I: $e^{\alpha}=e^{\alpha_0}\left[\frac{1-B/r}{1+B/r}\right]^{1/\lambda}$,

$e^{\beta}=e^{\beta_0}(1+B/r)^2\left[\frac{1-B/r}{1+B/r}\right]^{(\lambda-C-1)/\lambda}$,

$\phi=\phi_0\left[\frac{1-B/r}{1+B/r}\right]^{C/\lambda}$,

$\lambda^2\equiv(C+1)^2-C(1-\omega C/2)>0.$

II: $\alpha=\alpha_0+\frac{2}{\Lambda}\tan^{-1}(r/B)$,

$\beta=\beta_0-\frac{2(C+1)}{\Lambda}\tan^{-1}(r/B)-\ln[r^2/(r^2+B^2)]$,

$\phi=\phi_0 e^{2C/\Lambda\ \tan^{-1}(r/B)}$,

$\Lambda^2\equiv C(1-\omega C/2)-(C+1)^2>0.$

III: $\alpha=\alpha_0-r/B$,

$\beta=\beta_0-2\ln\frac{r}{B}+(C+1)\frac{r}{B}$

$\phi=\phi_0 e^{-Cr/B}$,

$C=\frac{-1\pm(-2\omega-3)^{\frac{1}{2}}}{\omega+2}.$

IV: $\alpha=\alpha_0-(1/Br)$,

$\beta=\beta_0+(C+1)/Br$,

$\phi=\phi_0 e^{-C/Br}$,

$C=\frac{-1\pm(-2\omega-3)^{\frac{1}{2}}}{\omega+2}.$

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Theorem on Crossing Relations and Its Application to KN and $\bar{K}N$ Scattering Processes*

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It is proved that the contributions of the intermediate state, n, of the process $B+\bar{B}\to A+\bar{A}$ to the AB and $A\bar{B}$ potentials are of the same magnitude and same sign (opposite signs), if the G parity of the state n is even (odd). This theorem is discussed in detail for the exchange of a pion pair in KN and $\bar{K}N$ scattering processes in conjunction with the application of the Mandelstam representation to these processes.

I. INTRODUCTION

IT has been known for a long time that the exchange of an even (odd) number of pions gives the same (opposite) contribution(s) to the nucleon-nucleon and antinucleon-nucleon potentials.[1] In the formulation of strong-interaction dynamics based on analyticity, unitarity, and crossing relations satisfied by the scattering amplitude,[2] it means that when the partial wave amplitudes of channel I, $N+N\to N+N'$, and of channel II, $N+\bar{N}'\to N'+\bar{N}$, are compared in the same angular momentum and isotopic spin states, the left-hand cuts of the two amplitudes associated with intermediate states of an even (odd) number of pions in channel III, $N'+\bar{N}'\to N+\bar{N}$, have the same magnitude and the same sign (opposite signs).

It is the purpose of this note to present a generalization of the above theorem. The generalization is actually twofold: The generalized form of the theorem is applicable to any elastic two-particle scattering processes on the one hand, and to "potentials" (or interactions, see Sec. II), associated with *any* intermediate states in channel III, on the other (Sec. II). In this sense, the proof to be given in Sec. II is a modified and detailed exposition of the remark recently made by Dalitz.[3] The proof we shall present here is elementary, instructive, and, we believe, rigorous.

An interesting example of this general theorem is afforded by the connection between the long-range forces arising from the exchange of two pions in K-nucleon and $\bar{K}$-nucleon scattering processes.[4–5] In a recent publication, an erroneous conclusion is arrived at

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[1] See, for example, J. S. Ball and G. F. Chew, Phys. Rev. **109**, 1385 (1958).
[2] G. F. Chew, Les Houches Lectures, 1960 (unpublished); University of California Radiation Laboratory Report UCRL-9289 (unpublished).
[3] R. H. Dalitz, Revs. Modern Phys. **33**, 471 (1961), p. 481, especially footnote 54.
[4] B. W. Lee, thesis, University of Pennsylvania, 1960 (unpublished); Phys. Rev. **121**, 1550 (1961). See also M. M. Islam, Nuovo cimento **3**, 546 (1961).
[5] F. Ferrari, G. Frye and M. Pusterla, Phys. Rev. **123**, 308, 315 (1961); Phys. Rev. Letters **4**, 615 (1960).

on this question.[5] We shall discuss the effects of the exchange of a pion pair on K-nucleon and $\bar{K}$-nucleon scattering in order to rectify the error, and to illustrate the theorem in a physically interesting case (Sec. III). We shall remark briefly on possible future investigations on KN and $\bar{K}N$ scattering in view of recent experimental developments and the theorem given herein (Sec. IV).

