Mach's Principle and a Relativistic Theory of Gravitation. II

C. H. BRANS*
Physics Department, Loyola University, New Orleans, Louisiana
(Received October 17, 1961)

Equations previously proposed are further analyzed in this paper. A locally measured gravitational constant is defined and evaluated in the theory. Boundary conditions and conservation laws are also discussed.

I. INTRODUCTION

In previous works, Dicke[1] and the present author[1,2] have proposed a modification of the general relativistic theory of gravitation in an attempt to be more consistent with Mach's principle and less reliant on absolute properties of space. The modification involves a violation of the "strong principle of equivalence"[3] on which Einstein theory is based. This was brought about by the introduction of a scalar function, ϕ, into the variational principle and field equations in a manner analogous to G^{-1} in Einstein theory. This was done in such a way, however, as to keep the Lagrangian and action for matter itself unchanged. This ensures continued satisfaction of the "weak principle of equivalence," that is, the statement that the paths of test particles in a gravitational field are independent of their masses. For convenience the field equations will be restated here.

$$\phi(R_{ij}-\tfrac{1}{2}g_{ij}R)=8\pi T_{ij}+\frac{\omega}{\phi}(\phi_{,i}\phi_{,j}-\tfrac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) +\phi_{;i;j}-g_{ij}\phi^{;k}{}_{;k}, \quad (1.1)$$

$$\phi^{;k}{}_{;k}=8\pi T/(2\omega+3). \quad (1.2)$$

Here ω is a dimensionless constant number and T_{ij} is the stress-energy tensor for matter itself. T_{ij} is assumed to have been derived from the Lagrangian for matter in the usual way. The equations satisfied by the matter variables themselves are formally the same as in standard general relativity.

Previous discussions[1] of (1.1) and (1.2) have included an analysis of the weak field equations, study of the "three standard tests," comparison with the work of Jordan,[4] discussions of boundary conditions for ϕ, investigations of cosmology and the general relationship to Mach's principle. This paper should be considered as a sequel to (1) and will be concerned with further analysis of (1.1) and (1.2).

The first question to ask about (1.1) and (1.2) is whether or not they result in a really observable dependence of a locally measured gravitational "constant" on universe structure. If so, what is the nature of this dependence? Further, how is ϕ related to such measurements? This is considered in Sec. II below, in which a precise definition of a locally measured gravitational constant is made and then evaluated from (1.1) and (1.2). The discussion of (1.1) and (1.2) in (1) was partially based on the assumption that ϕ^{-1} does indeed play the role of a gravitational "constant," not simply in the field equations themselves, but in actual observation. This is directly verified in Sec. II, where it is shown that in this theory, matter contributes to the locally measured gravitational constant in a manner consistent with the conjecture.[1,2]

$$G^{-1}\approx A\sum m/r, \quad (1.3)$$

where A is a dimensionless number. A similar analysis has been carried out[5] for Einstein theory and has shown that no such dependence as (1.3) occurs in standard general relativity, although there had been some expectations to the contrary.[6,7]

That the ϕ-field theory might lead to a relationship of the form (1.3) is suggested by (1.2). In fact, if ϕ does correspond to the reciprocal of the locally measured gravitational "constant" and if curvature effects can be neglected, (1.3) would provide a solution to (1.2) in the static case, if A is chosen properly. However, since (1.2) is a second order partial differential equation, there is an infinity of solutions to choose from. This indeterminacy must clearly be removed in any program which aims at a complete determination of the local gravitational constant by the structure of the universe. The most obvious procedure would be to impose boundary conditions, probably of a form such that $\phi\rightarrow 0$ outside all matter. The consequences of such boundary conditions for the case of a static spherically symmetric mass distribution are discussed in Sec. III. There it is shown that they would result in pressures of the same magnitude as densities for fluid matter within the shell.

* Based partially on work supported by a National Science Foundation Predoctoral Fellowship and done at Palmer Physical Laboratory, Princeton University.
[1] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
[2] C. Brans, Ph.D. thesis, Princeton University, Princeton, New Jersey, 1961 (unpublished).
[3] R. H. Dicke, Am. J. Phys. **28**, 344 (1960).
[4] P. Jordan, *Schwerkraft und Weltall* (Friedrich Viewag and Sohn, Braunschweig, 1955).
[5] C. Brans, Phys. Rev. **125**, 388 (1962).
[6] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1955), 5th ed., pp. 99-108.
[7] W. Davidson, Monthly Notices Roy. Astron. Soc. **117**, 212 (1957).

Finally, Sec. IV briefly discusses some aspects of conservation laws in the theory. It is found that there are really two types of such laws, one giving a conserved quantity having dimensions of length and the other a conserved quantity having dimensions of mass. For asymptotically flat space and the weak field approximation, the latter is found to be proportional to inertial mass.

II. DEFINITION AND EVALUATION OF THE LOCALLY MEASURED GRAVITATIONAL CONSTANT

In this section an attempt will be made to determine whether or not the field equations (1.1), (1.2) do lead to a really observable dependence of a locally measured gravitational constant on the mass distribution of the universe. As discussed in reference 5, there seems to be no such dependence in general relativity. Similarly here, even though (1.2) indicates that ϕ is related to the mass distribution, it is still necessary to relate ϕ to an observed gravitational constant. It is not adequate to say that since ϕ^{-1} enters the field equations analogously to G in Einstein theory, it will correspond to the actually measured gravitational constant. As in the case of general relativity, care must be taken to use only invariant definitions and to make sure no coordinate effects occur. Further, approximations used in obtaining solutions and evaluating equations of motion must not exclude effects looked for.

