Quantum Theory without Electromagnetic Potentials*

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It is pointed out that by a nonlocal phase transformation of the conventional wave function the quantum theory of the interaction between charged particles and the electromagnetic field can be reformulated solely in terms of the field strengths. The assertion of Aharonov and Bohm that the potentials are essential for expressing the laws of physics is therefore unfounded.

IN a recent paper[1] aimed at clarifying some issues raised in an earlier paper,[2] Aharonov and Bohm make the following statements: "It is well known that the potentials must appear in Schrödinger's equation, because there is no way in quantum mechanics to express the interaction of the electron with the electromagnetic field solely in terms of field quantities." And again: "It is clear that at least in the mathematical theory of quantum mechanics, the electromagnetic potentials (and not the fields) are what play a fundamental role in the expression of the laws of physics." And finally: "We must keep in mind that the quantum theory as it is now formulated requires that the interaction of electron with electromagnetic field must be a *local* one (i.e., the field can operate only where the charge is). Therefore, in the description of this interaction, only those quantities which differ from zero in the region accessible to the electron can account for observable physical effects on the electron. As a result, when the electron is confined to a multiply connected region, the fields in the excluded region···cease to be relevant for the problem under discussion." It is the purpose of this note to demonstrate that these assertions are false.

Although there can be no argument whatever about the experimentally verifiable results so ingeniously predicted by Aharonov and Bohm, it follows that the force of what one may call the "purely metaphysical" side of their arguments[3] is negligible.

The demonstration that quantum mechanics *can* be formulated solely in terms of field strengths is straightforward. Consider first the Schrödinger equation. The introduction of an electromagnetic field is customarily described by making the replacement

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x^\mu} - ieA_\mu, \quad (\hbar = c = 1), \tag{1}$$

wherever the differential operators $\partial/\partial x^\mu$ ($\mu = 0, 1, 2, 3$) occur in the Schrödinger equation for zero electromagnetic field.[4] Under the gauge transformation,

$$A_\mu' = A_\mu + \frac{\partial \Lambda}{\partial x^\mu}, \tag{2}$$

the wave function ψ must then transform according to

$$\psi' = \exp(ie\Lambda)\psi \tag{3}$$

in order that the Schrödinger equation remain invariant. The conventional wave function is, therefore, not gauge invariant. It is possible, however, to introduce a gauge-invariant wave function by the following device: One introduces four arbitrary *single-valued* differentiable functions $z^\mu(x,\xi)$ of the space-time coordinates x^μ and a parameter ξ, which are defined for all x^μ and for values of ξ in the interval $-\infty < \xi \leq 0$, and which satisfy the boundary conditions

$$z^\mu(x,0) = x^\mu, \tag{4}$$

$$\lim_{\xi \to -\infty} z^\mu(x,\xi) = \text{spatial infinity}. \tag{5}$$

The term "spatial infinity" here means any limit which is sufficiently remote in a space-like direction from x^μ that the electromagnetic field vanishes there, at which limit A_μ may without loss of generality be set equal to zero. The gauge-invariant wave function is then defined by[5]

$$\Psi \equiv \exp\left[-ie\int_{-\infty}^{0} A_\mu(z)\frac{\partial z^\mu}{\partial \xi}d\xi\right]\psi. \tag{6}$$

It is to be emphasized that Ψ, like ψ, is *single valued*, and since it differs from the latter only in phase it can equally well be used in the computation of quantum mechanical probabilities. Furthermore, by carrying out an integration by parts and using the boundary conditions (4) and (5), one may readily verify that Ψ may be used in place of ψ in the Schrödinger equation, provided that the operator replacement (1) is changed to

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x^\mu} - ie\int_{-\infty}^{0} F_{\nu\sigma}(z)\frac{\partial z^\nu}{\partial \xi}\frac{\partial z^\sigma}{\partial x^\mu}d\xi, \tag{7}$$

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[1] Y. Aharonov and D. Bohm, Phys. Rev. **123**, 1511 (1961).

[2] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[3] Especially their arguments (in reference 1) against the interpretations of M. Peshkin, I. Talmi, and L. Tassie, Ann. Phys. **12**, 426 (1961).

[4] Pauli magnetic moment terms are neglected here since they involve the field strengths alone from the beginning.