II. THEOREM

Consider the elastic scattering process of two particles A of mass m_A and B of mass m_B:

$$A(p,i,\alpha)+B(q,j,\beta)\to A(p',i',\alpha')+B(q',j',\beta'). \quad (1)$$

The entries in parentheses denote the asymptotic four-momentum, spin, and charge (isotopic spin) variable, in that order. The T matrix for process (1) may be written as

$$\langle A(p',i',\alpha'),B(q',j',\beta')|T|A(p,i,\alpha),B(q,j,\beta)\rangle \equiv \mathcal{T}_{i',j';i,j}{}^{\alpha',\beta';\alpha,\beta}(s,t), \quad (2)$$

where the Mandelstam variables[6] s and t are defined to be

$$s=-(p+q)^2=-(p'+q')^2=W^2,$$
$$t=-(p-p')^2=-(q-q')^2=-2k^2(1-\cos\theta),$$

W, k, and θ being, respectively, the c.m. total energy, the magnitude of the c.m. momentum, and the scattering angle of process (1). The partial wave amplitude $f_{J\Pi}{}^T(s)$ of the state of total angular momentum J, parity Π, and isotopic spin T can be obtained by the operation:

$$f_{J\Pi}{}^T(s)=\rho(s)\sum_{\alpha',\beta',\alpha,\beta} P_{\alpha',\beta';\alpha,\beta}{}^{AB}(T) \times\sum_{i,j',i,j}\int_{4\pi} d\Omega\, P_{i',j';i,j}(J\Pi,\Omega) \times\mathcal{T}_{i',j';i,j}{}^{\alpha',\beta';\alpha,\beta}[s,t(s,\Omega)], \quad (3)$$

where $\rho(s)$ is a suitably defined kinematical factor; $P_{\alpha',\beta';\alpha,\beta}{}^{AB}(T)$ is the projection operator for the state of isotopic spin T of the AB system;

$$P_{\alpha',\beta';\alpha,\beta}{}^{AB}(T)=\sum_{T_3}\langle A(\alpha'),B(\beta')|AB(T,T_3)\rangle \times\langle AB(T,T_3)|A(\alpha),B(\beta)\rangle, \quad (4)$$

and $P_{i',j';i,j}(J\Pi,\Omega)$ is the angular momentum projection operator:

$$P_{i',j';i,j}(J\Pi,\Omega)=\sum_{J_z}\langle(\hat{p}',i'),(\hat{q}',j')|J,J_z,\Pi\rangle \times\langle J,J_z,\Pi|(\hat{p},i),(\hat{q},j)\rangle,$$
$$\Omega=(\theta,\varphi),\quad \hat{p}'\cdot\hat{p}=\hat{q}\cdot\hat{q}'=\cos\theta.$$

A process related to process (1) by the crossing relation is

$$B(q,j,\beta)+\bar{B}(-q',-j',\beta_C')\to \bar{A}(-p,-i,\alpha_C)+A(p',i',\alpha'), \quad (5)$$

where we have denoted the charge conjugate state of $A(\alpha)$ by $\bar{A}(\alpha_C)$:

$$|\bar{A}(\alpha_C)\rangle=C|A(\alpha)\rangle,$$

and the negative spin in front of the spin variable means that the spin direction is reversed. The principle of crossing relations (substitution law[7]) and the Mandelstam representation[6] (or an equivalent statement on analyticity) imply that the T matrix for process (5) and that of Eq. (2) are boundary values of the same analytic function in the two variables s and t:

$$\langle\bar{A}(-p,-i,\alpha_C),A(p',i',\alpha')|T|B(q,j,\beta),\bar{B}(-q',-j',\beta_C')\rangle =\mathcal{T}_{i',j';i,j}{}^{\alpha',\beta';\alpha,\beta}(s,t), \quad (6)$$

where s and t denote, for process (5),

$$s=-k_i^2-k_f^2+2k_ik_f\cos\gamma,$$
$$t=4(k_i^2+m_B^2)=4(k_f^2+m_A^2),$$

k_ik_f, and γ being, respectively, the magnitudes of initial and final momenta, and the scattering angle.

The partial wave amplitude $f_{J\Pi}{}^T(s)$ of process (1) has left-hand cuts—the contributions from which we shall refer to as "interactions"—associated with real intermediate states in process (5). The discontinuities of the amplitude $f_{J\Pi}{}^T(s)$ across these cuts are given in terms of the absorptive part of the amplitude (6):

$$\mathrm{Abs}_{\mathrm{III}}\mathcal{T}_{i',j';i,j}{}^{\alpha',\beta';\alpha,\beta}(s,t)\equiv\frac{1}{2i}\langle\bar{A}(-p,-i,\alpha_C),A(p',i',\alpha')|T-T^\dagger|B(q,j,\beta),\bar{B}(-q',-j',\beta_C')\rangle$$
$$=\pi\sum_n\langle\bar{A}(-p,-i,\alpha_C),A(p',i',\alpha')|T|n\rangle\langle n|T|B(q,j,\beta),\bar{B}(-q',-j',\beta_C)\rangle\delta(p_n-p'+p), \quad (7)$$

where the subscript III refers to the third channel in the conventional terminology, and the delta function in the last line signifies that the summation is to be taken over energy-momentum conserving intermediate states, n.

