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Energy Loss of a Fast Ion in a Plasma

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The problem is considered of a fast charged particle injected into a plasma at an energy in excess of plasma thermal energies. Closed expressions are obtained for the behavior of the mean energy as a function of time, and also for the diffusion spread around this mean.

An expression is given for the characteristic speed, say u, of any plasma, defined such that for all test particle speeds in excess of u energy is lost predominantly to electrons, while the loss goes to the positive ions for speeds less than u. For fully ionized deuterium gas, u is 1/14 times the electron thermal speed; if the electron and ion temperatures are the same, u is only a factor of 4 greater than the ion thermal speed. (For convenience the "thermal speed" has been defined as $(\frac{2}{3})^{\frac{1}{2}}$ times the root-mean-square value).

1. INTRODUCTION

THE rate of energy loss to the electrons and positive ions of a plasma by a fast charged particle is of interest for many aspects of plasma and thermonuclear research.

This rate, determined by the Coulomb interaction, is often discussed in terms of estimates of relaxation times given by Spitzer[1] and Chandrasekhar.[2] The information often of interest, however, is the rate of energy loss for all times after injection, and a comparison of the relative energy loss rate to electrons and ions as a function of time. Moreover, the relaxation times of Spitzer[1] are determined for the energy region in which the speed of the test particle is either less than or not much greater than the thermal speeds of the plasma particles.

A quantitative treatment of the complete behavior of a fast particle, as it undergoes Coulomb scatterings and loses energy, involves the solution of a Fokker-Planck-type equation.[3] This requires lengthy numerical computation, and some special cases have been treated by Kranzer.[4] It is the purpose of this paper to show that a closed expression for the behavior of the mean energy as a function of time exists for almost all the range of interest. Diffusion spread around this mean can also be calculated.

An interesting result is that, apart from the thermal speeds w_e and w_i of electrons and positive ions respectively, there is an intermediate speed u characteristic of any plasma, which determines the relative energy losses to electrons and ions. If v represents the speed of the test particle, this particle loses energy predominantly to the positive ions only in the region $v \lesssim u$. For the remaining region $v \gtrsim u$ energy is lost predominantly to electrons.

For fully ionized deuterium gas $u \simeq w_e/14$; if the electron and ion temperatures are the same, we thus also have $u \simeq 4u_i$.

2. MEAN ENERGY LOSS PER COLLISION

Consider a test ion of mass M and charge Ze traveling with velocity $\mathbf{v}$ in the laboratory system, colliding with a field particle of mass m and charge ze traveling with velocity $\mathbf{w}$.

The cross section for scattering through an angle θ in the c.m. system is

$$\sigma(\theta)d\Omega = \left(\frac{Zze^2}{\frac{1}{2}[Mm/(M+m)](\mathbf{v}-\mathbf{w})^2}\right)^2 \times \frac{d\Omega}{[4\sin^2(\theta/2)]^2}, \quad \theta \geqslant \theta_0. \quad (1)$$

[1] L. Spitzer, *Physics of Ionized Gases* (Interscience Publishers, Inc., New York, 1950).

[2] S. Chandrasekhar, Astrophys. J. **93**, 285 (1941).

[3] M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. **107**, 1 (1957).

[4] H. C. Kranzer, Phys. Fluids 4, 214 (1961).

Here θ_0 is the minimum angle of scatter corresponding to an impact parameter of the order of the Debye screening length (see Sec. 7).

The velocity of the test ion in the c.m. system is initially $[m/(M+m)](\mathbf{v}-\mathbf{w})$, and this does not change magnitude in the collision. When one adds the velocity $(M\mathbf{v}+m\mathbf{w})/(M+m)$ of the center of mass, however, it is readily found that in the laboratory system the change ΔE of the kinetic energy of the test ion is

$$\Delta E=-\frac{mM}{(M+m)^2}\{2\sin^2(\theta/2)[(\mathbf{v}-\mathbf{w})\cdot(M\mathbf{v}+m\mathbf{w})] +\sin\theta\cos\phi|(\mathbf{v}-\mathbf{w})\times(M\mathbf{v}+m\mathbf{w})|\}. \quad (2)$$

Here θ is the angle of scattering in the c.m. system, and ϕ is the azimuthal angle of scattering in the c.m. system measured with respect to the plane of $\mathbf{v}$ and $\mathbf{w}$.

