of approximation as in Eqs. (3) and (9). That is, we put

$$u_0^2(q^2,r)\simeq\sum_{i=1}^{n} a_i(q^2)f_i(r). \tag{B.2}$$

Here the approximation need be accurate only in the region that is not too well approximated by $V_0(r)$—for instance, the region $r\lesssim 2\times 10^{-13}$ cm in high-energy nucleon-nucleon scattering, if $V_0(r)$ includes the one-pion exchange potential. In this latter problem $u_0^2(q^2,r)$ should be well approximated by the product of a normalization factor that depends only on q^2 and an expansion in powers of q^2, since the shape of $u_0^2(q^2,r)$ varies slowly with energy. Such an approximation has the form given by Eq. (B.2).

Substituting Eq. (B.2) into Eq. (B.1), we obtain the expression

$$\delta\simeq\delta_0-\tan^{-1}\left[2Mq^{-1}\sum_{i=1}^{n} a_i(q^2)\beta_i\right], \tag{B.3}$$

where

$$\beta_i=\int_0^\infty dr\; V_1(r)f_i(r). \tag{B.4}$$

We can evaluate the β_i at once if $V_1(r)$ is known; otherwise they can be determined from experiment. If n' parameters of $V_0(r)$ were also determined from experiment (say, from n' phase shifts), we would have n' additional conditions on the β_i [namely, that $(\delta-\delta_0)$ is zero at the corresponding energies].

The above method can be easily extended to perturbations in higher waves and tensor forces. Hard-core perturbations already involve only one parameter, namely the perturbation in the core radius, and are, to first order, independent of the other perturbations. The method can also be applied without decomposing into partial waves.

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Electromagnetism and Gravitation: an Action-at-a-Distance Confluence*

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It is shown how the distinct action-at-a-distance theories of electrodynamics (Wheeler-Feynman theory) and of gravitation (Einstein-Infeld-Hoffmann theory) may be brought together in the construction of a joint electromagnetic-gravitational Hamiltonian for the classical motions of interacting point particles. This is accomplished through a general scheme for finding a Hamiltonian for the Wheeler-Feynman theory proceeding by powers of e^2, in which the terms (involving only particle position and conjugate momentum variables) are all a species of functional of whatever arbitrary unperturbed (electrodynamically) Hamiltonian is stipulated. When the latter is in fact taken to be that of purely gravitating particles according to Einstein, Infeld, and Hoffmann, a well-defined joint Hamiltonian is produced. The lowest order electrodynamic-gravitational contribution to the Hamiltonian of a pair of particles is worked out explicitly as an example; it arises from the coupling of Newtonian gravitation to an electromagnetic interaction term of order $1/c^4$, which is the next in line after the Coulomb $(1/c^0)$ and Darwin-Breit $(1/c^2)$ interactions.

INTRODUCTION

THE attempts to ramify Einstein's general relativity, in particular to make it comprehensive of both electromagnetic and gravitational phenomena, have not been conspicuously satisfactory. Reliance has been placed for the most part on field-theoretic and geometro-physical notions. Yet in no ultimate sense has the quite antithetical view, that of action-at-a-distance, been foreclosed. In this viewpoint the focus of attention is on the question: What are the motions and the laws of motion of prescribed electrified and gravitating structureless points, solely in terms of dynamical variables relevant to the points and not to any auxiliaries such as fields? So to say, this extreme Newtonian position seeks a mechanization of geometry, in contrast to the general relativity trend toward geometrization of dynamics. It is peculiar to the breadth of the idea of general covariance that it is not forbidden so to make over the motions of particles along geodesics in the curved space-time into "flat" Newtonian motions under a dynamics with "forces." The apparent weakness of the Newtonian viewpoint stems not from its use of structureless points —whose difficulties are mainly answerable within action-at-a-distance but not within field theory—but in the lack, at first sight, of any rules for prescribing such things as masses and charges. In short the viewpoint is astronomical in character, seemingly devoid of any accounting of the existence of the particles whose motions are studied. Withal, "astronomy" has proven useful down to remarkably small distances, and to press

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it as far as is logically and practically possible would appear to be worthwhile.

