Production of Pion Pairs by a Photon in the Coulomb Field of a Nucleus*

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We have studied the possibility of measuring the π-π interaction through electromagnetic pion-pair production in the Coulomb field of a nucleus. The Mandelstam representation was set up for the process γ(virtual) $+\gamma \rightarrow \pi+\pi$, and partial-wave dispersion relations were derived. Solutions for the ensuing integral equations were obtained within the approximation of retaining only the one-pion exchange contribution to the unphysical cut. It is sufficient to consider only the π-π S-wave interaction, for which a current estimate with a virtual $I=0$ state near threshold is utilized. Our results indicate that the effects of the π-π final-state interaction on the Coulomb process may be observed in the presence of nuclear pair production, provided that the momentum transfers to the nucleus are sufficiently small, say smaller than 70 Mev/c in the illustrative case of an incident photon of 6 Bev.

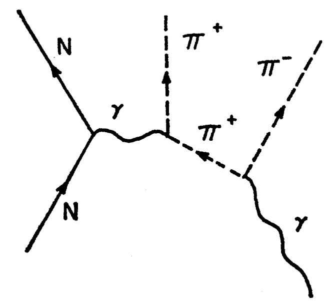
I. INTRODUCTION

It is well known that electron pair production in the Coulomb field of a nucleus involves small momentum transfers by high-energy photons, corresponding to a large classical impact parameter. The same situation exists for pion pair production by photons of sufficiently high energy, with the consequence that most of the pion pairs may be produced outside the nuclear radius. Thus, the effect of pion-pion final-state interactions may be observed without involving pion-nucleon interactions. This is the motivation for our studying the electromagnetic production of pion pairs.

It turns out that only even angular-momentum states occur in the center-of-momentum (c.m.) system of the two pions, so there can be no direct effect from the proposed π-π resonance in the P state.[1] We may, however, hope to observe S-wave interaction effects. Since the final state involves strong interactions, we propose to employ the double dispersion relation, or Mandelstam representation,[2] which has been applied to many processes with strongly interacting particles.[3–8] Here the process is $\gamma+\gamma \rightarrow \pi+\pi$, which has already been considered by Gourdin and Martin[9] and by Desai[10] for the case in which both photons are real. Our problem concerns one real and one virtual photon, the latter arising from the Coulomb field.

The double dispersion relation gives a general prescription for analytic continuation of scattering amplitudes into the complex plane as a function of both energy and momentum transfer variables. For this purpose we need Lorentz invariant amplitudes, satisfying all the known symmetry principles. Such amplitudes can be constructed by standard techniques. In Sec. II we write down and discuss the Mandelstam representation and we derive one-dimensional dispersion relations for the partial-wave helicity amplitude. In Sec. III we calculate the cross section for electromagnetic pion pair production in a high-energy approximation and compare it with nuclear production.

Without the complications due to strong interactions, the electromagnetic interaction will produce only charged pion pairs with the well-known Pauli-Weisskopf cross section,[11] which results from the diagram shown in Fig. 1. Even though electromagnetic pion pair production is smaller than nuclear pair production by a factor of e^4, the high power of momentum transfer in the denominator of the cross section, arising from the virtual



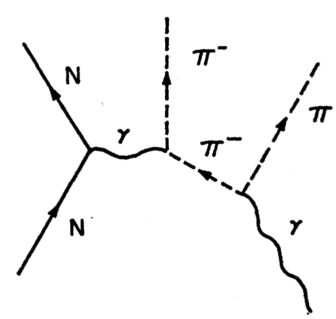


FIG. 1. The lowest order diagram for electromagnetic pion-pair production without π-π interaction.

