \frac{y - y_1}{y_2 - y_1} \right) G(q^2, y_2) \right] + \theta(y_2 - y) \left( \frac{y - y_2}{y_3 - y_2} \right)^2 \left\{ G(q^2, y_3) - \left[ \left( \frac{y_3 - y_2}{y_1 - y_2} \right) G(q^2, y_1) + \left( \frac{y_3 - y_1}{y_2 - y_1} \right) G(q^2, y_2) \right] \right\}. \quad (13)$$

<sup>5</sup> M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

<sup>6</sup> D. Amati, E. Leader, and B. Vitale, Nuovo cimento **17**, 68; 18, 409, 458 (1960).

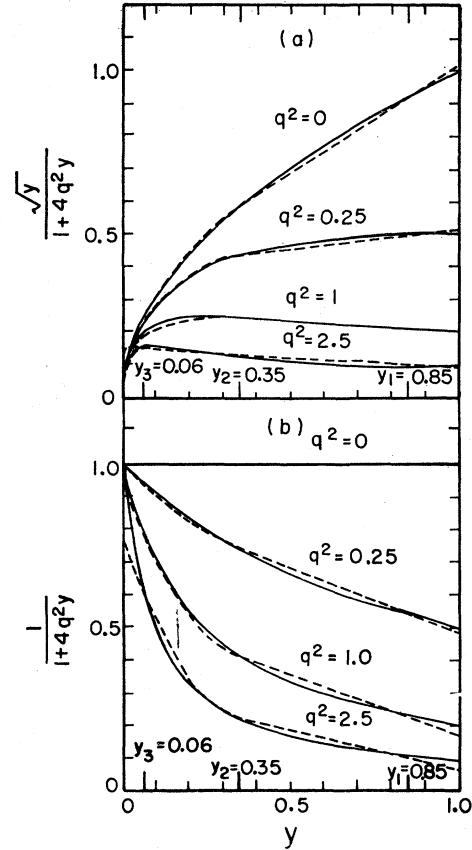


FIG. 1. Plots of (a)  $[y^1/(1+4q^2y)]$  and (b)  $[1/(1+4q^2y)]$  (solid lines) compared with the corresponding approximations having the forms of Eq. (13) (dashed lines).

We can make exactly the same approximation for  $H(q^2, y)$  [see Fig. 1(b)]. As can be seen from Fig. 1, the over-all error associated with this approximation is of the order of only several percent. This accuracy begins to deteriorate after 100 Mev, and so we shall not go beyond this energy.

Now the approximation (13) is an interpolation through fixed points, and so we obtain just Eq. (7). Since we can neglect Coulomb effects only for energies  $\gtrsim 40$  Mev in  $p$ - $p$  scattering, the  $\alpha_i$  were determined by fitting this formula to the  $n$ - $p$  scattering length and effective range, as well as the  $p$ - $p$  phase shift at 68.3 Mev. The results are shown in Table I.

#### V. EXACT TREATMENT OF KNOWN OUTER FORCES

In the last paragraph of Sec. III, a simple approximate method of treating known outer forces was given. We shall now give a method of treating these forces exactly, using a generalization of a method developed by Noyes,<sup>3</sup> who, however, used the usual pole approach in describing the inner forces.

Let

$$R(y) = R_0(y) + R'(y), \quad (14)$$

TABLE I. The  $^1S_0$  nucleon-nucleon phase shifts calculated from Eq. (7) with  $\alpha_1 = -6.78$ ,  $\alpha_2 = 23.10$ , and  $\alpha_3 = 0.43$  (which give a scattering length of  $23.7 \times 10^{-13}$  cm and an effective range of  $2.66 \times 10^{-13}$  cm).

Energy (Mev)	Phase shift (deg)	
	Calculated	Experimental <sup>a</sup>
39.4	41.9	44.0
68.3	34.6	34.5
95	30.4	27.0

<sup>a</sup> These values are taken from a recent phase-shift analysis made by H. P. Stapp, M. Moravcsik, and H. P. Noyes, Lawrence Radiation Laboratory (unpublished).