Our object is to point out a way for joining gravitation and electrodynamics, at a purely classical level, starting with distinct action-at-a-distance statements of the two theories. We thus refer advisedly to the "confluence" of the two and not to their unification in the sense in which this word has come to be used.

METHOD OF CONFLUENCE

Briefly the idea is as follows. The Wheeler-Feynman scheme of electrodynamics[1]—the only fully developed action-at-a-distance physical theory—can be cast into Hamiltonian form[2] (say for just two interacting charges e_1, e_2) according to

$$H=H_0+(e_1e_2)H_1+(e_1e_2)^2H_2+\cdots,$$

where H_k is $H_k(\mathbf{r}_i,\mathbf{p}_i)$, $\mathbf{r}$ and $\mathbf{p}$ denoting position and canonically conjugate momentum. The physical significance of this structure of electrodynamics is that $\mathbf{r}$, $\mathbf{p}$ are sufficient dynamical variables for the description of the motions of e_1, e_2, in spite of the fact that the equations of motion, when written in terms of instantaneously evaluated $\mathbf{r}_i(t)$, $\dot{\mathbf{r}}_i(t)$, $\ddot{\mathbf{r}}_i(t)$, $\cdots$ are of infinite order. They are sufficient under the requirement, due in essence to Bhabha,[3] that infinitesimally charged particles have motions contiguous to free-particle motions; the latter are characterized by $\mathbf{r}$, $\mathbf{p}$ alone (and not any infinitude of "free-field" plus "bare" free-particle variables), and this dictates that these same variables are all that are needed also for the contiguous motions. Bhabha's criterion, in a word, selects out of the infinite-parametered class of all motions just those, parametrized by $\mathbf{r}(0)$, $\mathbf{p}(0)$, that are continuously adjacent to free-particle ones and are taken to be the only physically admissible ones; the criterion is expressed by H's development in powers of $\epsilon\equiv e_1e_2$ with coefficients depending solely on positions and momenta.

Leaving aside details[2] for the moment, the proposed way to find H is as follows. The unambiguously defined energy of the system of two charges can be found from first principles in terms of their present positions and all their higher derivatives,

$$E=E(\mathbf{r}_i,\dot{\mathbf{r}}_i,\ddot{\mathbf{r}}_i,\cdots).$$

This comes simply from a well-known Taylor expansion of the ($\frac{1}{2}$-retarded)+($\frac{1}{2}$-advanced) Lienard-Wiechert potentials, by means of which the charges interact in the Wheeler-Feynman theory (there is no self-interaction and no meaning to "the" field). The Hamiltonizing procedure is to assume the existence of $H=H_0+\epsilon H_1+\cdots$; to use Poisson-brackets as the agent for computing time derivatives; and to require not only that H generate these derivatives but that it be the energy:

$$H_0+\epsilon H_1+\cdots=E=E(\mathbf{r}_i,(\mathbf{r}_i,H),((\mathbf{r}_i,H),H),\cdots).$$

In short, the Hamiltonian, as both energy and generator of motion, is made to tell itself its own structure. That this encompasses the primitive differential equations of motion has not been generally and conclusively proven, but it has been shown[2] that a truncated form of the equations of motion, up to fourth-order derivatives, is so encompassed, and a physical argument of complete generality has been adduced for the validity of the procedure. The algorithm gives a chain of partial differential equations for $H_k(\mathbf{r}_i,\mathbf{p}_i)$ in which H_k is coupled to itself and to its predecessors, starting from the presumably known H_0. The momentum $\mathbf{p}$ has at first the sole and sufficient meaning of being that which is canonically conjugate to position $\mathbf{r}$.

We can now summarize action-at-a-distance electrodynamics by stating

$$H=H_0+\epsilon H_1[H_0]+\epsilon^2H_2[H_0]+\cdots,$$

where $H_k[H_0]$ designates that H_k is a kind of differential functional of H_0. Everything is contingent on H_0 itself—the electrodynamics has a quite general form sitting on top of any statement whatever as to what may be the motions of *free* particles: *free, that is, of charge, not of gravitation.* Plainly we have to appeal to the deepest theory of motion, lying beyond electrodynamics, namely to general relativity, to say what H_0 must be.