The average value of $\sin^2(\theta/2)$ for such a collision is readily obtained from Eq. (1) (without recourse to small-angle approximations). We find

$$\langle\sin^2(\theta/2)\rangle_{\text{av}}=\frac{\pi}{2}\frac{1}{\sigma_t}\left(\frac{Zze^2}{\frac{1}{2}[Mm/(M+m)](\mathbf{v}-\mathbf{w})^2}\right)^2\ln\Lambda, \quad (3)$$

where σ_t is the total scattering cross section, and

$$\ln\Lambda=\ln(2/\theta_0).$$

The second term of Eq. (2) averages to zero.

Thus the average energy change of the test particle from such a collision is simply

$$\Delta E=-\frac{4\pi}{\sigma_t}\frac{(Zze^2)^2}{m}\times\ln\Lambda\left(\frac{(\mathbf{v}-\mathbf{w})\cdot[\mathbf{v}(1+m/M)-(m/M)(\mathbf{v}-\mathbf{w})]}{(\mathbf{v}-\mathbf{w})^4}\right). \quad (4)$$

3. LOSS RATE IN A PLASMA

We now consider energy transfers between the test particle and plasma particles of one type. Let the velocity distribution of the field particles under consideration be $f(\mathbf{w})d\mathbf{w}$ per unit volume, so that their number density $\rho=\int f(\mathbf{w})d\mathbf{w}$.

In a given small interval of time δt, the number of collisions with field particles having velocities between $\mathbf{w}$ and $\mathbf{w}+d\mathbf{w}$ is

$$\sigma_t|\mathbf{v}-\mathbf{w}|f(\mathbf{w})d\mathbf{w}\delta t.$$

Thus the rate of change of energy of the test ion, determined by plasma particles with velocity $\mathbf{w}$, is[5]

$$\left(\frac{dE}{dt}\right)_{(\mathbf{w})}=-\frac{4\pi}{m}(Zze^2)^2\times\ln\Lambda\left[\frac{(\mathbf{v}-\mathbf{w})\cdot[\mathbf{v}(1+m/M)-(m/M)(\mathbf{v}-\mathbf{w})]}{|\mathbf{v}-\mathbf{w}|^3}\right]\times f(\mathbf{w})d\mathbf{w}. \quad (5)$$

We now must simply integrate over $\mathbf{w}$ to find the total rate of energy change. If we take a Maxwellian distribution,

$$f(\mathbf{w})=f_0\exp[-(w/w_t)^2], \quad (6)$$

where w_t is the thermal speed of the plasma particles, the integrals are elementary[6]; we obtain

$$\frac{dE}{dt}=-\frac{8\sqrt{\pi}}{m}\frac{(Zze^2)^2}{w_t}F(v/w_t)\ln\Lambda, \quad (7)$$

where

$$F(x)=\frac{1}{x}\int_0^x\exp(-x^2)dx-\left(1+\frac{m}{M}\right)\exp(-x^2). \quad (8)$$

A more convenient form for Eq. (7) is found if we multiply both sides by $3v$, and write $dF/dt=Mvdv/dt$. We then obtain the equation,

$$\frac{dv^3}{dt}=-\frac{24\sqrt{\pi}}{mM}(zZe^2)^2\rho\ln\Lambda G(v/w_t), \quad (9)$$

where

$$G(x)=xF(x). \quad (10)$$

For $x\lesssim1$ we have

$$G(x)=-\frac{m}{M}x+\left(\frac{2}{3}+\frac{m}{M}\right)x^3+\cdots, \quad (11)$$

and for $x\gg1$ we have

$$G(x)\sim\tfrac{1}{2}\sqrt{\pi}. \quad (12)$$

It is to be noticed that $F(x)$ and $G(x)$ have a zero when $x^2\simeq3m/2M$, i.e., $\frac{1}{2}Mv^2=\frac{3}{4}mw_t^2$; according to our definition of w_t, this means that there is no average energy loss or gain when $\frac{1}{2}Mv^2=\frac{3}{2}kT$, where T is the temperature of the plasma particles. This is as it should be, for at this point the test particle is in kinetic thermal equilibrium. For $x=v/w_t>(3m/2M)^{\frac{1}{2}}$, $F(x)$ is positive and the test particle loses energy to the plasma; for $x<(3m/2M)^{\frac{1}{2}}$, $F(x)$ is negative.