[5] Bilinear invariants involving line integrals of the potentials have previously been suggested by P. G. Bergmann, Nuovo cimento **3**, 1177 (1956).

where

$$F_{\mu\nu}\equiv\frac{\partial}{\partial x^\mu}A_\nu-\frac{\partial}{\partial x^\nu}A_\mu. \tag{8}$$

Nonrelativistic particle mechanics as well as relativistic quantum field theories with an externally imposed electromagnetic field can therefore be formulated solely in terms of field strengths, *at the expense, however, of having the field strengths appear nonlocally in line integrals*. Needless to say, this formulation yields results identical to those of the conventional one for all observable effects. For example, the interference effects arising from the condition that the wave function be single-valued over multiply connected regions are expressed in terms of integrals of the field strengths over the surfaces swept out by the curves $z^\mu(x,\xi)$ as x makes a complete circuit of each such region. These surface integrals are identical to the line integrals $\oint A_\mu dx^\mu$ of the conventional formulation.

Consider next the general case in which the electromagnetic field is itself subject to quantization. If it is required that the theory be formulated in terms of a least-action principle, then the use of potentials is mandatory at least in the initial stages. Even this, however, is unnecessary. The introduction of an action functional is merely a device to obtain field equations which satisfy certain desired symmetries or lead to desired conservation laws. *The field equations are really sufficient by themselves to determine all the quantum properties of the system.* The commutation relations of the free electromagnetic field, for example, are uniquely determined by application of the uncertainty principle to test bodies in the manner of Bohr and Rosenfeld.[6] It is noteworthy that the potentials do not make a single appearance in the classic work of the latter authors.

In illustration of these remarks it is instructive to write the equations of quantum electrodynamics solely in terms of gauge-invariant quantities:

$$\gamma^\mu\left(\frac{\partial}{\partial x^\mu}+m\right)\Psi=ie\gamma^\mu\int_{-\infty}^{0}F_{\nu\sigma}(z)\frac{\partial z^\nu}{\partial\xi}\frac{\partial z^\sigma}{\partial x^\mu}d\xi\,\Psi, \tag{9}$$

$$\Box^2F_{\mu\nu}=-ie\frac{\partial}{\partial x^\mu}(\bar\Psi\gamma_\nu\Psi)+ie\frac{\partial}{\partial x^\nu}(\bar\Psi\gamma_\mu\Psi), \tag{10}$$

$$\frac{\partial}{\partial x^\mu}F_{\nu\sigma}+\frac{\partial}{\partial x^\nu}F_{\sigma\mu}+\frac{\partial}{\partial x^\sigma}F_{\mu\nu}=0. \tag{11}$$

Here Ψ is a gauge-invariant spinor and $\bar\Psi$ is its Pauli adjoint. A possible way of obtaining physical consequences from these equations (although by no means the only way) is to deal with the first two according to the Yang-Feldman prescription,[7] with an "incoming" spinor field satisfying the conventional free-spinor anticommutation relations and an "incoming" electromagnetic field satisfying the commutation relations of the Bohr-Rosenfeld analysis. At no stage do the potentials enter the picture. Equation (11) is an initial condition which is preserved by the other equations and which is needed only in the Fourier decomposition of the field into photon creation and annihilation operators.

None of the preceding remarks is intended to deny that the use of potentials is a great convenience in practice. Nevertheless, a formulation in terms of field strengths has several attractive features. In such a formulation one works from the beginning in the physical Hilbert space without ever needing to enlarge it artificially. The troublesome and essentially uninteresting conventional complications involving gauge conditions and the removal of longitudinal and scalar photons with the aid of an indefinite metric, etc., are completely avoided. Furthermore, one is enabled to introduce "gauge-invariant electrons." Thus, if for all points x on a space-like hypersurface Σ the curves $z^\mu(x,\xi)$ are chosen to be in Σ, then Ψ and $\bar\Psi$ will satisfy the same anticommutation relations on Σ as ψ and $\bar\psi$. Therefore, the operator $\bar\Psi$, like $\bar\psi$, may be regarded as an electron creation operator. It is, of course, true that because of the nonlocality of the interaction in terms of field strengths, the anticommutator of Ψ and $\bar\Psi$ will no longer generally vanish for *all* finite space-like separations. On the other hand, the causal propagators which enter into calculations of specific physical effects do not satisfy this condition anyway, and, although the advantage of working with gauge invariant propagators at first sight seems to be offset by the appearance of the functions $z^\mu(x,\xi)$ (or their Fourier transforms), it should be possible to use the very arbitrariness of these functions to re-establish the consequences of the conventional microcausality conditions as well as theorems related to Ward's identity. In this sense the $z^\mu(x,\xi)$ replace the arbitrary gauge parameters of the conventional theory.