The appeal is answered with considerable clarity in the achievement of Einstein, Infeld, and Hoffmann (E.I.H.)[4] and of Fock[5] and co-workers, who have shown that Einstein's field equations for the metric tensor contain within themselves, in virtue of their nonlinearity and of their interdependence, the equations of motion of gravitating particles. The E.I.H. viewpoint treats the particles as singularities of the metric, and thereby stands in fact and in conception as an action-at-a-distance theory of gravitation. This may be explicitly brought out by recalling the end result of a long calculation for the Lagrangian and then the energy of a pair

[1] J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **21** 425 (1949).

[2] E. H. Kerner, J. Math. Phys. **3**, 35 (1962). Background of recent work on action-at-a-distance: P. A. M. Dirac, Revs. Modern Phys. **21**, 392 (1949); L. H. Thomas, Phys. Rev. **85**, 868 (1952); L. H. Thomas and B. Bakamjian, Phys. Rev. **92**, 1300 (1953); P. Havas and J. Plebanski, Bull. Am. Phys. Soc. **5**, 433 (1960); B. Bakamjian, Phys. Rev. **121**, 1849 (1961); L. L. Foldy, Phys. Rev. **122**, 275 (1961).

[3] H. J. Bhabha, Phys. Rev. **70**, 759 (1946).

[4] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938); A. Einstein and L. Infeld, Ann. Math. **41**, 455 (1940); L. Infeld, Revs. Modern Phys. **29**, 398 (1957).

[5] V. Fock, *The Theory of Space, Time, and Gravitation*, translation, N. Kemmer (Pergamon Press, New York, 1959); Revs. Modern Phys. **29**, 325 (1957); A. Papapetrou, Proc. Phys. Soc. (London) **A64**, 57 (1951).

of particles,[5]

$$E_0^{\text{grav}}$$
$$=\tfrac{1}{2}m_1\mathbf{v}_1^2+\tfrac{3}{8}m_1\frac{\mathbf{v}_1^4}{c^2}+\cdots+\tfrac{1}{2}m_2\mathbf{v}_2^2+\tfrac{3}{8}m_2\frac{\mathbf{v}_2^4}{c^2}+\cdots$$
$$-\frac{\gamma m_1m_2}{r}+\frac{\gamma m_1m_2}{2c^2r}\left(3\mathbf{v}_1^2+3\mathbf{v}_2^2-7\mathbf{v}_1\cdot\mathbf{v}_2-\frac{\mathbf{v}_1\cdot\mathbf{r}\;\mathbf{v}_2\cdot\mathbf{r}}{r^2}\right)$$
$$+\frac{\gamma^2}{2c^2}\frac{m_1m_2(m_1+m_2)}{r^2}+\cdots,\quad(\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2).$$

The general structure of the series unfortunately is not yet known; γ stands for the Newtonian gravitational constant.

Our proposal for joining electrodynamics and gravitation is now simply stated:

$$H=H_0^{\text{grav}}+\epsilon H_1[H_0^{\text{grav}}]+\epsilon^2H_2[H_0^{\text{grav}}]+\cdots.$$

The electrodynamics is now decisively coupled to gravitation at a basic level.

In so calling upon general relativity we place it in the position of being first of all a theory of gravitation. But plainly, on the action-at-a-distance view being taken, this position does not detract from its central role, but rather heightens it in this sense—that only through it can equations of motion to begin with be understood at all to have other than an *ad hoc* origin and significance. Secondly, other types of interaction (typified above as electromagnetic) are utterly at the mercy of the basic H_0^{grav} insofar as they must be conceptually turned off to say what is their true nature when turned on, and turning them off is to turn for a final description of motion to what is universally left over—gravitation. That is, the relationship of gravitation to electrodynamics is wholly intransitive—the calculation of electrodynamic derivatives, Poisson-bracket style, being completely in thrall to gravitation, whose supposedly delicate conformations of particle orbits infuse with utmost subtlety into the dynamical ground fabric of electromagnetism.