First of all, a precise definition must be made of what is meant by the term "locally measured gravitational constant, G_E." This definition is, of course, arbitrary to a certain extent. However, it should represent a comparison of the observed acceleration of test particles near a gravitating mass to that predicted on the basis of classical Newtonian theory.

Of course, such acceleration observations should be restricted only to situations in which this classical theory might be expected to have some validity. For example, this would require that the test particles be instantaneously at rest relative to the gravitating mass and that this mass be sufficiently small. More precisely, the test particles must be far outside the Schwarzschild radius of the gravitating mass.[8] Further, the experiment should be done over a region of space that is so small that the components of the background metric can be considered almost constant. Finally, the length-time measurements involved must be proper to correspond to the results of real measurements. It is only under such conditions that Newtonian theory might be expected to be valid.

Consider then the following mathematical statement embodying the above restrictions. The effective, locally measured gravitational constant will be defined by

$$G_E \equiv -\lim_{r_p\to 0}\lim_{\mu\to 0} r_p^2 \partial A_p/\partial\mu. \tag{2.1}$$

Here A_p is the proper relative radial acceleration, $d^2r_p/d\tau^2$, of a test particle instantaneously at rest at a proper distance r_p (along $dt=0$) from a spherically symmetric inertial mass μ in a locally time orthogonal coordinate system. Equivalently, r_p could be defined as one-half the proper time (as measured on the test particle) of flight of a light ray to the gravitating mass and back with no coordinate conditions imposed. Of course, it is assumed that in defining such proper distances, the mass μ is represented by some smooth, extended density and is not simply a singularity. The partial derivative, $\partial A_p/\partial\mu$, is used since only that part of the proper relative acceleration due to the presence of μ is desired for such a definition.

Of course, the limits $r_p\to 0$ and $\mu\to 0$ are to be understood in a physical rather than a mathematical sense. More precisely, it is assumed that as r_p and μ are decreased below certain values, no physically observable change will occur in the results of measurements of the quantity to the right of the limit signs in (2.1).

This definition having been chosen, the next step is to find out what the results of applying it to various physical situations might be. As mentioned earlier, it has been conjectured that the gravitational constant might depend on universe structure. Such a result might then show up in an evaluation of (2.1). The dependence of G_E on mass distribution might be expected to be in general accord with a relation of the sort

$$G_E^{-1} \approx \sum m/r. \tag{2.2}$$

This would, of course, follow from (1.2) if ϕ could be identified with G_E^{-1} and if curvature effects could be neglected. Of course, such effects of the metric would not be globally negligible in a real situation. However, (2.2) might be interpreted as

$$G_E^{-1} = (G_E^0)^{-1} + A \sum_{\substack{\text{local}\\ \text{matter}}} m/r, \tag{2.3}$$

with A a positive dimensionless constant and G_E^0 the asymptotic value of G_E. If then each $G_E^0 m/r$ is very small compared to unity, it might be expected that local metric effects are small and that an approximation procedure is adequate to check (2.3). However, as has been discussed elsewhere,[5] first approximation is not enough and the equations of motion must be evaluated keeping terms of second order, i.e., $(G_E^0 m/r)^2$.

The method used here to investigate this effect is that due to Infeld.[9] In this method the internal structure

[8] Here, the Schwarzschild radius would have to be determined by using the gravitational constant, or its substitute, that appears in the field equations.

[9] L. Infeld, Revs. Modern Phys. **29**, 398 (1957).

of spherically symmetric masses is neglected and only their over-all behavior is considered. The matter tensor in this method is assumed to be representable by a "renormalized" δ function. In other words, it is assumed that all self-interaction effects are already contained in the adjustable parameters available for each mass. In this case, these parameters consist of a scalar μ and the "path" of the particle, represented by four coordinate functions of another parameter. In this paper, this type of matter tensor is used only as a source in the field equations. The real masses are not assumed to be actual singularities but rather to correspond to continuous densities spread over macroscopic dimensions. The use of the Infeld δ function is mainly for computational convenience since the matter tensor, considered as a source, only appears in integrals. The use of this δ function in such integrals essentially entails the assumption that the masses are nonspinning, spherically symmetric, far removed from each other compared to their dimensions and, in general, display over-all point particle type behavior but without the infinities usually associated with such. Apart from the last one, these are essentially the same assumptions as made by Papapetrou and Fock,[10] who use dilute fluid-type matter tensors. These two methods actually predict the same observable motion through second order in the case of the Einstein equations.[2,5] There is good reason to expect that they would also give the same results in the case of the Eqs. (1.1), (1.2). In fact, the introduction of ϕ alters the situation only by requiring the evaluation of the integral of the trace of the matter tensor. To the approximation required to obtain equations of motion through second order this integral gives the same number as the observed inertial mass, measured in charge units, for both methods.[2]