where  $R_0(y)$  is the known part and gives the entire contribution to  $R(y)$  for  $y > a$ . If we substitute Eq. (14) into expressions (1) and (2), and make the approximations (3) and (9) only within the integrals containing  $R'(y)$  (which means that these approximations have to be made accurately only for  $y < a$ ), we obtain

$$q \cot \delta = \frac{1 + \int_0^1 dy G(q^2, y) R_0(y) E(y) + \sum_{i=1}^N A_i(q^2) \alpha_i'}{\int_0^1 dy H(q^2, y) R_0(y) E(y) + \sum_{i=1}^N B_i(q^2) \alpha_i'}, \quad (15)$$

with

$$E(z) = 1 + \int_0^1 dy K(z, y) R_0(y) E(y) + \sum_{i=1}^N C_i(z) \alpha_i', \quad (16)$$

where

$$\alpha_i' = \int_0^a dy F_i(y) R'(y) E(y). \quad (17)$$

If we knew the  $\alpha_i'$ , we could solve Eq. (16) for  $E(y)$ , which would give us the phase shift through Eq. (15). However, since the  $\alpha_i'$  must be determined by experiment, it would be desirable to have to solve integral equations that do not contain any unknown parameters. This may be accomplished by writing

$$E(y) = f(y) + \sum_{i=1}^N g_i(y) \alpha_i', \quad (18)$$

where the  $g_i(y)$  are defined such that

$$g_i(z) = C_i(z) + \int_0^1 dy K(z, y) R_0(y) g_i(y). \quad (19)$$

Thus it follows from Eqs. (18) and (16) that  $f(y)$  must obey the equation

$$f(z) = 1 + \int_0^1 dy K(z, y) R_0(y) f(y). \quad (20)$$

Inserting expression (18) into Eq. (15), we obtain

$$q \cot \delta = \frac{1 + \int_0^1 dy G(q^2, y) R_0(y) f(y) + \sum_{i=1}^N \left[ \int_0^1 dy G(q^2, y) R_0(y) g_i(y) + A_i(q^2) \right] \alpha_i'}{\int_0^1 dy H(q^2, y) R_0(y) f(y) + \sum_{i=1}^N \left[ \int_0^1 dy H(q^2, y) R_0(y) g_i(y) + B_i(q^2) \right] \alpha_i'}. \quad (21)$$

Thus, if we solve the integral equations (19) and (20) (which do not contain any  $\alpha_i'$ ) we obtain  $g_i(y)$  and  $f(y)$  which, when inserted into the expression (21), give an explicit expression for the phase shift in terms of the  $\alpha_i'$ .

## VI. CONCLUSION

We have shown that it is possible to set up formulas for the phase shifts which depend on only a small number of parameters, which can be justified in an *a priori* manner, and whose accuracy can be easily estimated. Such a formula was found to approximately reproduce the  $^1S_0$  state in the 0- to 100-Mev range. In addition, if part of the interaction function is known, its contribution can be explicitly calculated, and only the unknown part need be expressed in a shape-independent manner.

The above methods can be applied to any problem in which unknown functions occur in integrals containing known kernels, and, specifically, to any partial-wave dispersion relations. (For an application to ordinary potential scattering, see Appendix B.) They should be particularly useful for treating inner forces, which cannot be

calculated with present techniques. In many problems, however, the outer forces themselves are not given explicitly, but must be calculated self-consistently.<sup>7</sup> In other words, in Eq. (14),  $R_0(y)$  itself depends on  $\delta$ , and so we have nonlinear integral equations which must be solved numerically (say, by iteration). Alternatively, we may use the method given in the last paragraph of Sec. III. This time, the coefficients of the  $\alpha_i$  in the last  $(N - N')$  of Eq. (11) depend on the  $\alpha_i$  (although in a known way). This means that we have nonlinear equations in the  $\alpha_i$ , which can be solved by standard numerical methods. In either case, at the same time we must put in the experimental data for determining the constants that represent the incalculable inner forces.

## ACKNOWLEDGMENTS

The author wishes to express his deepest gratitude to Professor G. F. Chew, Dr. D. Wong, and Dr. H. P. Noyes for some very helpful conversations and sug-

<sup>7</sup> This is the case, for instance, in pion-pion scattering. See G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

gestions, and to the latter for also supplying the phase shifts used here.