Before illustrating in a simple example the gravitational sway over electromagnetism, a few general points may be noticed. We have referred the Wheeler-Feynman electrodynamics to its most elemental aspect, the time-symmetrical interaction of a pair of charges. But the theory is complete only if the Wheeler-Feynman absorber is ultimately and explicitly reckoned with. For otherwise there is no accounting for radiation damping. This is to say that the rigorous action-at-a-distance standpoint makes a Mach-like view of motion compulsory on electrodynamic grounds at least; "absorber" can scarcely have any other final physical significance than "rest of the universe." Thence we are forced to look at the controlling gravitational understructure of electromagnetism cosmologically, too. Here, owing to the nonlinearity of Einstein's field equations, the absorber has a complexity far exceeding what it has purely electrodynamically. It is an interesting question whether and in what sense the gravitational absorber may be able to provide an account of gravitational radiation damping and so to elucidate the puzzles this has posed in the field-formulation of the problem. Whether the absorber may be as complete as is electrodynamically necessary also would seem to rely on the gravitational structure of the universe. We see already in E.I.H. and in Fock the use of so-called "standing-wave" types of solution of the field equations that correspond exactly to electrodynamical usage of ($\frac{1}{2}$-retarded)+($\frac{1}{2}$-advanced) potentials and whose hallmark is the existence of an energy integral of motion expanded in powers of $1/c^2$.

We observe next the basic role given to a Hamiltonian rendition of the equations of motion. This is not a mere mathematical incident that can be replaced easily by other equivalent mathematical statements, but is a virtually essential concomitant to the pursuit of action-at-a-distance to its logical conclusion. On the one hand, in electrodynamics it is effectively compelled through Bhabha's criterion for distinguishing the physically admissible motions and through the necessity of placing the study of motions of charged and uncharged particles on the same footing—that for which $\mathbf{r}$, $\mathbf{p}$ are the basic dynamical variables, the only visible ones for conducting anything more than the very crudest and quite inadequate analysis of the "perturbed" electrodynamic motions as extensions of the "unperturbed" ones. On the other hand, Hamiltonian methods succeed in floating electrodynamics onto the basic gravi-dynamics exactly in virtue of the Hamiltonian's prolific power, via Poisson brackets, to insinuate the dominating structure of the latter into the otherwise conditional structure of the former. The promise of quantization of this scheme of things will be evident.

Finally, the shortcomings of the action-at-a-distance viewpoint must be brought out. As noted before, no apparent rule for prescribing mass and charge seems available. This is in part the weakness Wheeler[6] has emphasized: The admission of singularities as to both strength and type appears at bottom to be an arbitrary procedure theoretically. Next, electrodynamics and gravitation can be fitted together, but they remain too distinct in physical conception; real unity is not achieved. The further pursuit of action-at-a-distance for other types of interactions and their composition with each other and with gravitation on the Hamiltonian basis is perhaps feasible and practically useful, but must result apparently in a patch-work of conceptually separate ideas, whose lack of over-all pattern

[6] J. A. Wheeler, Revs. Modern Phys. 33, 63 (1961).

would not be satisfactorily mitigated by any fineness of the seams. This is to say that the incomprehensibility of the existence of numbers of different particles and of interactions so far finds no more relief in action-at-a-distance than it does elsewhere outside of specifically fundamental particle theories. Merely this ultimate problem is heightened.

Granting the expedient of patch and sew, there remains the specifically action-at-a-distance difficulty that the nature of the series expansions employed—in electrodynamics a double expansion in ϵ and $1/c^2$, in gravi-dynamics an expansion in γ and $1/c^2$, in both together a multiple expansion in all three parameters—remains completely unknown. One must expect the early terms to be more or less useful; and the opportunity to gather the series in different ways, especially according to powers of $1/c^2$, with polynomial coefficients in ϵ and/or γ, which respects the universality of c in all physical interactions and relies less on any supposed smallness of interaction than on the largeness of c, would appear to have at least heuristic significance. But questions of convergence, or of analytic continuation beyond a too small domain of convergence, or of summability when convergence is lacking, lie beyond reach at this point. To disparage the use of formal methods is quite easy, but their power in other contexts has not been so slight as to call for their rejection out of hand in the present one.

CONCLUSION: ILLUSTRATIVE EXAMPLE

To conclude, we shall find now the leading term of the Hamiltonian that illustrates in the simplest way possible the conjunction by action-at-a-distance of electromagnetism to gravitation.