4. RELATIVE LOSS RATES TO ELECTRONS AND IONS

We now consider the relative energy loss rates to electrons and ions of a plasma. We denote the positive ion properties by subscripts i, and the electrons by subscripts e, and write the particle thermal speeds as w_i and w_e, respectively. Equation (9) now becomes

$$\frac{dv^3}{dt}=-\frac{24\sqrt{\pi}}{M}(Ze^2)^2\times\ln\Lambda\left(\frac{z_i^2\rho_i}{m_i}G(v/w_i)+\frac{\rho_e}{m_e}G(v/w_e)\right). \quad (13)$$

[5] As is well known, energy losses of a fast ion occur predominantly as the additive effect of large numbers of small-angle collisions each with small energy loss.

[6] It should also be possible to perform the integrations analytically for a wide class of distribution functions $f(\mathbf{w})$. For the above definition of w_t we have $\frac{1}{2}mw_t^2=kT$.

The relative magnitudes of the two terms on the right determine the relative loss rates to ions and electrons. In the high-energy region when both $v/w_i \gg 1$, and $v/w_e \gg 1$, we see that the loss rate to electrons is larger than to ions by the factor $(m_i/m_e)(\rho_e/z_i^2\rho_i)$.

It is of interest to determine the speed u of the test ion for which it is losing energy equally to ions and electrons. This is given by the solution of the equation

$$\frac{z_i^2\rho_i}{m_i}G(u/w_i)=\frac{\rho_e}{m_e}G(u/w_e).$$

The solution is in general such that u/w_e is small, but that u/w_i is still several times greater than unity. Thus we may use the forms (11) and (12) and find that

$$u^3 \simeq w_e^3\left(\frac{m_e}{m_i}\right)\left(\frac{3\sqrt{\pi}}{4}z_i^2\frac{\rho_i}{\rho_e}\right). \tag{14}$$

If the plasma electron and ion temperatures are the same, this is equivalent to

$$u^3 \simeq w_i^3\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}\left(\frac{3\sqrt{\pi}}{4}z_i^2\frac{\rho_i}{\rho_e}\right), \tag{15}$$

although Eq. (14) is independent of this assumption.

In the case of deuterium with $\rho_i=\rho_e$, $z_i=1$, and $m_i/m_e \simeq 3700$, we find $u \simeq w_e/14$; if the ion and electrons temperatures are the same this implies $u \simeq 4w_i$.

It is also perhaps of interest to note the relative rates at which a 3.2-Mev proton (say from a fusion reaction) gives its energy to electrons and ions of a deuterium plasma. If the temperature of the plasma is, say, 10^8 °C we have $v/w_i \simeq 24$, and $v/w_e \simeq 0.4$. The ratio of energy loss rate to electrons as compared to ions is approximately $(1/12\sqrt{\pi})m_i/m_e \simeq 170$. Thus only a small fraction of the energy of such protons is converted directly to plasma ion energies.

5. TIME-DEPENDENCE OF ENERGY LOSS

In the wide region $v/w_i \gg 1$ and $v/w_e \lesssim 1$, for which Eqs. (11) and (12) may be employed, the integration of Eq. (13) is trivial. If the test particle has speed V at time $t=0$, we obtain the result

$$[v(t)]^3=V^3-(V^3+u^3)(1-e^{-t/\tau_e}), \tag{16}$$

where

$$\frac{1}{\tau_e}=\left(\frac{(16\sqrt{\pi})Z^2e^4\rho\ \ln\Lambda}{Mm_e}\right)\frac{1}{w_e^3}, \tag{17}$$

and where we have put $\rho_i=\rho_e=\rho$.

There are two simple limits of Eq. (16):

(a) $V^3 \gg u^3$, $v^3 \gg u^3$:

$$v \simeq Ve^{-t/3\tau_e}. \tag{18}$$

In this region the loss is occurring predominantly to electrons.