## APPENDICES

### A. Effect of Oscillations

For small  $y$ , it appears that the well-known repulsive core exhibits itself as strong oscillations in the function  $R(y)$ ,<sup>8</sup> and hence in the integrands of Eqs. (1) and (3). In general, we may expect that, even if the oscillations grow rapidly for small  $y$ , the nearly exact cancellations would contribute little to the integrands compared with other contributions. This would be true whether or not we made the approximations (3) and (9). Let us assume, however, that the oscillations are so large that even the small remainders after an almost exact cancellation are large compared with the other contributions. At first sight, it may appear that in such cases the errors introduced by the approximations (3) and (9) would be large compared with the contributions themselves. We shall show, however, that if both the value and the slope

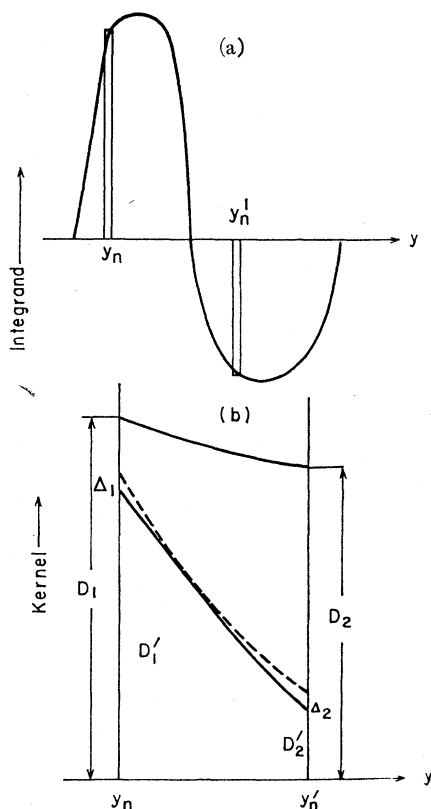


FIG. 2. (a) Two successive oscillations at the energy for which they cancel. The  $n$ th strips are shown in each case. (b) Values of the kernel (solid lines) and of the approximate kernel as given by Eq. (3) or Eq. (9) (dashed lines) between  $y_n$  and  $y'_n$  at two widely spaced energies in the range of interest. One of the energies is the same as in Fig. 2(a). There is no loss of generality in taking this to be the energy for the upper curve in this diagram.

<sup>8</sup> H. P. Noyes, Lawrence Radiation Laboratory, Livermore, California (private communication).

are well approximated by Eqs. (3) and (9), the errors will be small compared with the contribution, no matter how violent the oscillations become.

To show this, we first note that the accuracy of the phase shift is determined by the errors in the integrals compared with the values of those integrals at all energies in the range of interest, and not just these values at one energy. For instance, the mere fact that the value of an integral varying with energy passes through zero at a slightly different energy than it would if it were given correctly is obviously not going to affect the phase shift by much, although it is technically true that at the point where the integral passes through zero the relative error is infinite.

Suppose next that we have two successive oscillations which exactly cancel each other at some energy in the range of interest (see Fig. 2). If this never happens, we can always divide our integrand into an oscillating part for which this does happen, and into a nonoscillating part, which does not cause any difficulties. Dividing the two oscillations at that energy into infinitesimal strips of equal area  $A$ , we see that at the second energy of Fig. 2(b), the total contribution of the two  $n$ th strips of each oscillation will be  $A(D'_1/D_1 - D'_2/D_2)$ . This is large compared with the error  $A(\Delta_1/D_1 - \Delta_2/D_2)$  no matter how small  $(y'_n - y_n)$  is, if both the slope and the value are well approximated by Eqs. (3) and (9). Thus the error is small compared with the contribution. Since this is true of all such pairs of strips, it is also true of the oscillations.

### B. A Potential Theory of Phenomenological Perturbations

If a simple approximate potential  $V_0(r)$  is known in a potential scattering problem, the above techniques can be applied for expressing the deviations from the resulting zeroth-order solution in a shape-independent manner. Such a shape-independent approach does not depend on any special conditions of a problem (as is the usual low-energy shape-independent theory, for instance), and can be applied over wide energy ranges in a large class of problems.

Suppose  $u_0(q^2, r)$  is the radial wave function in the  $S$  state for the zeroth-order potential  $V_0(r)$ , normalized such that  $u_0(q^2, r) \rightarrow \sin(qr + \delta_0)$  as  $r$  approaches  $\infty$ , where  $\delta_0$  is the zeroth-order phase shift. If the correct potential is  $V(r) = V_0(r) + V_1(r)$ , then the phase shift  $\delta$  is given to first order by the standard perturbation-theory result<sup>9</sup>

$$\delta \simeq \delta_0 - \tan^{-1} \left[ 2Mq^{-1} \int_0^\infty dr V_1(r) u_0^2(q^2, r) \right]. \quad (\text{B.1})$$