We consider, next, the scattering process between the particle A and the antiparticle $\bar{B}$ characterized by

$$A(p,i,\alpha)+\bar{B}(q,j,\beta_G)\to A(p',i',\alpha')+\bar{B}(q',j',\beta_G'). \quad (8)$$

The state vector for $\bar{B}(\beta_G)$ is defined to be

$$|\bar{B}(\beta_G)\rangle=G|B(\beta)\rangle, \quad (9)$$

[6] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[7] J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), pp. 161–163.

where G is the product of the charge conjugation C, and the 180° rotation about the y axis in the isotopic spin space, $\exp(i\pi T_2)$. The initial and final states of process (8) have the same isotopic spin as well as the mechanical (momentum, spin) quantum numbers as those of process (1), since the state $|\bar{B}(\beta_G)\rangle$ transforms cogrediently with the state $|B(\beta)\rangle$ under the rotation in the isotopic spin space.[8] Therefore, we see that

$$P_{\alpha',\beta_G';\alpha,\beta_G}{}^{A\bar{B}}(T)=P_{\alpha',\beta';\alpha,\beta}{}^{AB}(T),$$

and

$$\bar{f}_{J\Pi}{}^{T}(s)=\rho(s)\sum_{\alpha',\beta',\alpha,\beta}P_{\alpha',\beta';\alpha,\beta}{}^{AB}(T)\sum_{i',j',i,j}\int_{4\pi}P_{i',j',i,j}(J\Pi;\Omega)d\Omega\ \bar{T}_{i',j';i,j}{}^{\alpha',\beta_G';\alpha,\beta_G}(s,t(s,\Omega)),\tag{10}$$

where $\bar{f}_{J\Pi}{}^{T}(s)$ is the partial wave amplitude for process (8) and

$$\bar{T}_{i',j';i,j}{}^{\alpha',\beta_G';\alpha,\beta_G}(s,t)=\langle A(p',i',\alpha'),\bar{B}(q'.j'\,\beta_G')|T|A(p,i,\alpha),\bar{B}(q,j,\beta)\rangle.\tag{11}$$

The interactions of the amplitude $\bar{f}_{J\Pi}{}^{T}(s)$ associated with real intermediate states in the process[9]:

$$\bar{B}(q,j,\beta_G)+B(-q',-j',\beta_{CG}')\to A(p',i',\alpha')+\bar{A}(-p,-i,\alpha_C),\tag{12}$$

are given in terms of the absorptive part of the T matrix for process (12):

$$\begin{aligned}\mathrm{Abs}_{\mathrm{III}}\bar{T}_{i',j';i,j}{}^{\alpha',\beta_G';\alpha,\beta_G}(s,t)&=\pi\textstyle\sum_n\langle A(p',i',\alpha'),\bar{A}(-p,-i,\alpha_C)|T|n\rangle\\&\quad\times\langle n|T|\bar{B}(q,j,\beta_G),B(-q',-j',\beta_{CG})\rangle\\&\quad\times\delta(p_n-p'+p).\end{aligned}\tag{13}$$

The theorem to be proved can now be stated precisely: *The interactions (contributions from discontinuities across the left-hand cuts) $[f_{J\Pi}{}^{T}(s)]_n$, and $[\bar{f}_{J\Pi}{}^{T}(s)]_n$, arising from the intermediate state, n, of process $B+\bar{B}\to A+\bar{A}$ have the same magnitude and the same (opposite) sign if the G parity of the state $|n\rangle$ is even (odd), i.e., if*

$$G|n\rangle=+|n\rangle(-|n\rangle).$$

To prove this statement it is sufficient to show that the contribution from the state n to the absorptive part of Eq. (7):

$$\langle A(p',i',\alpha'),\bar{A}(-p,-i,\alpha_C)|T|n\rangle\times\langle n|T|B(q,j,\beta),\bar{B}(-q',-j',\beta_C)\rangle,\tag{14}$$

and that to the absorptive part of Eq. (13):

$$\langle A(p',i',\alpha'),\bar{A}(-p,-i,\alpha_C)|T|n\rangle\times\langle n|T|\bar{B}(q,j,\beta_G),B(-q',-j',\beta_{CG})\rangle,\tag{15}$$

have the same magnitude and the same sign (opposite signs) if the G parity of the state n is even (odd), since the amplitudes of Eq. (14) and Eq. (15) appear, respectively, in $[f_{J\Pi}{}^{T}(s)]_n$ and $[\bar{f}_{J\Pi}{}^{T}(s)]_n$ with the same linear coefficient [see Eqs. (13) and (10).] Since the first factors of Eqs. (14) and (15) are the same, we need consider only the relation between the second factors of Eqs. (14) and (15):

$$\begin{aligned}\langle n|T|\bar{B}(q,j,\beta_G),B(-q',-j',\beta_{CG})\rangle&=\langle n|TG|B(q,j,\beta)\,\bar{B}(-q'\ -j',\beta_C)\rangle\\&=\langle n|GT|B(q,j,\beta),\bar{B}(-q',-j',\beta_C)\rangle\\&=\pm\langle n|T|B(q,j,\beta),\bar{B}(-q',-j',\beta_C)\rangle\end{aligned}\tag{16}$$

if

$$G|n\rangle=\pm|n\rangle,$$

where we have used the identities:

$$[G,C]=0,\quad [G,T]=0,\quad GG^\dagger=1.$$

This concludes the proof.

Special cases of the above theorem have been known to many physicists; for nucleon-nucleon and nucleon-antinucleon scattering processes, the theorem implies the statement in the Introduction, since the G parity of the state consisting of an even (odd) number of pions is even (odd). When applied to pion-pion or pion-nucleon (with $B=\pi$) problems, the theorem yields trivial, but certainly correct results. When the pion and the nucleon are taken to be the particles A and B, respectively, the theorem states that the interactions of pion-antinucleon scattering in the state characterized by T, J, Π are the same as those of pion-nucleon scattering in the state of the same quantum numbers T, J, Π.