The Wheeler-Feynman equations of motion of two charges e_1 and e_2, worked out as infinite-order differential equations after representing the fields in terms of derivatives of particle coordinates, are[2]

$$\frac{\partial L}{\partial \mathbf{r}_i}-D\frac{\partial L}{\partial \mathbf{v}_i}+D^2\frac{\partial L}{\partial \dot{\mathbf{v}}_i}-\cdots=0,$$

where, forgetting gravitation,

$$L=-m_1c^2\left(1-\frac{\mathbf{v}_1^2}{c^2}\right)^{\frac{1}{2}}-m_2c^2\left(1-\frac{\mathbf{v}_2^2}{c^2}\right)^{\frac{1}{2}}-e_1e_2\sum_{p=0}^{\infty}\frac{(-D_1D_2)^p}{2p!c^{2p}}\left(1-\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right)r^{2p-1}.$$

Here, D_1 and D_2 are differential operators that differentiate particle-1 and particle-2 variables separately with respect to time; $D\equiv D_1+D_2$ is the operator of total time differentiation. From this, or by other routes, is deducible the conserved energy

$$E=\left\{\tfrac{1}{2}m_1\mathbf{v}_1^2+\tfrac{3}{8}m_1\frac{\mathbf{v}_1^4}{c^2}+\cdots+\tfrac{1}{2}m_2\mathbf{v}_2^2+\tfrac{3}{8}m_2\frac{\mathbf{v}_2^4}{c^2}+\cdots\right\}+\frac{\epsilon}{r}+\frac{\epsilon}{2c^2}\left(\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{r}+\frac{\mathbf{v}_1\cdot\mathbf{r}\ \mathbf{v}_2\cdot\mathbf{r}}{r^3}\right)+\frac{\epsilon}{2c^4}(D_1^2+D_2^2-D_1D_2)\mathbf{v}_1\cdot\mathbf{v}_2r+\cdots,$$

where only the opening terms in the series in $1/c^2$ have been written out, corresponding to fourth-order Lagrangian equation of motion, from retention of only the first two terms in the $\sum_p$.

To a sufficient approximation we may keep just the Newtonian terms in the curly bracket and then introduce gravitation into it also in Newtonian approximation:

$$\{\ \}\to\tfrac{1}{2}m_1\mathbf{v}_1^2+\tfrac{1}{2}m_2\mathbf{v}_2^2-\frac{\gamma m_1m_2}{r}\equiv E_0.$$

Now to win the Hamiltonian form, we take $H=H_0+\epsilon H_1+\cdots=E(\mathbf{r}_i,\mathbf{v}_i,\dot{\mathbf{v}}_i,\dot{\mathbf{v}}_i)$, and in the latter calculate $\mathbf{v}$, $\dot{\mathbf{v}}$, $\cdots$ as Poisson brackets, to give

$$H_0=\tfrac{1}{2}m_1\mathbf{r}_{10}^2+\tfrac{1}{2}m_2\mathbf{r}_{20}^2-\frac{\gamma m_1m_2}{r}$$

$$H_1=m_1\mathbf{r}_{10}\cdot\mathbf{r}_{11}+m_2\mathbf{r}_{20}\cdot\mathbf{r}_{21}+\frac{1}{r}+\frac{1}{2c^2}\left(\frac{\mathbf{r}_{10}\cdot\mathbf{r}_{20}}{r}+\frac{\mathbf{r}_{10}\cdot\mathbf{r}\ \mathbf{r}_{20}\cdot\mathbf{r}}{r^3}\right)+\frac{1}{2c^4}[((\mathbf{r}_{10}\cdot\mathbf{r}_{20}r,H_0),H_0)-3(\mathbf{r}_{10}\cdot\mathbf{r}_{20}r;H_0,H_0)]$$

$$H_2=m_1(\mathbf{r}_{10}\cdot\mathbf{r}_{12}+\tfrac{1}{2}\mathbf{r}_{11}^2)+m_2(\mathbf{r}_{20}\cdot\mathbf{r}_{22}+\tfrac{1}{2}\mathbf{r}_{21}^2)+\frac{1}{2c^2}\left(\frac{\mathbf{r}_{10}\cdot\mathbf{r}_{21}+\mathbf{r}_{11}\cdot\mathbf{r}_{20}}{r}+\frac{\mathbf{r}_{10}\cdot\mathbf{r}\ \mathbf{r}_{21}\cdot\mathbf{r}+\mathbf{r}_{11}\cdot\mathbf{r}\ \mathbf{r}_{20}\cdot\mathbf{r}}{r^3}\right)+\cdots,\ \text{etc}$$