It should further be noted that the convenience of using potentials is restricted to electrodynamics. There exist theories, notably that of the Yang-Mills field[8] and Einsteinian gravitation, which are invariant under *non-Abelian* infinite dimensional invariance groups and for which the use of potentials is much less satisfactory. These theories share with electrodynamics the feature that coupling is introduced by changing an ordinary derivative into a "covariant" derivative based on a law of affine connection relative either to physical space-time itself or to another product space. The wave-function transformation (6) is merely a special case of a general transformation applicable to all these theories, in which Ψ is defined to be the result of effecting a "parallel" displacement of the original ψ along the curve $z^\mu(x,\xi)$ to the chosen reference point at $\xi=-\infty$. It is equivalent to "viewing" ψ in a special "coordinate system" based on and "propagated" from an "observer" located in-

[6] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab Selskab., Mat.-fys. Med. **XII**, 8 (1933).

[7] C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950).

[8] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

stantaneously at the reference point in question. In gravitation theory the world line of the "observer" need not even be located in a field-free region, and the space-like character of the curves $z^\mu(x,\xi)$ may be ensured by choosing them to be geodesics with an initial space-like direction. In all cases the "observer" provides a special reference frame with the aid of which an invariant Ψ may be constructed. It will be observed that the space-like character of the curves $z^\mu(x,\xi)$ removes any difficulties in the construction which might otherwise arise in connection with the ordering of operators, e.g., the $A_\mu(z)$ in Eq. (6).

Note added in proof. In a reply to the present note in the following paper, this issue, Aharonov and Bohm[9] make the valid and significant point that the nonlocal formulation of electrodynamics is transformable back into a local one with potentials. The question is: Does this give the potentials physical significance? And if so, what sort of significance? The author disagrees with the effort of Aharonov and Bohm to tie this question to the question of complete sets of observables. In the author's opinion the significance of the existence of a local formulation of a theory is that it permits a simple causal description of the propagation of small disturbances in the system, in terms, for example, of retarded and advanced Green's functions. There are two things which occur simultaneously in such theories, however. Not only do the "potentials" provide a local formulation of the theory but they also provide a linear representation of a certain infinite dimensional invariance group (the gauge group in the case of electrodynamics). Only the group invariants, not the potentials themselves, are observable by a measuring apparatus. It is *not* true in the case of electrodynamics that the potentials provide a complete set of commuting observables on each space-like hypersurface. The potentials can be made to perform this duty only if the Hilbert space of the theory is extended in a nonphysical way, in which case they constitute an *overcomplete* set.

Within the framework of the formalism suggested here a complete set of commuting or anticommuting operators is easily obtained. One may choose, for example, the magnetic field together with the gauge-invariant spinor Ψ taken over a space-like hypersurface in which the curves $z^\mu(x,\xi)$ are made to lie. Are these quantities local? Yes, in the sense that each is associated with a definite space-time point. No, in the sense that they are involved in nonlocal field equations.

Which is more significant, the fact that nonlocal formulations of causal theories exist which deal only with observables, or the fact that in all known cases local formulations in terms of "potentials" also exist? There may even be a theorem (which would be very interesting if true) that any nonlocal theory formulated solely with observables, which satisfies certain causality requirements, also has a local form related to the nonlocal form via an infinite dimensional invariance group. But would this imply that the potentials which provide the linear representation of the group have in themselves a special physical significance transcending that of the observables? Not obviously.

In a similar vein the author disagrees with the assertion of Aharonov and Bohm that quantum electrodynamics is ultimately determined by the requirement that it be expressible in a local form. Quantum electrodynamics is really determined by experiment. Maxwell derived his classical equations ultimately from the experiments of Faraday and others. The fact that two of his equations, from a purely mathematical viewpoint, proved to be statements of the necessary and sufficient conditions that the field strengths be expressible in terms of potentials is certainly interesting, but may well be more a consequence of the demands of causality than of any physical significance to be attributed directly to the potentials. The flexibility of being able to work in various gauges is an admitted mathematical convenience, but all of the post-Maxwellian theory—Hamiltonian, quantization, etc.—*could* have been worked out in the special gauge

$$A_\mu = \int_{-\infty}^{0} F_{\nu\sigma}(z) \frac{\partial z^\nu}{\partial \xi} \frac{\partial z^\sigma}{\partial x^\mu} d\xi.$$

The author is happy to acknowledge a stimulating correspondence with Professor Bohm and, although maintaining a different viewpoint, wishes to express his wholehearted agreement with the effort to shift the controversy over the significance of potentials to the arena of local vs nonlocal theories.

[9] Y. Aharonov and D. Bohm, following paper [Phys. Rev. 125, 2192 (1962)].