It is clear that the exchange of an even (odd) number of pions between the K meson and nucleon gives rise to forces of the same (opposite) sign(s) in KN and $\bar{K}N$ scattering processes. The application of this theorem to KN and $\bar{K}N$ elastic scattering processes will be discussed in conjunction with the double-dispersion representation in detail in the next section.

III. TWO-PION EXCHANGE IN KN AND $\bar{K}N$ SCATTERING PROCESSES

In this section we shall discuss the effect of two-pion exchange in KN and $\bar{K}N$ scattering. Our discussions will be based on the Mandelstam analyticity[6] and crossing relations. While it is clear from the result of the

[8] Anticipating the subsequent application, let us record the transformation of the K meson under G:

$$G\binom{K^\dagger}{K^0}=\binom{\bar{K}^0}{-K^-}.$$

In general, in the representation space of $SU(2)$, the spinor (a^*,b^*) transforms contragrediently to (a,b), while $(b^*,-a^*)$ transforms cogrediently to (a,b). See, for example, B. L. van der Waerden, *Die Gruppentheoritische Methode in der Quantum-Mechanik* (Verlag Julius Springer, Berlin, 1932), Chap. III.

[9] The order in which the subscripts G, C appear is irrelevant, since $[G,C]=0$, so that

$$|\beta_{GC}\rangle=CG|\beta\rangle=GC|\beta\rangle=|\beta_{CG}\rangle.$$

preceding section that the two-pion exchange mechanism gives rise to the interactions of the same sign and magnitude in KN and $\bar{K}N$ scattering processes, the manner in which this theorem manifests itself in the application of the Mandelstam representation has yet to be demonstrated in detail.

We consider the following three processes (channels) simultaneously:

$$\begin{aligned}
&(\mathrm{I}) \quad N(p_1,\alpha_1)+K(q_1,\beta_1)\to N(p_2,\alpha_2)+K(q_2,\beta_2),\\
&(\mathrm{II}) \quad N(p_1,\alpha_1)+\bar{K}(-q_2,-\beta_2)\to\\
&\qquad\qquad N(p_2,\alpha_2)+\bar{K}(-q_1,-\beta_1),\\
&(\mathrm{III}) \quad K(q_1,\beta_1)+\bar{K}(-q_2,-\beta_2)\to\\
&\qquad\qquad \bar{N}(-p_1,-\alpha_1)+N(p_2,\alpha_2),
\end{aligned} \tag{17}$$

where the p and q denote momenta, and the α and β the third components of the isotopic spin ($\frac{1}{2}$ or $-\frac{1}{2}$). The invariant T matrix for the above three channels may be written[10]

$$\begin{aligned}
\mathcal{T}_{\alpha_2,\beta_2;\alpha_1,\beta_1}{}^{\mathrm{I}}(s,t)&=\sum_{T=0,1}\left[-A^{(T)}(s,t)+i\gamma\frac{q_1+q_2}{2}B^{(T)}(s,t)\right]P_{\alpha_2,\beta_2;\alpha_1,\beta_1}{}^{NK},\\
\mathcal{T}_{\alpha_2,-\beta_1;\alpha_1,-\beta_2}{}^{\mathrm{II}}(\bar{s},t)&=\sum_{T=0,1}\left[-\bar{A}^{(T)}(\bar{s},t)+i\gamma\frac{-q_2-q_1}{2}\bar{B}^{(T)}(\bar{s},t)\right]P_{\alpha_2,-\beta_1;\alpha_1,-\beta_2}{}^{N\bar{K}},\\
\mathcal{T}_{-\alpha_1,\alpha_2;\beta_1,-\beta_2}{}^{\mathrm{III}}(t,s)&=\sum_{T=0,1}\left[-C^{(T)}(t,s)+i\gamma\frac{q_1+q_2}{2}D^{(T)}(t,s)\right]P_{-\alpha_1,\alpha_2;\beta_1,-\beta_2}{}^{\bar{N}N;K\bar{K}}.
\end{aligned} \tag{18}$$

where

$$s=-(p_1+q_1)^2,\qquad t=-(p_1-p_2)^2,\qquad \bar{s}=-(p_2-q_1)^2.$$

The isotopic spin protection operators can be explicitly exhibited;

$$\begin{aligned}
P_{\alpha_2,\beta_2;\alpha_1,\beta_1}{}^{NK}(T)&=\tfrac{1}{2}(\delta_{\alpha_1,\alpha_2}\delta_{\beta_1,\beta_2}\mp\delta_{\alpha_1,\beta_2}\delta_{\alpha_2,\beta_1}), && T=\begin{pmatrix}0\\1\end{pmatrix};\\
P_{\alpha_2,-\beta_1;\alpha_1,-\beta_2}{}^{N\bar{K}}(T)&=\tfrac{1}{2}\delta_{\alpha_1,\beta_2}\delta_{\alpha_2,\beta_1}, && T=0,\\
&=\delta_{\alpha_1,\alpha_2}\delta_{\beta_1,\beta_2}-\tfrac{1}{2}\delta_{\alpha_1,\beta_2}\delta_{\alpha_2,\beta_1}, && T=1;\\
P_{-\alpha_1,\alpha_2;\beta_1,-\beta_2}{}^{\bar{N}N;K\bar{K}}(T)&=\textstyle\sum_{T_3}\langle\bar{N}(-\alpha_1),N(\alpha_2)\,|\,\bar{N}N(T,T_3)\rangle\langle K\bar{K}(T,T_3)\,|\,K(\beta_1),\bar{K}(-\beta_2)\rangle\\
&=(\tfrac{1}{2})\delta_{\alpha_1,\alpha_2}\delta_{\beta_1,\beta_2}, && T=0,\\
&=\delta_{\alpha_1,\beta_2}\delta_{\alpha_2,\beta_1}-\tfrac{1}{2}\delta_{\alpha_1,\alpha_2}\delta_{\beta_1,\beta_2}, && T=1.
\end{aligned} \tag{19}$$