The result of an application of Infeld's method to (1.1) and (1.2) will now be stated. Assume n particles are interacting and let them be labeled by a, b, c, $\cdots$ so that their inertial masses are μ_a, μ_b, $\mu_c\cdots$ and their coordinate positions are a^i, b^i, $c^i\cdots(i=1,2,3)$. Further define

$$\dot{f}\equiv df/dt;\quad v_a\equiv\Big[\sum_{i=1}^{3}(\dot{a}^i)^2\Big]^{\frac{1}{2}},$$

$$r_a\equiv\Big[\sum_{i=1}^{3}(x^i-a^i)^2\Big]^{\frac{1}{2}};\quad r_{ab}\equiv\Big[\sum_{i=1}^{3}(a^i-b^i)^2\Big]^{\frac{1}{2}}, \tag{2.4}$$

$$M_a\equiv G_0\mu_a;\quad G_0\equiv[(2\omega+4)/(2\omega+3)]G;\quad G\equiv\lim_{r\to\infty}1/\phi(r).$$

A lengthy but straightforward calculation then results in the following expression for the coordinate acceleration of the ath particle.[2]

$$\begin{aligned}\ddot{a}^i=\sum_{b\neq a}M_b\bigg\{&\frac{\partial}{\partial a^i}\frac{1}{r_{ab}}+\bigg[v_a^2\Big(\frac{\omega+1}{\omega+2}\Big)+v_b^2\Big(\frac{3\omega+4}{2\omega+4}\Big)\\&-4\dot{a}^j\dot{b}^j\Big(\frac{2\omega+3}{2\omega+4}\Big)-2\frac{M_b}{r_{ab}}\Big(\frac{2\omega+3}{\omega+2}\Big)\\&-\frac{M_a}{r_{ab}}\Big(\frac{5\omega+8}{\omega+2}\Big)\bigg]\frac{\partial}{\partial a^i}\frac{1}{r_{ab}}+\bigg[-4\dot{b}^i\dot{b}^k\Big(\frac{2\omega+3}{2\omega+4}\Big)\\&+4\dot{a}^k\dot{b}^i\Big(\frac{2\omega+3}{2\omega+4}\Big)+\dot{a}^i\dot{b}^k\Big(\frac{3\omega+4}{\omega+2}\Big)\\&-2\dot{a}^i\dot{a}^k\Big(\frac{2\omega+3}{\omega+2}\Big)\bigg]\frac{\partial}{\partial a^k}\frac{1}{r_{ab}}+\tfrac{1}{2}\dot{b}^k\dot{b}^l\frac{\partial^3 r_{ab}}{\partial a^i\partial a^k\partial a^l}\\&+\tfrac{1}{2}\sum_{\substack{c\neq b\\c\neq a}}M_c\bigg[\frac{1}{r_{ab}}\frac{\partial}{\partial b^i}\frac{1}{r_{bc}}\Big(\frac{7\omega+10}{\omega+2}\Big)-\frac{2}{r_{bc}}\frac{\partial}{\partial a^i}\frac{1}{r_{ab}}\\&-4\Big(\frac{2\omega+3}{\omega+2}\Big)\frac{1}{r_{ac}}\frac{\partial}{\partial a^i}\frac{1}{r_{ab}}\\&+(b^j-a^j)\Big(\frac{\partial}{\partial a^i}\frac{1}{r_{ab}}\Big)\Big(\frac{\partial}{\partial b^j}\frac{1}{r_{bc}}\Big)\bigg]\bigg\}.\end{aligned}\tag{2.5}$$

As a check, it is easily observed that in the limit $|\omega|\to\infty$, i.e., in the Einstein case, this expression is identical with that given by Papapetrou.[10]

The main interest that (2.5) holds for the work of this paper is in its application to an evaluation of the locally measured gravitational constant G_E as given in (2.1). For this purpose let b correspond to the gravitating particle in the lab and a to the test particle ($\mu_a=0$) instantaneously at rest relative to it. Further, for simplicity, assume b and all the other masses are also instantaneously at rest. (2.5) then yields

$$\ddot{a}^i=G_0\mu_b\bigg[1-\Big(\frac{5\omega+8}{\omega+2}\Big)\frac{G_0M}{R}\bigg]\frac{\partial}{\partial a^i}\frac{1}{r_{ab}}+\text{order}(r_{ab}^{-1},r_{ab},\cdots),\tag{2.6}$$

where

$$M/R\equiv\sum_{\substack{c\neq b\\c\neq a}}\frac{\mu_c}{r_{ac}},$$

and where the added terms on the right, order $(r_{ab}^{-1},r_{ab},\cdots)$, do not contribute to the definition (2.1) because of the limit $r_{ab}\to 0$ involved in it. Thus, converting to proper units,[2] an evaluation of (2.1) gives

$$G_E=[1-G_0M/(\omega+2)R]G_0.\tag{2.7}$$

Clearly, as $|\omega|\to\infty$, $G_E\to G_0\to G$, independent of M and R as in Einstein theory.[5] For finite ω, however, (2.7) plainly demonstrates the violation of the strong principle of equivalence. The effect of the "rest of the

[10] A. Papapetrou, Proc. Phys. Soc. (London) **A64**, 57 (1951); V. Fock, J. Phys. (USSR) **1**, 81 (1939); *The Theory of Space, Time and Gravitation* (Pergamon Press, New York, 1959).

universe" on local, proper gravitational experiments cannot be even approximately transformed away.

