Flux Quantization and the Current-Carrying State in a Superconducting Cylinder*

H. J. LIPKIN,† M. PESHKIN, AND L. J. TASSIE‡
Argonne National Laboratory, Argonne Illinois

(Received August 22, 1961; revised manuscript received December 5, 1961)

London's phenomenological theory is modified to include quantization of angular momentum in the boson gas model of a superconductor. The modified London equation is solved in ring geometry to find the magnetic field in the penetration region. The unit of flux quantization is obtained for all thicknesses of the superconductor. The relation between angular momentum and kinetic energy in the current-carrying state is discussed briefly.

1. INTRODUCTION

THE quantization of the trapped magnetic flux that threads a superconducting ring has been observed experimentally[1,2] and has been discussed theoretically from several points of view. The first treatments[3,4] considered thick superconductors and obtained flux quantization in integral multiples of $hc/2e$ (Gaussian units). Other treatments[5,6] show that the flux quantization unit is smaller when the ring thickness is less than the penetration depth of the superconductor.

We present here a simple description of the current-carrying state in a ring superconductor of arbitrary thickness, based on London's phenomenological theory[7] and on the free-boson gas model of Schafroth, Butler, and Blatt.[8] That means we replace the electrons by a condensed gas of bosons of charge $2e$ and mass $2m$. We assume that the internal structure of the bosons is unaffected by the fields which are present, and that the electrostatic interactions (boson-boson and boson-ion) maintain uniform boson density. We do not investigate here the applicability of the boson gas model to real superconductors. Bardeen[6] gives some justification for its use near the transition temperature. The question of the density has been studied more carefully by Bloch and Rorschach,[9] who find uniform density to good approximation.

The effect of introducing the boson gas model and quantizing the canonical angular momentum of the bosons is to add an inhomogeneous term to the usual London equation. We give an explicit solution in the ring geometry, to obtain the field and current distribution in the superconductor and the value of the flux quantization unit for any thickness.[10]

2. CALCULATIONS

Consider a long, hollow cylinder of inner radius R_1 and outer radius R_2. Introduce cylindrical coordinates z, r, θ. We wish to describe a state in which current flows only in the θ direction, so that the magnetic field is in the z direction and depends only on r. We can then choose the gauge to make the vector potential have only the component $A_\theta(r)$. In this gauge, the Hamiltonian exhibits the cylindrical symmetry, so that the canonical angular momentum component L_z is a good quantum number. In the condensed state, each boson is taken to contribute $l\hbar$ to L_z, where l is an integer. Then the canonical angular momentum density is equal to $\frac{1}{2}\rho l\hbar$, where $\frac{1}{2}\rho$ is the boson number density.

The current density $\mathbf{j}(\mathbf{r})$ has only the component $j_\theta(r)$, which is determined by

$$\mathbf{j}(\mathbf{r})=\tfrac{1}{2}\rho\frac{2e}{2m}\left(\mathbf{p}-\frac{2e}{c}\mathbf{A}\right), \tag{1}$$

where c is the velocity of light, $2e$ and $2m$ are the boson charge and mass, respectively, and $\frac{1}{2}\rho\mathbf{p}$ is the canonical momentum density.

$$j_\theta=r^{-1}(\mathbf{r}\times\mathbf{j})_z,$$
$$j_\theta=\frac{\rho e}{m}\left(\frac{l\hbar}{2r}-\frac{e}{c}A_\theta\right). \tag{2}$$

Equation (2) is the same as London's equation relating j_θ and A_θ, except for the inhomogeneous term $\frac{1}{2}l\hbar/r$.

A second relation between j_θ and A_θ is obtained from

$$\nabla\times\nabla\times\mathbf{A}=4\pi\mathbf{j}/c, \tag{3}$$

which gives for the θ components

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rA_\theta)\right]-\frac{4\pi}{c}j_\theta=0. \tag{4}$$

* Sponsored in part by the U. S. Atomic Energy Commission and in part by the Air Force Office of Scientific Research of the Air Research and Development Command, United States Air Force, through its European Office.
† On leave from the Weizmann Institute of Science.
‡ Fulbright Scholar on leave from Australian National University.
[1] B. S. Deaver and W. M. Fairbank, Phys. Rev. Letters **7**, 43 (1961).
[2] R. Doll and M. Näbauer, Phys. Rev. Letters **7**, 51 (1961).
[3] N. Byers and C. N. Yang, Phys. Rev. Letters **7**, 46 (1961).
[4] L. Onsager, Phys. Rev. Letters **7**, 50 (1961).
[5] J. M. Blatt, Phys. Rev. Letters **7**, 82 (1961).
[6] J. Bardeen, Phys. Rev. Letters **7**, 162 (1961).
[7] F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1950).
[8] M. R. Schafroth, S. T. Butler, and J. M. Blatt, Helv. Phys. Acta **30**, 93 (1957).
[9] F. Bloch (private discussion).
[10] *Note added in proof.* Similar considerations are reported by J. B. Keller and B. Zumino, Phys. Rev. Letters **7**, 162 (1961).