The principle of crossing relations and the Mandelstam analyticity imply that the T matrices of Eq. (18) are boundary values of the same analytic matrix function (in the Dirac γ space) of two variables, s and t, say:

$$\begin{aligned}
\mathcal{T}_{\alpha_2,\beta_2;\alpha_1,\beta_1}{}^{\mathrm{I}}(s,t)&=\mathcal{T}_{\alpha_2,-\beta_1;\alpha_1,-\beta_2}{}^{\mathrm{II}}(\bar{s},t)\\
&=\mathcal{T}_{-\alpha_1,\alpha_2;\beta_1,-\beta_2}{}^{\mathrm{III}}(t,s),
\end{aligned} \tag{20}$$

$$s+\bar{s}+t=2(m_N^2+m_K^2)\equiv\Sigma.$$

Combining Eqs. (18)–(20) we obtain[11]

$$\begin{bmatrix}A^{(T)}(s,t)\\B^{(T)}(s,t)\end{bmatrix}=\sum_{T'}\beta_{T,T'}\begin{bmatrix}C^{(T')}(t,s)\\D^{(T')}(t,s)\end{bmatrix}, \tag{21}$$

where

$$\beta_{T,T'}=\tfrac{1}{2}\begin{pmatrix}1, & -3\\1, & 1\end{pmatrix}$$

(the columns give the values for $T'=0$, 1 and the rows $T=0$, 1, respectively);

$$\begin{bmatrix}\bar{A}^{(T)}(\bar{s},t)\\\bar{B}^{(T)}(\bar{s},t)\end{bmatrix}=\sum_{T'}\bar{\beta}_{T,T'}\begin{bmatrix}C^{(T')}(t,s)\\-D^{(T')}(t,s)\end{bmatrix}, \tag{22}$$

where

$$\bar{\beta}_{T,T'}=\tfrac{1}{2}\begin{pmatrix}1, & 3\\1, & -1\end{pmatrix};$$

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[10] G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu, Phys. Rev. **106**, 1337 (1957). To avoid confusion, their A and B are denoted by a and b here.

[11] These relations have been noted previously. See, for example, M. Amati and B. Vitali, Nuovo cimento **6**, 1013 (1957); **7**, 190 (1958). In reference 4, $\bar{\beta}_{T,T'}$ is taken to be

$$\begin{pmatrix}1 & -3\\1 & 1\end{pmatrix}.$$

This is due to the fact that (1) a different normalization is used; (2) $(\bar{K}^0,K^-)$ are taken to be particles (rather than antiparticles). The erroneous conclusion reached there seems to have arisen from the failure to realize that (K^+,K^0) and $(\bar{K}^0,K^-)$ transform contragrediently, so that independent of the phase convention, $\beta_{T,0}=\bar{\beta}_{T,0}$, $\beta_{T,1}=-\bar{\beta}_{T,1}$ [Eq. (34)]. Once this is realized, the correct relation [Eq. (36′), p. 2207] can be deduced from their argument.

and

$$\begin{bmatrix} A^{(T)}(s,t) \\ B^{(T)}(s,t) \end{bmatrix} = \sum_{T'} \alpha_{T,T'} \begin{bmatrix} \bar{A}^{(T')}(\bar{s},t) \\ \bar{B}^{(T')}(\bar{s},t) \end{bmatrix}, \tag{23}$$

where

$$\alpha_{T,T'} = \tfrac{1}{2}\begin{pmatrix} -1, & 3 \\ 1, & 1 \end{pmatrix}.$$

The crossing matrices α, β, $\bar{\beta}$ satisfy

$$\begin{aligned} \textstyle\sum_{T''} \alpha_{T,T''}\alpha_{T'',T'} &= \delta_{T,T'}, \\ \textstyle\sum_{T''} \alpha_{T,T''}\bar{\beta}_{T'',T'} &= \beta_{T,T'}. \end{aligned} \tag{24}$$

A choice of partial wave amplitude which is devoid of kinematical singularities is that of Frazer and Fulco[12]:

$$h_l(W) = \frac{1}{16\pi k^{2l}} \Big\{ A_l(s) + (W-m_N)B_l(s) + \frac{(E-m_N)^2}{k^2} \times [-A_{l+1}(s) + (W+m_N)B_{l+1}(s)] \Big\}, \tag{25}$$

(channel I);

$$\bar{h}_l(\bar{W}) = \frac{1}{16\pi \bar{k}^{2l}} \Big\{ \bar{A}_l(\bar{s}) + (\bar{W}-m_N)\bar{B}_l(\bar{s}) + \frac{(\bar{E}-m_N)^2}{\bar{k}^2} \times [-A_{l+1}(\bar{s}) + (\bar{W}+m_N)\bar{B}_{l+1}(\bar{s})] \Big\}, \tag{25'}$$