(b) $V^3 \ll u^3$:

$$v \simeq V\left\{1-\left[1+\left(\frac{u}{V}\right)^3\right]\frac{t}{\tau_e}+\cdots\right\}^{\frac{1}{3}}$$

$$=V\left(1-\frac{t}{\tau_i}-\cdots\right)^{\frac{1}{3}},$$

with

$$\tau_i=\frac{\tau_e}{1+(u/V)^3} \simeq \left(\frac{V}{u}\right)^3\tau_e. \tag{19}$$

In this region the loss is occurring predominantly to ions. For the particular case in which the injected and plasma ions are deuterons, we have

$$\tau_i=\frac{1.6\times 10^{11}}{\ln\Lambda}\times\frac{E_0^{\frac{3}{2}}}{\rho},$$

where E_0 is the injection energy in kev. If we take the usual value $\ln\Lambda \simeq 20$, we have

$$\tau_i=8\times 10^9 \times E_0^{\frac{3}{2}}/\rho.$$

It should be noted however that it is only in the region $v<u$ that energy loss occurs predominantly to ions, and for which the above time τ_i is applicable.

6. DIFFUSION AROUND MEAN

The equation for the energy loss rate—Eq. (7)—really refers to the average energy of the test particle at time t, say $\bar{E}(t)$. Similarly, Eq. (16), for example, refers to the average speed of the particle $\bar{v}(t)$ at time t. Clearly, for completeness we should investigate the diffusion of energies around the average.

A true distribution function $g(E,t)$ would give as its average for E at time t the value $\bar{E}(t)$ determined by Eq. (7). To investigate the diffusion, that is, the spread of the distribution about the mean, we must calculate the difference $\langle(\Delta E)^2\rangle_{\text{av}}-(\langle\Delta E\rangle_{\text{av}})^2$ for each collision. From Eqs. (1) and (2) it can readily be found that for a collision between a test ion of velocity $\mathbf{v}$ and a field particle of velocity $\mathbf{w}$ we have

$$\begin{aligned}&[\langle(\Delta E)^2\rangle_{\text{av}}-(\langle\Delta E\rangle_{\text{av}})^2](\mathbf{w})\\&=\frac{\pi}{\sigma_t}\left(\frac{Zze^2}{\frac{1}{2}[Mm/(M+m)](\mathbf{v}-\mathbf{w})^2}\right)^2\left(\frac{mM}{(M+m)^2}\right)^2\\&\quad\times\{[(\mathbf{v}-\mathbf{w})(M\mathbf{v}+m\mathbf{w})]^2[1-(\theta_0\ln\Lambda)^2]\\&\qquad+[(\mathbf{v}-\mathbf{w})\times(M\mathbf{v}+m\mathbf{w})]^2(\ln\Lambda-\tfrac{1}{2})\}.\end{aligned} \tag{20}$$

In a short time δt, the number of collisions with field particles of velocity between $\mathbf{w}$ and $\mathbf{w}+d\mathbf{w}$ is $\sigma_t|\mathbf{v}-\mathbf{w}|f(\mathbf{w})d\mathbf{w}\delta t$, so that in time δt the total mean square deviation is

$$\delta t\int d\mathbf{w}\ f(\mathbf{w})\,|\mathbf{v}-\mathbf{w}|\,\sigma_t[\langle(\Delta E)^2\rangle_{\text{av}}-(\langle\Delta E\rangle_{\text{av}})^2](\mathbf{w}). \tag{21}$$

If we substitute (20) into (21), and use the Maxwellian distribution (6) for $f(\mathbf{w})$, we find once again that all integrals are elementary, though tedious. By far the largest contribution comes from the last term of (20), as it is the dominant term proportional to $\ln\Lambda$. [The number $(\theta_0 \ln\Lambda)^2$ is extremely small.] Although the complete result can be written down, the contribution from the last term above gives the mean square deviation accurate to better than 5% for $\ln\Lambda \simeq 20$, and we therefore use this result.