Equation (2.7) might have been anticipated from first order approximation theory. The effect of $\mu_c(c\neq b;c\neq a)$ on local calculations is to replace the boundary value G^{-1} by $G^{-1}[1+G_0M/(\omega+2)R]$. This follows from a weak-field analysis of (1.1), (1.2).[1]

Actually, the restrictions on the velocities of the particles turn out to be unnecessary, as long as a is instantaneously at rest relative to b as is required in the definition of G_E. In fact, a glance at (2.5) shows that the only velocities that would contribute (2.1) would be those of b and a. A simple calculation shows that if these velocities are equal, the terms involving them are independent of ω. A direct calculation then shows that these terms are precisely those that would appear in a Lorentz transformation (after transforming the background metric to the Minkowskian) of the relation (2.6) between acceleration and distance for both gravitating and test particle instantaneously at rest. In particular, if all the velocities are the same, this shows that G_E as defined by (2.1) is Lorentz invariant through order v^2 and GM/R.

As an example, the maximum variation of G_E measured on earth due to its varying distance from the sun would be of the order

$$\frac{G_E{}^{\max}}{G_E{}^{\min}}\approx 1+\frac{G_0M_{\text{sun}}}{\omega+2}\left(\frac{1}{R_{\min}}-\frac{1}{R_{\max}}\right)\approx 1+\frac{3\times10^{-10}}{\omega+2}. \tag{2.8}$$

Similarly, the variation to be expected in a measurement of G_E at a height 10 km above the earth compared to the value on earth would be approximately

$$G_E{}^{+}/G_E{}^{-}\approx 1+10^{-12}/(\omega+2). \tag{2.9}$$

III. BOUNDARY CONDITIONS

As is well known, an equation of the form (1.2) does not completely specify a solution. One of the most common methods for removing this indeterminacy is the imposition of boundary conditions. Mach's principle might suggest that $G^{-1}\rightarrow 0$ at infinity. Hence $\phi\rightarrow 0$ as $r\rightarrow\infty$ for such a universe might appear as an appropriate boundary condition. Further, this together with (1.2) would predict that the value of G inside a static mass shell of mass M and radius R in otherwise empty universe with a flat metric is consistent with the conjecture,

$$GM/R\approx 1. \tag{3.1}$$

There is another possibility, namely, $\phi\rightarrow 0$ *somewhere* outside all matter, not necessarily at infinity. In this case, (3.1) would be replaced by

$$GM/R-GM/r_0\approx 1, \tag{3.2}$$

where r_0, assumed greater than R, is the radius at which $\phi=0$. However, the presence of matter does curve space so that the metric in such a universe is not flat and the argument that $\phi\rightarrow 0$ as $r\rightarrow\infty$ (or $r\rightarrow r_0$) together with (1.2) leads to (3.1) or (3.2) greatly oversimplifies the situation. In fact, it will be the purpose of this section to show that in a reasonable static mass shell universe $\phi\rightarrow 0$ anywhere outside all matter requires that the pressures in the shell be of the order of densities. Thus the relation of the M in (3.1) or (3.2) to the ordinary idea of mass may be quite tenuous.

The most appropriate coordinates in which to study this problem are isotropic.[11] Hence, the metric will be assumed to have the form

$$ds^2=-e^{2\alpha}dt^2+e^{2\beta}[dr^2+r^2d\Omega^2], \tag{3.3}$$

with $d\Omega=$ element of solid angle. Further, α, β, ϕ and all matter variables will be assumed to be functions of r only, differentiation being denoted by a dash. The matter variables themselves are assumed to correspond to a universe containing only a static spherically symmetric mass shell between $r=R_1$ and $r=R_2>R_1$ together with relatively small masses near the origin. This last assumption is needed only to determine the signature of α', β', ϕ', and ϕ at the inner edge of the shell (i.e., at R_1).

Specifically, if the masses inside produce a field for which the weak-field approximation is valid, then if $(R_1-r)/R_1$ is a small positive number,

$$\begin{gathered} e^{2\alpha}\approx 1-\frac{2G_0m}{r};\quad e^{2\beta}\approx 1+\left(\frac{2\omega+2}{\omega+2}\right)\frac{G_0m}{r},\\ \phi\approx\frac{1}{G}\left(1+\frac{G_0m}{r(\omega+2)}\right);\quad G>0,\\ m\geq 0;\quad 0\leq\frac{G_0m}{R_1}\ll 1. \end{gathered} \tag{3.4}$$

Again, it should be noted that (3.4) will be used only to determine reasonable signatures for α', β', ϕ', and ϕ at R_1. That is, (3.4) is used only to justify the assumption that at R_1 the following inequalities are valid,

$$\begin{gathered} \phi>0;\quad \alpha'\geq 0,\\ \beta'\leq 0;\quad (2\omega+3)\phi'\leq 0,\\ (2\omega+3)(\alpha'+\beta')\geq 0. \end{gathered} \tag{3.5}$$

For the matter within the shell itself, i.e., for $R_1<r<R_2$, assume a fluid-type matter tensor $T_\alpha{}^\beta$ which is diagonal in these coordinates, i.e., $T_\alpha{}^\beta=0$ if $\alpha\neq\beta$, vanishes outside $R_1<r<R_2$ and satisfies

$$T_{\alpha\beta}\geq 0;\quad T_\alpha{}^\alpha\leq 0. \tag{3.6}$$

These conditions are equivalent to requiring not only that the pressures and density be non-negative but also that the sum of the pressures in all three directions

[11] This choice was suggested to the author by C. Misner.

not exceed the mass density. Further, in order to avoid large discrepancies in the deflection of light and perihelion rotation observations, assume

$$|\omega|>2. \tag{3.7}$$

The main purpose of the following argument will then be to show that $\phi\to 0$ anywhere outside all matter violates either (3.6) or (3.7) if the signatures of α', β', ϕ', and ϕ are consistent with (3.5).