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[1] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959).
[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1959); **115**, 1741 and 1752 (1959).
[3] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[4] H. Wong, Phys. Rev. Letters **5**, 70 (1960; Phys. Rev. **121**, 289 (1961).
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[6] J. S. Ball, Lawrence Radiation Laboratory Report UCRL-9172, 1960 (unpublished); Phys. Rev. Letters **5**, 73 (1960).
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[10] B. Desai, Phys. Rev. **124**, 1248 (1961).
[11] W. Pauli and V. Weisskopf, Helv. Phys. Acta **7**, 709 (1934).

photon in Fig. 1, enhances the former production at high energy, making it comparable to the latter for sufficiently small momentum transfer. The electromagnetic pion pair production can be further enhanced by a factor of Z^2 if a nucleus of high atomic number is used as the source of electromagnetic field, while the competing nuclear cross section should go approximately as $A^{\frac{1}{3}}$, since only surface nucleons are effective. All these considerations will be combined in Sec. III to allow an estimate of the experimental conditions needed for unambiguous measurement.

II. ONE-DIMENSIONAL DISPERSION RELATIONS FOR PARTIAL-WAVE HELICITY AMPLITUDES

In the study of pion pair production by the dispersion relation method, we need to know the most general form of the physical amplitude that satisfies all the symmetry principles known to be obeyed by strong and electromagnetic interactions. This amplitude turns out to be

$$\begin{aligned}T^{\mu\nu}=&A(K_1^{\mu}K_2^{\nu}-K_1\cdot K_2 g^{\mu\nu})\\&+B\left(K_1^{\mu}P^{\nu}+\frac{K_1\cdot P}{K_2\cdot P}P^{\mu}K_2^{\nu}-\frac{K_1\cdot K_2}{K_2\cdot P}P^{\mu}P^{\nu}-K_1\cdot P\text{—}g^{\mu\nu}\right)\\&+C\left(\frac{1}{K_1\cdot K_2+\lambda^2}\right)\bigg(K_1\cdot K_2K_2^{\mu}P^{\nu}-\lambda^2P^{\mu}K_2^{\nu}\\&\qquad-P\cdot K_1K_2^{\mu}K_2^{\nu}-\lambda^2\frac{K_1\cdot K_2}{K_2\cdot P}P^{\mu}P^{\nu}\bigg).\end{aligned}\tag{2.1}$$

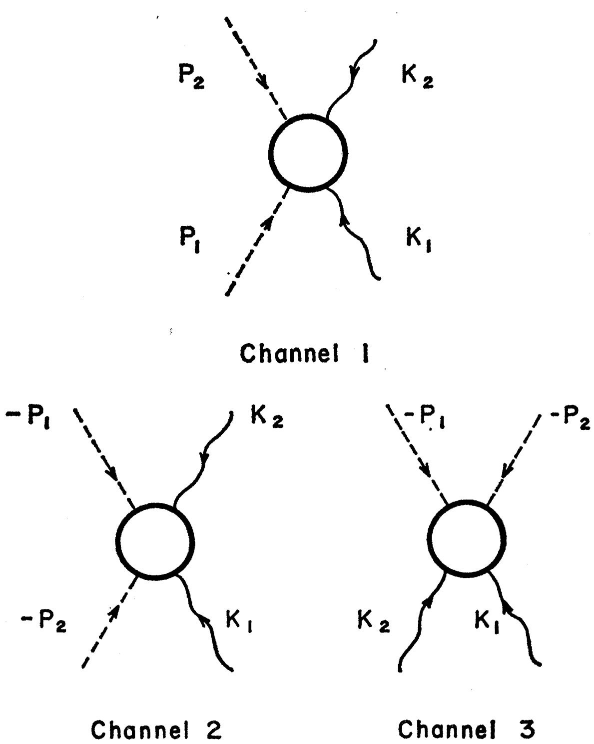

FIG. 2. The three channels of $\gamma+\gamma$ (virtual) $\rightarrow 2\pi$ problem.