(channel II), where

$$[A_l(s), B_l(s)] = \int_{-1}^{1} dz\, P_l(z) \times [A(s, -2k^2(1-z)), B(s, -2k^2(1-z))], \tag{26}$$

W and k are defined as in the last section, E is the c.m. nucleon energy, $E=(k^2+m_N^2)^{\frac{1}{2}}=(s+m_N^2-m_K^2)/2\sqrt{s}$, and barred quantities are those of channel II. The isotopic spin indices have been suppressed for the moment. The amplitude $h_l(W)$ $[\bar{h}_l(\bar{W})]$ is analytic in a cut W $(\bar{W})$ plane. The locations of singularities have been discussed in detail elsewhere.[4,12]

Let us concentrate on the interaction associated with two-pion intermediate states in channel III. The amplitudes $A^{(T)}(s,t)$, $\bar{A}^{(T)}(\bar{s},t)$, $B^{(T)}(s,t)$, and $\bar{B}^{(T)}(\bar{s},t)$ may be written as[6]

$$\begin{aligned} \begin{bmatrix} A^{(T)}(s,t) \\ B^{(T)}(s,t) \end{bmatrix} &= \sum_{T'} \alpha_{T,T'} \frac{1}{\pi}\int_{(m_\Lambda+m_\pi)^2}^{\infty} \frac{d\bar{s}'}{\bar{s}'-\bar{s}} \operatorname{Abs}\begin{bmatrix} \bar{A}^{(T')}(\bar{s}', \Sigma-s-\bar{s}') \\ -\bar{B}^{(T')}(\bar{s}', \Sigma-s-\bar{s}') \end{bmatrix} \\ &\quad + \sum_{T'} \beta_{T,T'} \frac{1}{\pi}\int_{(2m_\pi)^2}^{\infty} \frac{dt'}{t'-t} \operatorname{Abs}\begin{bmatrix} C^{(T')}(t',s) \\ D^{(T')}(t',s) \end{bmatrix} + \text{pole terms}; \\ \begin{bmatrix} \bar{A}^{(T)}(\bar{s},t) \\ \bar{B}^{(T)}(\bar{s},t) \end{bmatrix} &= \sum_{T'} \alpha_{T,T'} \frac{1}{\pi}\int_{(m_\Lambda+m_\pi)^2}^{\infty} \frac{ds'}{s'-s} \operatorname{Abs}\begin{bmatrix} A^{(T')}(s', \Sigma-s'-\bar{s}) \\ -B^{(T')}(s', \Sigma-s'-\bar{s}) \end{bmatrix} \\ &\quad + \sum_{T'} \bar{\beta}_{T,T'} \frac{1}{\pi}\int_{(2m_\pi)^2}^{\infty} \frac{dt'}{t'-t} \operatorname{Abs}\begin{bmatrix} C^{(T')}(t', \Sigma-\bar{s}-t') \\ -D^{(T')}(t', \Sigma-\bar{s}-t') \end{bmatrix} + \text{pole terms}, \end{aligned} \tag{27}$$

where the symbol Abs means "the absorptive part of." We shall denote by $\{h_l(W)\}_{\pi\pi}$ the part of $h_l(W)$ arising from the absorptive part associated with two-pion intermediate states in channel III, $\operatorname{Abs}_{\pi\pi}[C(t,s); D(t,s)]$. From Eqs. (25)–(27), we see that

$$\begin{aligned} \{h_C(W)\}_{\pi\pi} &= -\frac{1}{16\pi k^{2l}} \sum_{T'} \beta_{T,T'} \frac{1}{\pi}\int_{(2m_\pi)^2}^{\infty} \frac{dt'}{k^2} \Big\{ Q_l\Big(1+\frac{t'}{2k^2}\Big)[\operatorname{Abs}_{\pi\pi}C^{(T')}(t',s) + (W-m_N)\operatorname{Abs}_{\pi\pi}D^{(T')}(t',s)] \\ &\quad + \frac{(E-m_N)^2}{k^2} Q_{l+1}\Big(1+\frac{t'}{2k^2}\Big)[-\operatorname{Abs}_{\pi\pi}C^{(T')}(t',s) + (W+m_N)\operatorname{Abs}_{\pi\pi}D^{(T')}(t',s)] \Big\}; \\ \{\bar{h}_l(\bar{W})\}_{\pi\pi} &= -\frac{1}{16\pi \bar{k}^{2l}} \sum_{T'} \bar{\beta}_{T,T'} \frac{1}{\pi}\int_{(2m_\pi)^2}^{\infty} \frac{dt'}{\bar{k}^2} \Big\{ Q_l\Big(1+\frac{t'}{2\bar{k}^2}\Big) \\ &\quad \times [\operatorname{Abs}_{\pi\pi}C^{(T')}(t', \Sigma-\bar{s}-t') - (\bar{W}-m_N)\operatorname{Abs}_{\pi\pi}D^{(T')}(t', \Sigma-\bar{s}-t')] \\ &\quad + \frac{(\bar{E}-m_N)^2}{\bar{k}^2} Q_{l+1}\Big(1+\frac{t'}{2\bar{k}^2}\Big)[-\operatorname{Abs}_{\pi\pi}C^{(T')}(t', \Sigma-\bar{s}-t') - (\bar{W}+m_N)\operatorname{Abs}_{\pi\pi}D^{(T')}(t', \Sigma-\bar{s}-t')] \Big\}, \end{aligned} \tag{28}$$