First of all, let us write the field equations in isotropic coordinates.

$$\phi e^{-2\beta}\left[2\beta''+\frac{4\beta'}{r}+(\beta')^2\right]=e^{-2\beta}\left[\phi'\alpha'-\frac{\omega(\phi')^2}{2\phi}\right]+8\pi\left(T_0{}^0-\frac{T}{2\omega+3}\right), \tag{3.8}$$

$$\phi e^{-2\beta}\left[2\alpha'\beta'+(\beta')^2+\frac{2(\alpha'+\beta')}{r}\right]=e^{-2\beta}\left[\phi''-\phi'\beta+\frac{\omega(\phi')^2}{2\phi}\right]+8\pi\left(T_1{}^1-\frac{T}{2\omega+3}\right), \tag{3.9}$$

$$\phi e^{-2\beta}\left[\alpha''+(\alpha')^2+\frac{\alpha'+\beta'}{r}+\beta''\right]=e^{-2\beta}\left[\phi'\beta'+\frac{\phi'}{r}-\frac{\omega(\phi')^2}{2\phi}\right]+8\pi\left(T_2{}^2-\frac{T}{2\omega+3}\right), \tag{3.10}$$

$$e^{-2\beta}\left[\phi''+\phi'\left(\alpha'+\beta'+\frac{2}{r}\right)\right]=\frac{8\pi T}{2\omega+3}, \tag{3.11}$$

$$T_\alpha{}^\beta{}_{;\beta}=0. \tag{3.12}$$

The exact vacuum solution, $T_\alpha{}^\beta=0$, to these equations has been obtained.[2] For convenience the various branches of this solution are stated in the Appendix to the present paper. It should be noticed that from (2.7) positive contribution of matter to $G_E{}^{-1}$ requires $\omega>0$ which together with $|\omega|>2$ permits only form I of the solution. This particular form was discussed in (1), relative to the boundary condition problem, but here all four forms will be permitted.

It is immediately seen, however, that $\phi\to 0$ outside the shell eliminates forms II and IV, leaving only I or III. This then requires a range of C and ω consistent with $(C+1)^2\geq C(1-\omega C/2)$.

Consider now $\phi\to 0$ as $r\to\infty$. This eliminates form I, leaving only III and specifies C to be one of the roots

$$C=[1\pm(-2\omega-3)^{\frac{1}{2}}]/(\omega+2). \tag{3.13}$$

These roots (and thus the solution), can be real only if $2\omega+3\leq 0$. As mentioned above, however, this already contradicts positive contribution of matter to $G_E{}^{-1}$. However, it is also inconsistent with the assumption that (3.5), (3.6), and (3.7) are all valid. In fact, from (1.2) it follows that for $r>R_1$

$$e^{-2\beta}(-g)^{\frac{1}{2}}\phi'=\frac{8\pi}{2\omega+3}\int_{R_1}^{r}(-g)^{\frac{1}{2}}T\,dr+e^{-2\beta(R_1)}\phi'(R_1)[-g(R_1)]^{\frac{1}{2}}. \tag{3.14}$$

Hence, as long as no singularities occur, (3.14) and (3.5) give

$$(2\omega+3)\phi'(r)\leq(2\omega+3)\phi'(R_1)\leq 0, \tag{3.15}$$

Thus, since for this solution $2\omega+3\leq 0$, (3.15) together with (3.7) gives

$$\phi'(r)\geq 0;\quad \phi(r)>0, \tag{3.16}$$

for all $r>R_1$. This, of course, prevents $\phi\to 0$ as $r\to\infty$, and completes the proof that this boundary condition is inconsistent with (3.5), (3.6), and (3.7).

The remaining possibility is form I, with $(2\omega+3)\geq 0$. Using the assumption $|\omega|>2$, this reduces to $\omega>2$. Thus (3.15) gives in this case

$$\phi'(r)\leq\phi'(R_1)\leq 0, \tag{3.17}$$

for $r>R_1$. This seems hopeful since $\phi>0$ at R_1, so that a negative derivative will be necessary to bring ϕ to zero somewhere. However, the following argument will show that provided ϕ remains positive in the shell itself, the vanishing of ϕ outside still violates at least one of (3.5), (3.6), and (3.7). The method will be to find what restrictions the field equations and (3.5), (3.6), place on the signatures of α', β', ϕ', and ϕ at R_2, i.e., at the outer edge of the shell. These restrictions are stated in (3.20). The next step will then be to show that the range of constants, B and λ, in form I of the exterior solution imposed by (3.20) is inconsistent with $\phi\to 0$ for some $r>R_2$.

The first step will be to obtain some information about the signature of $(\alpha'+\beta')$ at R_2. This can be done by adding (3.9) to (3.10) and defining $z\equiv\alpha'+\beta'$. This gives

$$z'+z^2+\frac{3z}{r}+\frac{z\phi'}{\phi}=\frac{8\pi e^{2\beta}}{\phi}(T_1{}^1+T_2{}^2)-\frac{8\pi e^{2\beta}T}{2\omega+3}-\frac{\phi'}{r\phi}. \tag{3.18}$$

It is easily seen that under the above assumptions, the right side of (3.18) is non-negative so that at every zero of z, $z'\geq 0$. This, together with the fact that $z\geq 0$ at R_1 implies that $z\geq 0$ for all $r>R_1$, assuming, of course, that no singularities occur and that r remains in a region for which z is single valued and continuous.