Here P_2, P_1 and K_2, K_1 are the four-momenta of the two pions and the real and virtual photons, respectively. $P\equiv\frac{1}{2}(P_2-P_1)$, $K_2^2=\lambda^2$,[12] and the coefficient functions A, B, and C are the Lorentz invariants depending on the scalar variables that can be formed from the momentum vectors, such as

$$s\equiv-(K_1-P_1)^2,\quad t\equiv-(K_1+K_2)^2,\quad\text{and}\quad \bar{s}\equiv-(K_1-P_2)^2,$$

which are the squares of the total energies in the barycentric system of each channel shown in Fig. 2. The quantity $g^{\mu\nu}$ is the metric tensor.

In dispersion theory, the helicity amplitudes introduced by Jacob and Wick[13] are often employed because of their simple analyticity properties. We shall adopt this approach and find the relations between the helicity amplitudes $f_{\lambda\lambda'}$ and A, B, and C as defined in formula (2.1):

$$\begin{aligned}f_{++}=f_{--}=&\frac{e^{i\phi}}{8\pi}\left(\frac{1}{t+\lambda^2}\right)^{\frac{1}{2}}\left(\frac{t-4}{t}\right)^{\frac{1}{4}}\\&\times\left(\tfrac{1}{2}(t+\lambda^2)A(s,\bar{s},t)-\frac{\lambda^2}{4(t+\lambda^2)}(s-\bar{s})B(s,\bar{s},t)\right.\\&\qquad\left.-\frac{(t+\lambda^2)(t-4)}{4}\frac{1}{(s-\bar{s})}B(s,\bar{s},t)\right);\\f_{+-}=f_{-+}=&\frac{e^{i\phi}}{8\pi}\left(\frac{1}{t+\lambda^2}\right)^{\frac{1}{2}}\left(\frac{t-4}{t}\right)^{\frac{1}{4}}\\&\times\left(-\frac{t(s-\bar{s})}{4(t+\lambda^2)}+\frac{(t+\lambda^2)(t-4)}{4(s-\bar{s})}\right)B(s,\bar{s},t);\\f_{+0}=-f_{-0}=&\frac{e^{i\phi}}{8\pi}\left(\frac{1}{t+\lambda^2}\right)^{\frac{1}{2}}\left(\frac{t-4}{t}\right)^{\frac{1}{4}}\\&\times\left(\frac{\lambda^2(t/2)^{\frac{1}{2}}[(t-4)(t+\lambda^2)^2-t(s-\bar{s})^2]^{\frac{1}{2}}}{(t-\lambda^2)(t+\lambda^2)}\right)\\&\times\left[B(s,\bar{s},t)+\left(\frac{t+\lambda^2}{t-\lambda^2}\right)C(s,\bar{s},t)\right].\end{aligned}\tag{2.2}$$

Here, ϕ is an arbitrary phase that can be chosen to be zero in our problem. The subscripts $+$, $-$, and 0 denote right- and left-circular and longitudinal polarizations, respectively. It may be noted that the helicity amplitudes f_{+0} and f_{-0}, involving the longitudinal polarization vanish, when the mass of virtual photon ap-

[12] The mass of a pion is set equal to 1.

[13] M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1958).

proaches zero. In this case, there exist only two amplitudes, the result known to be correct for two real photons.

In accordance with the general method given in the paper of Jacob and Wick,[13] we can expand the helicity amplitudes $f_{\lambda\lambda'}$ in terms of eigenamplitudes $T_{\lambda\lambda'}{}^J(t)$, where J is the total angular momentum of the two pions in their barycentric system:

$$f_{++}=f_{--}=\frac{1}{k}\sum_{J=0}^{\infty}(J+\tfrac{1}{2})T_{++}{}^J(t)d_{00}{}^J(\cos\theta),$$

$$f_{+-}=f_{-+}=\frac{1}{k}\sum_{J=2}^{\infty}(J+\tfrac{1}{2})T_{+-}{}^J(t)d_{20}{}^J(\cos\theta),\tag{2.3}$$

and

$$f_{+0}=-f_{-0}=\frac{1}{k}\sum_{J=0}^{\infty}(J+\tfrac{1}{2})T_{+0}{}^J(t)d_{10}{}^J(\cos\theta).$$