Here we have used the notation $\mathbf{r}_{ik}\equiv(\mathbf{r}_i,H_k)$ and have expressed the operator $D_1^2+D_2^2-D_1D_2$ as $D^2-3D_1D_2$. Then D acting on a function F of canonical variables is $DF=(F,H)$; while D_1D_2F is[2]

$$D_1D_2F=D_2D_1F\equiv(F;H,H)=\frac{\partial^2F}{\partial\xi_i\partial x_j}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial\rho_i}-\frac{\partial^2F}{\partial\xi_i\partial p_j}\frac{\partial H}{\partial\rho_i}\frac{\partial H}{\partial x_j}-\frac{\partial^2F}{\partial\rho_i\partial x_j}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial\xi_i}+\frac{\partial^2F}{\partial\rho_i\partial p_j}\frac{\partial H}{\partial x_j}\frac{\partial H}{\partial\xi_i},$$

where here $\mathbf{r}_1=(x_1,x_2,x_3)$, $\mathbf{p}_1=(p_1,p_2,p_3)$, $\mathbf{r}_2=(\xi_1,\xi_2,\xi_3)$, $\mathbf{p}_2=(\rho_1,\rho_2,\rho_3)$, and where the summation convention is used. This "semi-bracket" is a specially devised kind of Poisson bracket brought in simply to save the explicit introduction of the D_i into operands to the point where they may be called D and so to allow the direct calculation of the important D_1D_2 operator.

We have now for H_0,

$$H_0=\tfrac{1}{2}m_1\left(\frac{\partial H_0}{\partial \mathbf{p}_1}\right)^2+\tfrac{1}{2}m_2\left(\frac{\partial H_0}{\partial \mathbf{p}_2}\right)^2-\frac{\gamma m_1m_2}{r},$$

which, of course, is satisfied by the known

$$H_0=\frac{\mathbf{p}_1^2}{2m_1}+\frac{\mathbf{p}_2^2}{2m_2}-\frac{\gamma m_1m_2}{r},$$

and for H_1,

$$\begin{aligned}H_1={}&\mathbf{p}_1\cdot\frac{\partial H_1}{\partial \mathbf{p}_1}+\mathbf{p}_2\cdot\frac{\partial H_1}{\partial \mathbf{p}_2}\\&+\left\{\frac{1}{r}+\frac{1}{2c^2m_1m_2}\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r}+\frac{\mathbf{p}_1\cdot\mathbf{r}\,\mathbf{p}_2\cdot\mathbf{r}}{r^3}\right)\right.\\&+\frac{1}{2c^4m_1m_2}\left[\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{m_1^2}\left(\frac{\mathbf{p}_1^2}{r}-\frac{(\mathbf{p}_1\cdot\mathbf{r})^2}{r^3}\right)\right.\\&\left.\left.+\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{m_2^2}\left(\frac{\mathbf{p}_2^2}{r}-\frac{(\mathbf{p}_2\cdot\mathbf{r})^2}{r^3}\right)+\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{m_1m_2}\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r}-\frac{\mathbf{p}_1\cdot\mathbf{r}\mathbf{p}_2\cdot\mathbf{r}}{r^3}\right)\right]\right\}\\&+\left\{\frac{\gamma}{2c^4}\left[\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left(\frac{\mathbf{p}_1\cdot\mathbf{r}\,\mathbf{p}_2\cdot\mathbf{r}}{r^4}-\frac{2\mathbf{p}_1\cdot\mathbf{p}_2}{r^2}\right)\right.\right.\\&\left.-\frac{4}{r^4}\left(\frac{(\mathbf{p}_1\cdot\mathbf{r})^2}{m_1}+\frac{(\mathbf{p}_2\cdot\mathbf{r})^2}{m_2}\right)+\frac{1}{r^2}\left(\frac{\mathbf{p}_1^2}{m_1}+\frac{\mathbf{p}_2^2}{m_2}\right)\right]\\&\left.+\frac{\gamma^2}{2c^4}\frac{m_1m_2}{r^3}\right\}.\end{aligned}$$