[12] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960).

where Q_l is the Legendre function of the second kind. It should be noted that $\{h_l(W)\}_{\pi\pi}$ has cuts arising from singularities in s of $\mathrm{Abs}[C^{(T)}(t',s),D^{(T)}(t',s)]$ for fixed t, in addition to the "$\pi\pi$ cuts" whose locations and discontinuities are the same as those of $[h_l(W)]_{\pi\pi}$. Now, unitarity in channel II gives[4]

$$-\mathrm{Abs}_{\pi\pi}C^{(T)}(t,s)+i\gamma\frac{q_1+q_2}{2}\mathrm{Abs}_{\pi\pi}D^{(T)}(t,s)$$
$$=-\frac{\pi}{(2\pi)^3}\int\frac{d^3\kappa_1 d^3\kappa_2}{4\kappa_{10}\kappa_{20}}\left[-a^{(T)}(s',t)+i\gamma\frac{\kappa_1-\kappa_2}{2}b^{(T)}(s',t)\right]\times F^{(T)*}(t,s'')\delta(\kappa_1+\kappa_2-q_1+q_2), \quad (29)$$

where $a^{(T)}(s',t)$ and $b^{(T)}(s',t)$ are the customary invariant amplitudes for pion-nucleon scattering[10]

$$\pi(\kappa_1)+N(p_1)\rightarrow\pi(\kappa_2)+N(p_2),$$

with

$$s'=-(p_1+\kappa_1)^2=-(p_2+\kappa_2)^2,$$

and $F^{(T)}(t,s'')$ is the invariant amplitude for the process $\pi+\pi\rightarrow K+\bar{K}$[13,14]:

$$\tfrac{1}{2}(8\kappa_0\kappa_{20}q_{20})^{\frac{1}{2}}\langle\bar{K}(-q_2)|j_K{}^\dagger|\pi(\kappa_1)\pi(\kappa_2)^{\mathrm{in}}\rangle =\textstyle\sum_T F^{(T)}(t,s'')P^{K\bar{K};\pi\pi}(T),\quad s''=-(q_2+\kappa_2)^2. \quad (30)$$

We define the partial wave amplitude $F_J{}^{(T)}(t)$ for the process $\pi+\pi\rightarrow K+\bar{K}$ by

$$F^{(T)}(t,s'')=\textstyle\sum_J(2J+1)(\kappa k_i)^J F_J{}^{(T)}(t)P_J(y''), \quad (31)$$

where κ, k_i, and y are given by

$$4(\kappa^2+m_\pi{}^2)=4(k_i{}^2+m_K{}^2)=t,$$
$$s''=-\kappa^2-k_i{}^2+2\kappa k_i y''.$$

The amplitudes $a^{(T)}$, $b^{(T)}$ can be conveniently decomposed into the Jacob-Wick helicity amplitudes[15] for the process $\pi+\pi\rightarrow N+\bar{N}$. When these partial wave amplitudes are inserted into Eq. (29), the integrations over κ_1 and κ_2 can be performed, and there results

$$\mathrm{Abs}_{\pi\pi}C^{(T)}(t,s)=\tfrac{1}{4}\left(\frac{t-4m_\pi{}^2}{t}\right)^{\frac{1}{2}}\sum_J(2J+1)F_J{}^{(T)*}(t)\kappa^{2J}\frac{(k_i{}^*k_f)^J}{k_f{}^2}\times\left[f_+{}^{J(T)}(t)P_J(y)-\frac{m_N}{[J(J+1)]^{\frac{1}{2}}}f_-{}^{J(T)}(t)yP_J'(y)\right]\theta(t-4m_\pi{}^2);$$
$$\mathrm{Abs}_{\pi\pi}D^{(T)}(t,s)=-\tfrac{1}{4}\left(\frac{t-4m_\pi{}^2}{t}\right)^{\frac{1}{2}}\sum_J(2J+1)F_J{}^{(T)*}(t)\kappa^{2J}(k_i{}^*k_f)^{J-1}\left(\frac{k_i{}^*}{k_i}\right)\times[J(J+1)]^{-\frac{1}{2}}f_-{}^{J(T)}(t)P_J'(y)\theta(t-4m_\pi{}^2), \quad (32)$$

where

$$k_f{}^2=\tfrac{1}{4}(t-4m_N{}^2),\quad y=\frac{\Sigma-t-2s}{4k_ik_f}=-\frac{\Sigma-t-2\bar{s}}{4k_ik_f}. \quad (33)$$