Here the angular functions $d_{m\mu}{}^J(\cos\theta)$ are defined by

$$\begin{aligned}
d_{00}{}^J(\cos\theta)&=P_J(\cos\theta);\\
d_{10}{}^J(\cos\theta)&=-d_{-10}{}^J(\cos\theta)\\
&=-[J(J+1)]^{-\frac{1}{2}}\sin\theta P_J{}'(\cos\theta);\\
d_{20}{}^J(\cos\theta)&=[(J-1)J(J+1)(J+2)]^{-\frac{1}{2}}\\
&\quad\times[2P_{J-1}{}'(\cos\theta-J(J-1)P_J(\cos\theta)];
\end{aligned}\tag{2.4}$$

where $P_J(\cos\theta)$ is the usual Legendre polynomial of order J, and the prime stands for the derivative with respect to $\cos\theta$. The orthogonality of the functions $d^J(\cos\theta)$ allows one to express $T_{\lambda\lambda'}{}^J(t)$ in terms of the invariant physical amplitudes A, B, and C.

Now we assume the Mandelstam representation for the invariant functions A, B, and C such that

$$\begin{aligned}
A(s,\bar{s},t)=e^2F_\pi(\lambda^2)\left(\frac{1}{s-1}+\frac{1}{\bar{s}-1}\right)&+\frac{1}{\pi}\int_4^\infty\frac{\rho_{a1}(s')}{s'-s}ds'\\
&+\frac{1}{\pi}\int_4^\infty\frac{\rho_{a2}(\bar{s}')}{\bar{s}'-\bar{s}}d\bar{s}'+\frac{1}{\pi}\int_4^\infty\frac{\rho_{a3}(t')}{t'-t}dt'\\
&+\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty d\bar{s}'\frac{a_{12}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}\\
&+\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty dt'\frac{a_{13}(s',t')}{(s'-s)(t'-t)}\\
&+\frac{1}{\pi^2}\int_4^\infty d\bar{s}'\int_4^\infty dt'\frac{a_{23}(\bar{s},t')}{(\bar{s}'-\bar{s})(t'-t)};
\end{aligned}\tag{2.5a}$$

$$\begin{aligned}
B(s,\bar{s},t)=2e^2F_\pi(\lambda^2)&\left(\frac{1}{\bar{s}-1}-\frac{1}{s-1}\right)\\
&-\frac{1}{\pi}\int_4^\infty\frac{\rho_{b1}(s')}{s'-s}ds'+\frac{1}{\pi}\int_4^\infty\frac{\rho_{b2}(\bar{s}')}{\bar{s}'-\bar{s}}d\bar{s}'\\
&+\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty d\bar{s}'\frac{b_{12}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}\\
&+\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty dt'\frac{b_{13}(s',t')}{(s'-s)(t'-t)}\\
&+\frac{1}{\pi^2}\int_4^\infty d\bar{s}'\int_4^\infty dt'\frac{b_{23}(\bar{s}',t')}{(\bar{s}'-\bar{s})(t'-t)},
\end{aligned}\tag{2.5b}$$

and

$$\begin{aligned}
C(s,\bar{s},t)=&-\frac{1}{\pi}\int_4^\infty\frac{\rho_{c1}(s')}{s'-s}ds'+\frac{1}{\pi}\int_4^\infty\frac{\rho_{c2}(\bar{s}')}{\bar{s}'-\bar{s}}d\bar{s}'\\
&+\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty d\bar{s}'\frac{c_{12}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}\\
&+\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty dt'\frac{c_{13}(s',t')}{(s'-s)\ (t'-t)}\\
&+\frac{1}{\pi^2}\int_4^\infty d\bar{s}'\int_4^\infty dt'\frac{c_{23}(\bar{s}',t)}{(\bar{s}'-\bar{s})(t'-t)},
\end{aligned}\tag{2.5c}$$

where e is the rationalized electronic charge such that $e^2/4\pi=1/137$ and $F_\pi(\lambda^2)$ is the pion form factor normalized to one at $\lambda^2=0$.[14]