Since  $u_0^2(q^2, r)$  is known, we may make the same type

<sup>9</sup> P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Publishing Company, New York, 1953), Part II, p. 1694.

of approximation as in Eqs. (3) and (9). That is, we put where

$$u_0^2(q^2, r) \simeq \sum_{i=1}^n a_i(q^2) f_i(r). \quad (\text{B.2})$$

Here the approximation need be accurate only in the region that is not too well approximated by  $V_0(r)$ —for instance, the region  $r \lesssim 2 \times 10^{-13}$  cm in high-energy nucleon-nucleon scattering, if  $V_0(r)$  includes the one-pion exchange potential. In this latter problem  $u_0^2(q^2, r)$  should be well approximated by the product of a normalization factor that depends only on  $q^2$  and an expansion in powers of  $q^2$ , since the shape of  $u_0^2(q^2, r)$  varies slowly with energy. Such an approximation has the form given by Eq. (B.2).

Substituting Eq. (B.2) into Eq. (B.1), we obtain the expression

$$\delta \simeq \delta_0 - \tan^{-1} \left[ 2Mq^{-1} \sum_{i=1}^n a_i(q^2) \beta_i \right], \quad (\text{B.3})$$

$$\beta_i = \int_0^\infty dr V_1(r) f_i(r). \quad (\text{B.4})$$

We can evaluate the  $\beta_i$  at once if  $V_1(r)$  is known; otherwise they can be determined from experiment. If  $n'$  parameters of  $V_0(r)$  were also determined from experiment (say, from  $n'$  phase shifts), we would have  $n'$  additional conditions on the  $\beta_i$  [namely, that  $(\delta - \delta_0)$  is zero at the corresponding energies].

The above method can be easily extended to perturbations in higher waves and tensor forces. Hard-core perturbations already involve only one parameter, namely the perturbation in the core radius, and are, to first order, independent of the other perturbations. The method can also be applied without decomposing into partial waves.

## Electromagnetism and Gravitation: an Action-at-a-Distance Confluence\*

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It is shown how the distinct action-at-a-distance theories of electrodynamics (Wheeler-Feynman theory) and of gravitation (Einstein-Infeld-Hoffmann theory) may be brought together in the construction of a joint electromagnetic-gravitational Hamiltonian for the classical motions of interacting point particles. This is accomplished through a general scheme for finding a Hamiltonian for the Wheeler-Feynman theory proceeding by powers of  $c^2$ , in which the terms (involving only particle position and conjugate momentum variables) are all a species of functional of whatever arbitrary unperturbed (electrodynamically) Hamiltonian is stipulated. When the latter is in fact taken to be that of purely gravitating particles according to Einstein, Infeld, and Hoffmann, a well-defined joint Hamiltonian is produced. The lowest order electrodynamic-gravitational contribution to the Hamiltonian of a pair of particles is worked out explicitly as an example; it arises from the coupling of Newtonian gravitation to an electromagnetic interaction term of order  $1/c^4$ , which is the next in line after the Coulomb ( $1/c^0$ ) and Darwin-Breit ( $1/c^2$ ) interactions.

### INTRODUCTION

THE attempts to ramify Einstein's general relativity, in particular to make it comprehensive of both electromagnetic and gravitational phenomena, have not been conspicuously satisfactory. Reliance has been placed for the most part on field-theoretic and geometro-physical notions. Yet in no ultimate sense has the quite antithetical view, that of action-at-a-distance, been foreclosed. In this viewpoint the focus of attention is on the question: What are the motions and the laws of motion of prescribed electrified and gravitating structureless points, solely in terms of dynamical variables relevant to the points and not to any auxiliaries such as

fields? So to say, this extreme Newtonian position seeks a mechanization of geometry, in contrast to the general relativity trend toward geometrization of dynamics. It is peculiar to the breadth of the idea of general covariance that it is not forbidden so to make over the motions of particles along geodesics in the curved space-time into "flat" Newtonian motions under a dynamics with "forces." The apparent weakness of the Newtonian viewpoint stems not from its use of structureless points—whose difficulties are mainly answerable within action-at-a-distance but not within field theory—but in the lack, at first sight, of any rules for prescribing such things as masses and charges. In short the viewpoint is astronomical in character, seemingly devoid of any accounting of the existence of the particles whose motions are studied. Withal, "astronomy" has proven useful down to remarkably small distances, and to press

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