Because of the Pauli principle applied to the state of two pions, the sum in Eq. (32) extends over even (odd) J's for $T=0(1)$. Since $\mathrm{Abs}_{\pi\pi}C^{(T)}$ is even (odd) and $\mathrm{Abs}_{\pi\pi}D^{(T)}$ is odd (even) in y for $T=0(1)$, we see from Eqs. (32) and (33) that

$$\mathrm{Abs}_{\pi\pi}C^{(T)}(t,\Sigma-\bar{s}-t)=(\pm)\,\mathrm{Abs}_{\pi\pi}C^{(T)}(t,\bar{s}),$$
$$\mathrm{Abs}_{\pi\pi}D^{(T)}(t,\Sigma-\bar{s}-t)=(\mp)\,\mathrm{Abs}_{\pi\pi}D^{(T)}(t,\bar{s}), \quad (34)$$

for

$$T=\binom{0}{1}.$$

When Eq. (34) and

$$\beta_{T,T'}=(\pm)\bar{\beta}_{T,T'}\qquad\text{for}\qquad T'=\binom{0}{1} \quad (35)$$

is substituted into the second of Eq. (28), we obtain

$$\{\bar{h}_l(\bar{W})\}_{\pi\pi}=-\frac{1}{16\pi\bar{k}^{2l}}\sum_{T'}\beta_{T,T'}\frac{1}{\pi}\int_{(2m_\pi)^2}^\infty\frac{dt'}{\bar{k}^2}\left\{Q_l\left(1+\frac{t'}{2\bar{k}^2}\right)[\mathrm{Abs}_{\pi\pi}C^{(T')}(t',\bar{s})+(\bar{W}-m_N)\,\mathrm{Abs}_{\pi\pi}D^{(T')}(t',\bar{s})]\right.$$
$$\left.+\frac{(\bar{E}-m_N)^2}{\bar{k}^2}Q_{l+1}\left(1+\frac{t'}{2\bar{k}^2}\right)[-\mathrm{Abs}_{\pi\pi}C^{(T')}(t',\bar{s})+(\bar{W}+m_N)\,\mathrm{Abs}_{\pi\pi}D^{(T')}(t',\bar{s})],\right.$$

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[13] B. W. Lee, Phys. Rev. **120**, 325 (1960). The normalization of $F^{(T)}(t,s)$ is such that $F^{(T)}(t,s)=-4\pi B^{(T)}$ where $B^{(T)}$ is the invariant amplitude so designated in the above reference.

[14] We *define* the state vector of the two-pion system as

$$|\pi(\kappa_1,\alpha)\pi(\kappa_2,\beta)\rangle=a_\alpha{}^\dagger(\kappa_1)a_\beta{}^\dagger(\kappa_2)|\mathrm{vac}\rangle$$

rather than with a factor of $2^{-\frac{1}{2}}$ on the right-hand side, where $a_\alpha{}^\dagger(\kappa_1)$ is the creation operator for the pion of momentum κ_1 and isotopic spin index α.

[15] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959); the normalization of $f_+{}^{J(T)}$ is chosen in accord with W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).

so that

$$\{\bar{h}_l(\bar{W})\}_{\pi\pi}=\{h_l(\bar{W})\}_{\pi\pi}, \tag{36}$$

and, therefore,

$$[\bar{h}_l(\bar{W})]_{\pi\pi}=[h_l(\bar{W})]_{\pi\pi}. \tag{36'}$$

IV. CONCLUDING REMARKS

We have shown in the previous section that the two-pion exchange interactions have the same sign and magnitude in KN and $\bar{K}N$ scattering processes. Since the two-pion exchange gives rise to the longest range interaction in these processes,[4] it is presumably responsible for the energy-dependence of the K-matrix elements for these processes in the low-energy region. Indeed, this assumption is the basis of analysis of Lee,[3] Ferrari *et al.*,[5] and Islam.[4] Based on the assumption that the exchange of two pions resonating in the $T=J=1$ state is responsible for the energy dependence of the S-wave K matrix in KN scattering, Lee and Islam conclude that their model supports the D solution of Chinowsky *et al.*,[16] and, from Eq. (36), that the magnitude of the interaction is possibly smaller by a factor of 2 than that of Ferrari *et al.*

Since the completion of the aforementioned analyses, there have been some experimental developments which call for a re-examination of the previous analyses. These are:

(1) The location of the $T=J=1$ resonance. The experiments of Pickup *et al.*,[17] and Erwin *et al.*[18] indicate the resonance energy of about 700$\sim$800 Mev, which is close to the value, 660 Mev, deduced by Bowcock *et al.*[19] In the analyses of references 3 and 4, the lower value, $\approx$500 Mev, of Frazer and Fulco[20] were used. This would mean that the cut arising from the exchange of a dipion ($T=J=1$) is farther removed from the threshold.

(2) Existence of the Y^* isobar state.[21] If the spin of the isobar should turn out to be $\frac{1}{2}$, the a_+ solution of Dalitz and Tuan,[22] on which the work of Ferrari *et al.*, is based, would be ruled out. Instead the a_- solution will be the likely candidate. It would be interesting to reperform the analysis of Ferrari *et al.*, assuming the a_- scattering lengths.

(3) Low-energy K^+p scattering data.[23] The new data from Berkeley indicate a rapid variation of the K^+p cross section below 60 Mev. This is very puzzling, since the energy dependence is enormous, and casts some doubt about the validity of the results of reference 3. The long-range force should be re-evaluated based on the refined data.

Finally, if there is a bound or resonance state ($T=0$, $J=1$)[24] of three pions, this can conceivably give rise to large interactions of opposite signs, but of the same magnitude in KN and $\bar{K}N$ processes. At this juncture, it may be remarked[25] that the theorem shown in this paper applies equally well to the exchange of vector bosons of the type discussed by Sakurai,[26] and Zachariasen and Gell-Mann,[27] and Glashaw and Gell-Mann.[28]

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[27] F. Zachariasen and M. Gell-Mann (to be published).

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