From the double dispersion relations (2.5) and the formulas (2.2) and (2.3), one can derive one-dimensional dispersion relations for the partial-wave helicity amplitudes $T_{\lambda\lambda}{}^J(t)$. We shall be concerned mainly with the S-wave part and consider a modified function

$$\mathcal{T}_{++}{}^0(t)=16\pi\left(\frac{t}{t-4}\right)^{\frac{1}{4}}[t(t+\lambda^2)]^{\frac{1}{2}}T_{++}{}^0(t),$$

which has simpler analytic properties. We can show that the singularities consist of a series of branch cuts arising from the vanishing of the denominators in the Mandelstam representation and all singularities lie on the real axis. On the physical cut one branch runs from 4, the lowest intermediate state, to ∞, and the next from 16, etc., each associated with possible intermediate states in the channel $\gamma+\gamma\rightarrow\pi+\pi$. On the unphysical cut we have branch cuts from 0 to t_1, and from t_2 to

[14] A detailed derivation of the above formulas can be found in Y. Kim, Lawrence Radiation Laboratory Report UCRL 9734 (unpublished).

$-\infty$, where

$$t_1=\frac{-[s_0^2-2s_0-1+\lambda^2(s_0+1)]+\{[(s_0-1)^2+\lambda^2(s_0+1)]^2-4\lambda^4 s_0\}^{\frac{1}{2}}}{2s_0},$$

and

$$t_2=\frac{-[s_0^2-2s_0-1+\lambda^2(s_0+1)]-\{[(s_0-1)^2+\lambda^2(s_0+1)]^2-4\lambda^4 s_0\}^{\frac{1}{2}}}{2s_0}, \tag{2.6}$$

if s_0 is a threshold value of s for physical intermediate states in channels 1 and 2 (4, 9 etc.). Therefore, we may write down the one-dimensional dispersion relation for the partial-wave helicity amplitudes as follows

$$\mathcal{T}_{\lambda\lambda'}{}^0(t)=\mathcal{T}_{\lambda\lambda'}{}^0(t)_{1\pi}+\frac{1}{\pi}\int_{-\infty}^{t_2}+\int_{t_1}^{0}\frac{\operatorname{Im}\mathcal{T}_{\lambda\lambda'}{}^0(t')}{t'-t}dt' +\frac{1}{\pi}\int_4^\infty\frac{\operatorname{Im}\mathcal{T}_{\lambda\lambda'}{}^0(t')}{t'-t}dt', \tag{2.7}$$

where $\mathcal{T}_{\lambda\lambda'}{}^0(t)_{1\pi}$ is the S-wave projection of the pole terms. Now we need to know the imaginary part of the amplitudes on the physical as well as the unphysical cut. On the former we use the unitarity conditions:

$$2\operatorname{Im}\langle\lambda_2\lambda_2'\Omega|T|\lambda_1\lambda_1'0\rangle =\sum_J\left(\frac{2J+1}{4\pi}\right)e^{-i(\mu_2-\mu_1)\phi}d_{\mu_1\mu_2}{}^J(\theta) \times\langle\lambda_2\lambda'_2|T^J|0\rangle\langle 0|T^{J+}|\lambda_1\lambda'_1\rangle, \tag{2.8}$$

where $\mu_1=\lambda_1-\lambda'_1$ and $\mu_2=\lambda_2-\lambda'_2$. If we retain only the two-pion state as usual, we have

$$2\operatorname{Im}\mathcal{T}_{\lambda\lambda'}{}^{J,I}(t)=\mathcal{T}_{\lambda\lambda'}{}^{J,I}(t)T_{\pi\pi}{}^{J,I}(t)^*, \tag{2.9}$$

where I stands for the isotropic spin of the two pions and the charged and neutral pion-production amplitudes are to be represented with the proper combination of the above amplitudes. Here the asterisk denotes complex conjugate, and

$$T_{\pi\pi}{}^{J,I}(t)\,(\text{equaling})\,(1/i)[S_{\pi\pi}{}^{J,I}(t)-1]$$

stands for the π-π scattering amplitude in the state of angular momentum J and isotopic spin I.