Similarly, setting $x\equiv\beta'$, (3.8) can be written

$$x'+\frac{2x}{r}+\frac{x^2}{2}+\frac{x\phi'}{2\phi}=\frac{e^{2\beta}}{2\phi}\left[8\pi T_0{}^0-\frac{8\pi T}{2\omega+3}+e^{-2\beta}\left(\phi'z-\frac{\omega(\phi')^2}{2\phi}\right)\right]. \tag{3.19}$$

This time the right side is non-positive so that $x'\leq 0$ at every zero of x. Thus, since $x\leq 0$ at R_1, $x\leq 0$ for all $r>R_1$. Again, this is subject to necessary continuity restrictions.

In summary, for a reasonable mass distribution between R_1 and R_2, (3.5), (3.6), (3.7) yield

$$\begin{gathered}\phi(R_2)>0;\quad \alpha'(R_2)\geq 0;\quad \beta'(R_2)\leq 0,\\ \alpha'(R_2)+\beta'(R_2)\geq 0;\quad \phi'(R_2)\leq 0.\end{gathered}\tag{3.20}$$

The final step in the argument is to assume a range of values of B and λ in form I of the solution which will permit $\phi\rightarrow 0$ for some $r_0>R_2$. From $\phi\propto[(r-B)/(r+B)]^{C/\lambda}$ this is easily seen to reduce to

$$R_2<|B|;\quad BC/\lambda>0.\tag{3.21}$$

(3.21) will now be shown to be inconsistent with (3.20).

First of all notice that

$$\phi'/\phi\leq 0;\quad \alpha'\geq 0;\quad \phi'/\phi=C\alpha'\tag{3.22}$$

imply $C\leq 0$. This together with (3.21) requires $B/\lambda<0$. On the other hand, $(\alpha'+\beta')\geq 0$ at R_2 requires

$$\frac{\lambda-C}{\lambda}\left(\frac{2B}{R_2{}^2-B^2}\right)\geq\frac{2B}{R_2(R_2+B)}.\tag{3.23}$$

There are now two choices. First, if $B>0$, $\lambda<0$, (3.23) becomes

$$\left(1-\frac{C}{\lambda}\right)\left(\frac{1}{R_2-B}\right)\geq\frac{1}{R_2}>0,\tag{3.24}$$

so that

$$1-C/\lambda\leq 0;\quad C^2\geq\lambda^2.\tag{3.25}$$

Second, if $B<0$, $\lambda>0$ (3.23) yields

$$C/\lambda\leq B/R_2<-1.\tag{3.26}$$

Thus, in either case

$$C^2\geq\lambda^2.\tag{3.27}$$

However, from the definition of λ this requires

$$0\geq C^2\omega/2+C+1.\tag{3.28}$$

Such a range for real C will exist only if $1-2\omega\geq 0$, which is inconsistent with $\omega>2$.

In summary, for a spherically symmetric mass distribution between R_1 and R_2, the following assumptions imply that there exists a constant $L>0$ such that $\phi>L$ for all $r>R_2$:

(1) $|\omega|>2$;

(2) all field quantities are time independent, spherically symmetric, single valued, and have continuous first derivatives;

(3) as r approaches R_1 from the left, ϕ, the metric components and their first derivatives have signatures consistent with positive mass and a positive gravitational constant;

(4) for r between R_1 and R_2 the matter tensor, $T_\alpha{}^\beta$ is diagonal, $\phi>0$, and $T_{00}\geq 0$, $T_\alpha{}^\alpha\leq 0$.

IV. CONSERVATION LAWS

Methods have already been given in Sec. II above for obtaining a "mass" associated with certain solutions to the field equations. More precisely, ways were discussed for obtaining experimentally measurable numbers from constants appearing in the field variables. Other possible numbers can be obtained from conservation laws.[12] These will give constants associated with certain types of solutions which might be called "total masses" or "total energies." Of course, these "masses" are less precisely defined in the sense that they are not as directly related to experimentally measured numbers as are the inertial and active gravitational masses discussed above.

In this section the structure of conservation laws associated with (1.1) and (1.2) will be studied. In this case there are really two possible approaches. In the first, the field equations (1.1) are divided by ϕ to give an Einstein-type equation but with a modified matter tensor. Procedures used in the Einstein case are then applicable. Hence the conserved quantity, when expressed as a function of the metric only, will be formally identical with that in general relativity. The resultant conserved total "mass" has units of length, however, corresponding to some averaged gravitational constant times total mass, and should more properly be called a "total Schwarzschild radius."

An alternate procedure yields a conserved quantity having true units of mass and to which $T_i{}^j{}_{\mathrm{matter}}$ instead of $\phi^{-1}T_i{}^j{}_{\mathrm{matter}}$ contributes directly. This "total mass" is found to equal to the "total Schwarzschild radius" times the asymptotic value of ϕ. Further, both are linearly proportional to the inertial mass associated with the solution. Hence, the not-too-surprising result is obtained that for an isolated system inertial mass is conserved as well as the "total Schwarzschild radius."

The first and most straightforward procedure for obtaining conservation laws is based on the assumption that ϕ never vanishes. After division by ϕ, (1.1) can be considered as Einstein equations having for source not the matter tensor itself, but a modification of it containing contributions from ϕ. (1.2) can then be considered simply as part of the equations to be satisfied by the modified "matter" variables. Hence, the standard procedure, as described for example by Møller,[13] may be used. In this method a quantity $\mathfrak{T}_i{}^j$

[12] For an extensive discussion of conservation laws and their uses see J. Fletcher, Revs. Modern Phys. **32**, 65 (1960).