The first curly bracket arises from the purely "free" part $\mathbf{p}_1^2/2m_1+\mathbf{p}_2^2/2m_2$ of H_0; the second, from the gravitational component $-\gamma m_1m_2/r$, expresses differentially, so to say, the effect of the Keplerian gravitational orbit percolating into the electrodynamic derivatives D_1, D_2. Owing to the simple structure of the inhomogeneous part of this differential equation for H_1, we readily find, discarding solutions of the homogeneous equation (which is entirely detached from the physics of the problem, and whose role is simply to give leave to add to H_1 a linear function of momenta, corresponding to a contact transformation redefining momentum conjugate to position compatibly with leaving the coordinates themselves unchanged),

$$\begin{aligned}H_1={}&\frac{1}{r}-\frac{1}{2c^2m_1m_2}\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r}+\frac{\mathbf{p}_1\cdot\mathbf{r}\,\mathbf{p}_2\cdot\mathbf{r}}{r^3}\right)\\&-\frac{1}{6c^4m_1m_2}\left[\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{m_1^2}\left(\frac{\mathbf{p}_1^2}{r}-\frac{(\mathbf{p}_1\cdot\mathbf{r})^2}{r^3}\right)\right.\\&\left.+\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{m_2^2}\left(\frac{\mathbf{p}_2^2}{r}-\frac{(\mathbf{p}_2\cdot\mathbf{r})^2}{r^3}\right)+\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{m_1m_2}\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r}-\frac{\mathbf{p}_1\cdot\mathbf{r}\,\mathbf{p}_2\cdot\mathbf{r}}{r^3}\right)\right]\\&-\frac{\gamma}{2c^4}\left[\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left(\frac{\mathbf{p}_1\cdot\mathbf{r}\,\mathbf{p}_2\cdot\mathbf{r}}{r^4}-\frac{2\mathbf{p}_1\cdot\mathbf{p}_2}{r^2}\right)\right.\\&\left.-\frac{4}{r^4}\left(\frac{(\mathbf{p}_1\cdot\mathbf{r})^2}{m_1}+\frac{(\mathbf{p}_2\cdot\mathbf{r})^2}{m_2}\right)+\frac{1}{r^2}\left(\frac{\mathbf{p}_1^2}{m_1}+\frac{\mathbf{p}_2^2}{m_2}\right)\right]+\frac{\gamma^2}{2c^4}\frac{m_1m_2}{r^3}.\end{aligned}$$

The first two terms are the Coulomb and Darwin interactions; the third a further purely electromagnetic extension of the first two. The last term, in γ^2, is properly to be joined to a similar term that would come from the γ^2 part of E_0^{grav}, and is of the nature of a higher order gravitational interaction occurring because of the existence of the electromagnetic one. Finally the fourth term provides the leading joint electromagnetic-gravitational contribution to the full Hamiltonian,

$$\begin{aligned}\frac{-\gamma e_1e_2}{2c^4}&\left[\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left(\frac{\mathbf{p}_1\cdot\mathbf{r}\,\mathbf{p}_2\cdot\mathbf{r}}{r^4}-\frac{2\mathbf{p}_1\cdot\mathbf{p}_2}{r^2}\right)\right.\\&\left.-\frac{4}{r^4}\left(\frac{(\mathbf{p}_1\cdot\mathbf{r})^2}{m_1}+\frac{(\mathbf{p}_2\cdot\mathbf{r})^2}{m_2}\right)+\frac{1}{r^2}\left(\frac{\mathbf{p}_1^2}{m_1}+\frac{\mathbf{p}_2^2}{m_2}\right)\right].\end{aligned}$$

Its genesis is in the electromagnetic $1/c^4$ interaction, just beyond the previously known Darwin ($1/c^2$) term; it is very small, of the order of magnitude

$$\frac{\text{electromagnetic radius}\times\text{gravitational radius}}{(\text{distance of separation})^2}\times\text{Newtonian kinetic energy.}$$

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