The discontinuity of $\mathcal{T}_{\lambda\lambda'}{}^J$ on the unphysical cut ($-\infty<t\leqslant 0$), is given by projection from formula (2.2) in terms of the absorptive parts for channels 1 and 2, analytically continued to unphysical angles. These absorptive parts are a sum of contribution from 1π, 2π, 3π, etc. intermediate states, but the 2π term is very small compared to the 1π term, as shown by Desai in the case of two real photons,[10] and at present we do not know how to estimate the 3π and higher terms. Therefore, we make the approximation of retaining only the 1π term on the unphysical cut. With this approximation, we obtain

$$\mathcal{T}_{\lambda\lambda'}{}^{J,I}(t)=\mathcal{T}_{\lambda\lambda'}{}^{J,I}(t)_{1\pi}+\frac{1}{\pi}\int_4^\infty\frac{dt'}{t'-t} \times(\tfrac{1}{2}\mathcal{T}_{\lambda\lambda'}{}^{J,I}(t)T_{\pi\pi}{}^{J,I}(t)^*). \tag{2.10}$$

This integral equation was solved by Omnes[15]; we shall use a variation of his solution based on the N/D representation of the π-π scattering amplitude[3,16]

$$\mathcal{T}_{\lambda\lambda'}{}^{J,I}(t)=\mathcal{T}_{\lambda\lambda'}{}^{J,I}(t)_{1\pi}+\frac{1}{\pi}\frac{1}{D_J{}^I(t)}\int_4^\infty\frac{dt'}{t'-t}\left(\frac{t'-4}{t'}\right)^{\frac{1}{2}} \times\mathcal{T}_{\lambda\lambda'}{}^{J,I}(t')_{1\pi}N_J{}^I(t'), \tag{2.11}$$

where

$$\tfrac{1}{2}T_{\pi\pi}{}^{J,I}=\left(\frac{t-4}{t}\right)^{\frac{1}{2}}\frac{N_J{}^I(t)}{D_J{}^I(t)}=\exp[i\delta_J{}^I(t)]\sin\delta_J{}^I(t),$$

and

$$D_J{}^I(t)=\exp\left(-\frac{(t-t_0)}{\pi}\int_4^\infty dt'\frac{\delta_J{}^I(t')}{(t'-t)(t'-t_0)}\right).$$

The second term on the right-hand side of (2.11) is sometimes referred to as "rescattering." Our solutions for $\mathcal{T}_{++}{}^0$ and $\mathcal{T}_{--}{}^0$ are similar to the ones obtained by Desai for the case of two real photons.[10]

III. EVALUATION OF THE CROSS SECTION FOR PION-PAIR PRODUCTION IN THE COULOMB FIELD

A. General Cross-Section Formula for Pion-Pair Production

In practical application of the amplitudes studied in the preceding sections, we shall consider pion-pair production at high energies where the momentum transfer to the nucleus can be made small. For this purpose we require first a general cross-section formula for production in a fixed Coulomb field in terms of the partial helicity amplitudes. Prior to writing down the general formula we make three observations: (a) The amplitude in a fixed Coulomb field may be obtained from the general invariant amplitude Eq. (2.1) by retaining only the time-like component of the polarization; (b) with the approximation of retaining only the 1π contribution to the unphysical cut, the invariant amplitude C vanishes; and (c) most of the π-π re-

[15] R. Omnes, Nuovo cimento **8**, 316 (1958).

[16] G. F. Chew and S. Mandelstam, Nuovo cimento **19**, 752 (1961).

scattering occurs in the S wave, because the high power of the momentum transfer in the denominator of the cross-section formula Eq. (3.1a) favors low values of t. A simple calculation shows that $\lambda_{\min}=t/2k$, and for large k, we estimate an average $t\sim 10\mu^2$. If we remember that only even angular-momentum states appear in the π-π rescattering, it seems safe to retain only the S-wave part in our formula.