[13] C. Møller, *The Theory of Relativity* (Oxford University Press, New York, 1960).

is defined by

$$\mathfrak{T}_i{}^j = s_i{}^{jk}{}_{,k}, \tag{4.1}$$

where

$$s_i{}^{jk} \equiv (-g)^{\frac{1}{2}}\big[-g^{lj}(-\Gamma_l{}^k{}_i+\tfrac{1}{2}\delta_l{}^k\Gamma_i{}^n{}_n+\tfrac{1}{2}\delta_i{}^k\Gamma_l{}^n{}_n) + \tfrac{1}{2}\delta_i{}^j g^{ln}(-\Gamma_l{}^k{}_n+\delta_l{}^k\Gamma_n{}^m{}_m)\big]. \tag{4.2}$$

The "conservation law" is then

$$\mathfrak{T}_i{}^j{}_{,j}=0. \tag{4.3}$$

The quantity $\mathfrak{T}_i{}^j$ is an affine tensor and the quantity

$$P_i \equiv \int d^3x \mathfrak{T}_i{}^0 \tag{4.4}$$

is an affine vector and has been called "total momentum." Further, P_i is constant for an isolated system, that is, one for which the integral of $T_i{}^\alpha$ $(\alpha=1, 2, 3)$ over a bounding surface is always negligible.

An evaluation of P_i for form I of the static spherically symmetric solution given in the Appendix gives

$$\begin{aligned} &P_\alpha=0 \quad \text{for} \quad \alpha\neq 0, \\ &P_0=2B(C+1)/\lambda. \end{aligned} \tag{4.5}$$

This is the form of the solution which is physically significant, corresponding to large positive ω. An interpretation of the constants, B, C, and λ can be obtained by comparing an expansion of this form of the solution to that obtained from the Infeld approximation procedure described above in Sec. II. The result is[2]

$$P_0=-\frac{2\omega+2}{2\omega+3}G\mu, \tag{4.6}$$

where G is the asymptotic value of ϕ^{-1} and where

$$\mu=-\int d^3x T_0{}^0{}_{\text{matter}}. \tag{4.7}$$

As shown in Sec. II, μ turns out to be the inertial mass.

Thus, for an isolated system, the "total Schwarzschild radius" is constant. From (4.6) this means that for a single particle the inertial mass times the asymptotic value of ϕ^{-1} remains constant.

It would be interesting to find out whether a quantity having units of mass is also conserved. Noting that the direct contribution of matter to $\mathfrak{T}_i{}^j$ in (4.1) is through a term of the form $\phi^{-1}(-g)^{\frac{1}{2}}T_i{}^j{}_{\text{matter}}$ it seems that $\mathfrak{T}_i{}^j$ must somehow be multiplied by ϕ. However, for nonconstant ϕ this would destroy the conservation equation (4.3). To get around this, construct the conserved quantity as the divergence of an antisymmetric affine tensor. The latter can then be multiplied by ϕ and the divergence of the resulting quantity is again identically conserved. Specifically, following a method proposed by Freud,[14] it is noted that $\mathfrak{T}_i{}^j$ can be written

$$\mathfrak{T}_i{}^j=\mathfrak{U}_i{}^{jk}{}_{,k}, \tag{4.8}$$

where

$$\mathfrak{U}_i{}^{kj}=-\mathfrak{U}_i{}^{jk}\equiv\tfrac{1}{2}(-g)^{\frac{1}{2}}\{\delta_i{}^j(g^{ln}\Gamma_l{}^k{}_n-g^{kl}\Gamma_l{}^n{}_n) +\delta_i{}^k(g^{jl}\Gamma_l{}^n{}_n-g^{ln}\Gamma_l{}^j{}_n)+g^{kl}\Gamma_l{}^j{}_i-g^{jl}\Gamma_l{}^k{}_i\}. \tag{4.9}$$

Hence, for the ϕ-field theory, define a quantity

$$\bar{\mathfrak{T}}_i{}^j\equiv(\phi\mathfrak{U}_i{}^{jk})_{,k}. \tag{4.10}$$

Thus

$$\bar{\mathfrak{T}}_i{}^j{}_{,j}=0, \tag{4.11}$$

and $\bar{\mathfrak{T}}_i{}^j$ is also an affine tensor and

$$\bar{P}_i\equiv\int d^3x\bar{\mathfrak{T}}_i{}^0 \tag{4.12}$$

is an affine vector. As in the case of P_i above, $\bar{P}_i$ is constant for an isolated system. Here, however, $\bar{P}_0$ will have units of mass. Further, for the static case at any rate, it is clear that

$$\bar{P}_0=\int\bar{\mathfrak{T}}_0{}^0d^3x=P_0\lim_{r\to\infty}\phi=G^{-1}P_0, \tag{4.13}$$

with P_0 defined in (4.4) above. Again, an application to the case corresponding to form I of the solution in the Appendix shows that $\bar{P}_0$ is proportional to the inertial mass in the case of a single particle and is constant.

This result might have been expected from the following argument. The first conservation law, (4.3), gives a constant "total Schwarzschild radius" for an isolated system. However, for such a system, G, the asymptotic value of ϕ^{-1}, might also be expected to be constant. If this were true, $G^{-1}P_0$, having dimensions of mass, would also be constant. This line of reasoning is borne out in the argument leading to (4.13) above.