These results are combined to eliminate j_θ, giving

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rA_\theta)\right]+\frac{4\pi\rho e}{mc}\left[\frac{l\hbar}{2r}-\frac{e}{c}A_\theta\right]=0, \tag{5}$$

which again differs from London's corresponding equation by the inhomogeneous term.

Since we assume constant ρ, Eq. (5) is easily solved. A solution for given l is

$$A_\theta=\frac{l\hbar c}{2e}\frac{1}{r}, \tag{6a}$$

$$j_\theta=0, \tag{6b}$$

which gives zero current and magnetic field in the superconductor, and quantized flux

$$\Phi=l\Phi_u=lhc/e \tag{7}$$

inside the cylinder. Solution (6) describes the interior of the superconductor and is equivalent to the simplest discussion of a thick superconductor.[4] However, Eqs. (2)–(5) describe the surface region of the superconductor as well as the interior, and the correct solution is obtained by inserting appropriate boundary conditions. The general solution of (5) is obtained by adding to (6a) the general solution of the homogeneous equation. Since that is just London's equation, the added part corresponds to London's solution, a surface current which decreases exponentially beyond the penetration depth. The general solution of (5) is then given by

$$A_\theta=\frac{l\hbar c}{2e}\frac{1}{r}+C_1I_1(\beta r)+C_2K_1(\beta r), \tag{8}$$

where I_1 and K_1 are the modified Bessel functions,[11] and

$$\beta=(4\pi\rho e^2/mc^2)^{\frac{1}{2}} \tag{9}$$

is the reciprocal of the penetration depth.

The constants C_1 and C_2 are determined from the boundary conditions at R_1 and R_2, assuming a uniform field B_0 outside the cylinder. Within the cylinder will be another uniform field whose magnitude is fixed by the boundary conditions. The results for the flux Φ_{in} contained within the inner radius R_1 and the total flux Φ_T contained within the outer radius R_2 are given by

$$\Phi_{\text{in}}=l\Phi_u\left\{1-\frac{2}{\beta R_1D}[K_1(\beta R_1)I_0(\beta R_2)+I_1(\beta R_1)K_0(\beta R_2)]\right\}+2\Phi_0(\beta^2R_2^2D)^{-1}, \tag{10}$$

$$\Phi_T=l\Phi_u\left(1-\frac{2}{\beta^2R_1^2D}\right)+\Phi_0\{1-D^{-1}[K_2(\beta R_1)I_2(\beta R_2)-I_2(\beta R_1)K_2(\beta R_2)]\}, \tag{11}$$

$$D=K_0(\beta R_1)I_0(\beta R_2)-I_0(\beta R_1)K_0(\beta R_2)+\frac{2}{\beta R_1}[K_1(\beta R_1)I_0(\beta R_2)+I_1(\beta R_1)K_0(\beta R_2)]. \tag{12}$$

The quantity

$$\Phi_0=\pi R_2^2B_0 \tag{13}$$

represents the flux which is produced by the external field B_0 within radius R_2. Expressions (10) and (11) go over into the known results in the limit of very thick or very thin superconductors.[6]

3. DISCUSSION

Understanding of the current-carrying state in a superconductor has long been obscured by the idea that such a state must have high energy because the quantum of angular momentum of a condensed state is of the order of the density times the quantum for a single particle. The crucial feature of this and the previous discussions of flux quantization is the difference between canonical and kinetic angular momentum in the presence of a magnetic field, even when the charged particles do not penetrate the field. This difference permits the construction of states having high canonical angular momentum but small circulating current and small kinetic energy. Analogous remarks apply to the linear case, where the necessity for a return current insures that there will be a trapped flux. The quantization of canonical angular momentum in the presence of magnetic fields has been discussed in detail elsewhere.[12,13]

ACKNOWLEDGMENTS

We have enjoyed and profited from discussions with Professor J. Bardeen and Professor F. Bloch, who have communicated their unpublished results to us.

[11] W. Magnus and F. Oberhettinger, *Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949), p. 19.

[12] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959); Y. Aharonov and D. Bohm, *ibid.* **123**, 1511 (1961).

[13] M. Peshkin, I. Talmi, and L. J. Tassie, Ann. Phys. **12**, 426 (1961); L. J. Tassie and M. Peshkin, *ibid.* **16**, 177 (1961).