After time δt this mean square deviation is found to be

$$\delta t[(8\sqrt{\pi})(Zze^2)^2 \rho w_t F_1(v/w_t) \ln\Lambda], \tag{22}$$

where

$$F_1(x) = \frac{1}{x}\int_0^x \exp(-x^2)dx - \exp(-x^2). \tag{23}$$

It is to be noticed that unless the test particle is very close to being thermal, the function $F_1(x)$ is essentially identical to the function $F(x)$ of Eq. (8). It is thus immediately clear from Eq. (7) that (22) may be written in terms of $d\bar{E}/dt$, which is the rate of change of the mean energy. We have simply

$$\text{Mean square deviation in } \delta t = -mw_t^2(d\bar{E}/dt)\delta t. \tag{24}$$

But mw_t^2 is a constant; it is the thermal energy of the plasma field particles. Hence we can simply integrate to find the total mean square deviation in energy after time t—say, $\epsilon^2(t)$. We have

$$\epsilon^2(t) = -mw_t^2 \int_0^t \left(\frac{d\bar{E}}{dt}\right) dt = mw_t^2(E_0 - \bar{E}), \tag{25}$$

where E_0 is the injection energy.

It is to be observed that $\epsilon^2(t)$ is independent of the nature of the field particles, but is dependent simply on their temperature; thus whether the test particle loses energy predominantly to ions or electrons is irrelevant in the determination of the diffusion of energies around the mean. If electrons and ions are at the same temperature T such that $\frac{1}{2}m_i w_i^2 = \frac{1}{2}m_e w_e^2 = kT$, the diffusion spread is

$$\epsilon^2(t) = 2kT(E_0 - \bar{E}).$$

In general, therefore, the energy E of the test particle will be given by $\bar{E}$ to within $\pm\epsilon$, i.e.

$$\begin{aligned} E &= \bar{E} \pm \epsilon \\ &= \bar{E} \pm [mw_t^2(E_0 - E)]^{\frac{1}{2}}. \end{aligned}$$

The diffusion spread in energy will become of the same order as E itself only when

$$\bar{E} \sim (mw_t^2 E_0)^{\frac{1}{2}}.$$

If, for example, test particles are injected with an energy 200 times greater than the plasma thermal energy, they will lose 90% of their energy, and be only 20 times thermal, before the diffusion spread becomes of the same order as the average remaining energy E.

Except for energies close to thermal, the distribution function $g(E,t)$ for the test particle may thus be considered to have the approximate form

$$\begin{aligned} g(E,t) &= C \exp\left(-\frac{[E - \bar{E}(t)]^2}{2\epsilon^2(t)}\right) \\ &= C \exp\left(-\frac{[E - \bar{E}(t)]^2}{2mw_t^2[E_0 - \bar{E}(t)]}\right). \end{aligned} \tag{26}$$

The normalization constant C is a function of t only, chosen such that for any t the integral over all E is unity.

7. ČERENKOV RADIATION

In the preceding sections we have confined our attention to the test particle itself, and its rate of loss of energy. It is also possible, of course, to investigate the manner in which the energy is taken up by the plasma. We shall not do this in detail here, except to remark that for $v > w_t$ the test particle will leave a wake or Mach cone whose opening angle θ is given in the usual way by $\sin(\theta/2) \simeq v/w_t$. One is thus led to ask whether the binary collision description which we have used in this paper includes what is commonly referred to as the "Čerenkov radiation."[7]

This question has been investigated by Salpeter,[8] who has shown that the binary collision treatment does accurately include the "Čerenkov radiation" effects, provided that for $v > w_t$ the screening distance, or maximum collision impact parameter, be taken as $\lambda_D v/w_t$, where λ_D is the Debye length. Impact parameters less than λ_D produce what are normally considered the binary collisions, and the range of impact parameters from λ_D to $\lambda_D v/w_t$ produces the "Čerenkov radiation."

For $v < w_t$, the maximum impact parameter is the usual Debye length λ_D. Thus for $v > w_t$ the value of $\ln\Lambda$ is increased by the (in general) small term $\ln(v/w_t)$.

At this point it is appropriate to mention also that, in the average energy loss per collision, the second term of Eq. (2) has zero contribution only when the cross section is ϕ independent.[9] It has been shown by Salpeter[8] that when the screening effects are accurately considered this term does make a small contribution to the rate of energy loss, but that it is at most of order unity compared to $\ln\Lambda$.

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[7] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).
[8] E. E. Salpeter (private communication).
[9] This term yields the dominant contribution in the diffusion of energies around the mean (see Sec. 6).