The author is indebted to R. H. Dicke and C. W. Misner for their helpful suggestions on various aspects of these problems.

APPENDIX

This Appendix contains a statement of the exact static spherically symmetric vacuum solution to (1.1), (1.2) expressed in isotropic coordinates. This was derived in (2) with slightly different notation.

There are four forms for this solution corresponding to different ranges of the arbitrary constants available. In the following these will be denoted by α_0, β_0, ϕ_0, C, and B and have values in the indicated ranges. The metric is assumed to be of the form (3.3).

---

[14] P. Freud, Ann. Math. 40, 417 (1939).

I: $$e^{\alpha}=e^{\alpha_0}\left[\frac{1-B/r}{1+B/r}\right]^{1/\lambda},$$

$$e^{\beta}=e^{\beta_0}(1+B/r)^2\left[\frac{1-B/r}{1+B/r}\right]^{(\lambda-C-1)/\lambda},$$

$$\phi=\phi_0\left[\frac{1-B/r}{1+B/r}\right]^{C/\lambda},$$

$$\lambda^2\equiv(C+1)^2-C(1-\omega C/2)>0.$$

II: $$\alpha=\alpha_0+\frac{2}{\Lambda}\tan^{-1}(r/B),$$

$$\beta=\beta_0-\frac{2(C+1)}{\Lambda}\tan^{-1}(r/B)-\ln[r^2/(r^2+B^2)],$$

$$\phi=\phi_0e^{2C/\Lambda\tan^{-1}(r/B)},$$

$$\Lambda^2\equiv C(1-\omega C/2)-(C+1)^2>0.$$

III: $$\alpha=\alpha_0-r/B,$$

$$\beta=\beta_0-2\ln\frac{r}{B}+(C+1)\frac{r}{B}$$

$$\phi=\phi_0e^{-Cr/B},$$

$$C=\frac{-1\pm(-2\omega-3)^{\frac{1}{2}}}{\omega+2}.$$

IV: $$\alpha=\alpha_0-(1/Br),$$

$$\beta=\beta_0+(C+1)/Br,$$

$$\phi=\phi_0e^{-C/Br},$$

$$C=\frac{-1\pm(-2\omega-3)^{\frac{1}{2}}}{\omega+2}.$$

PHYSICAL REVIEW VOLUME 125, NUMBER 6 MARCH 15, 1962

Theorem on Crossing Relations and Its Application to KN and $\bar{K}N$ Scattering Processes*

BENJAMIN W. LEE†
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania

(Received August 3, 1961)

It is proved that the contributions of the intermediate state, n, of the process $B+\bar{B}\rightarrow A+\bar{A}$ to the AB and $A\bar{B}$ potentials are of the same magnitude and same sign (opposite signs), if the G parity of the state n is even (odd). This theorem is discussed in detail for the exchange of a pion pair in KN and $\bar{K}N$ scattering processes in conjunction with the application of the Mandelstam representation to these processes.

I. INTRODUCTION

It has been known for a long time that the exchange of an even (odd) number of pions gives the same (opposite) contribution(s) to the nucleon-nucleon and antinucleon-nucleon potentials.[1] In the formulation of strong-interaction dynamics based on analyticity, unitarity, and crossing relations satisfied by the scattering amplitude,[2] it means that when the partial wave amplitudes of channel I, $N+N\rightarrow N+N'$, and of channel II, $N+\bar{N}'\rightarrow N'+\bar{N}$, are compared in the same angular momentum and isotopic spin states, the left-hand cuts of the two amplitudes associated with intermediate states of an even (odd) number of pions in channel III, $N'+\bar{N}'\rightarrow N+\bar{N}$, have the same magnitude and the same sign (opposite signs).

It is the purpose of this note to present a generalization of the above theorem. The generalization is actually twofold: The generalized form of the theorem is applicable to any elastic two-particle scattering processes on the one hand, and to "potentials" (or interactions, see Sec. II), associated with *any* intermediate states in channel III, on the other (Sec. II). In this sense, the proof to be given in Sec. II is a modified and detailed exposition of the remark recently made by Dalitz.[3] The proof we shall present here is elementary, instructive, and, we believe, rigorous.

An interesting example of this general theorem is afforded by the connection between the long-range forces arising from the exchange of two pions in K-nucleon and $\bar{K}$-nucleon scattering processes.[4–5] In a recent publication, an erroneous conclusion is arrived at

* Supported in part by the National Science Foundation and the Atomic Energy Commission.
† Now on leave of absence at the Institute for Advanced Study, Princeton, New Jersey.
[1] See, for example, J. S. Ball and G. F. Chew, Phys. Rev. **109**, 1385 (1958).
[2] G. F. Chew, Les Houches Lectures, 1960 (unpublished); University of California Radiation Laboratory Report UCRL-9289 (unpublished).
[3] R. H. Dalitz, Revs. Modern Phys. **33**, 471 (1961), p. 481, especially footnote 54.
[4] B. W. Lee, thesis, University of Pennsylvania, 1960 (unpublished); Phys. Rev. **121**, 1550 (1961). See also M. M. Islam, Nuovo cimento **3**, 546 (1961).
[5] F. Ferrari, G. Frye and M. Pusterla, Phys. Rev. **123**, 308, 315 (1961); Phys. Rev. Letters **4**, 615 (1960).