With the above considerations, we are ready to write down the cross section. For charged-pion pairs, we have

$$\frac{d^3\sigma}{d\epsilon_1 d\Omega_1 d\Omega_2}=\frac{Z^2\alpha^3 p_1 p_2}{8\pi^2 k^3\lambda^4}F_N(\lambda^2)^2 F_\pi(\lambda^2)^2 \times\{T_{1\pi}{}^2+2T_{1\pi}\,\mathrm{Re}T'+|T'|^2\}, \quad (3.1a)$$

where

$$T_{1\pi}{}^2=4\left\{\left(\frac{\epsilon_2 p_1\sin\theta_1}{\epsilon_1-p_1\cos\theta_1}\right)^2+\left(\frac{\epsilon_1 p_2\sin\theta_2}{\epsilon_2-p_2\cos\theta_2}\right)^2 +\frac{2\epsilon_1\epsilon_2 p_1 p_2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\right\}, \quad (3.1b)$$

and

$$T_{1\pi}\,\mathrm{Re}T'=-\frac{k^2}{\sqrt{3}}\mathrm{Re}\left\{\mathcal{T}_{++}{}^{00}(t)'+\frac{1}{\sqrt{2}}\mathcal{T}_{++}{}^{0,2}(t)'\right\} \times\left\{\frac{\epsilon_2 p_1{}^2\sin^2\theta_1}{\epsilon_1-p_1\cos\theta_1}+\frac{\epsilon_1 p_2{}^2\sin^2\theta_2}{\epsilon_2-p_2\cos\theta_2} +p_1p_2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) \times\left(\frac{\epsilon_2}{\epsilon_1-p_1\cos\theta_1}+\frac{\epsilon_1}{\epsilon_2-p_2\cos\theta_2}\right)\right\}; \quad (3.1c)$$

$$|T(t)'|^2=(k^3/12)|\mathcal{T}_{++}{}^{0,0}(t)'+(1/\sqrt{2})\mathcal{T}_{++}{}^{0,2}(t)'|^2 \times\{p_1{}^2\sin^2\theta_1+p_2{}^2\sin^2\theta_2 +2p_1p_2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\}. \quad (3.1d)$$

The primed quantities denote the rescattering terms as given in Eq. (2.11) after the pion form factor is taken out:

$$\mathcal{T}_{++}{}^{0,(0,2)}(t)' =-(1,1/\sqrt{2})\frac{8}{(3\pi)^{\frac{1}{2}}}\frac{1}{D_0{}^{(0,2)}(t)_{\pi\pi}}\int_4^\infty\frac{dt'}{t'-t}\left\{\frac{4+\lambda^2}{(t'+\lambda^2)^2}\right\} \times\ln\left(\frac{\sqrt{t'}+\sqrt{t'-4}}{\sqrt{t'}-\sqrt{t'-4}}\right)N_0{}^{(0,2)}(t)_{\pi\pi}. \quad (3.2)$$

The cross section for neutral-pion production is also given by an expression of the form (3.1a), except that the terms $T_{1\pi}{}^2$ and $2T_{1\pi}\,\mathrm{Re}T'$ are absent and $|T'|^2$ has a factor $|\mathcal{T}_{++}{}^{0,0}(t)'-\sqrt{2}\mathcal{T}_{++}{}^{0,2}(t)'|^2$. In formula (3.1) angles are measured with respect to the direction of the incident photon; θ_1 and θ_2 are the pion zenith angles, and ϕ_1 and ϕ_2 are the azimuthal angles. The nuclear charge is Z. The square of the momentum transfer to the nucleus is given by

$$\lambda^2=(\mathbf{K}-\mathbf{p}_1-\mathbf{p}_2)^2=k^2+p_1{}^2+p_2{}^2-2kp_1\cos\theta_1 -2Kp_2\cos\theta_2+2p_1p_2\{\cos\theta_1\cos\theta_2 +\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2\},$$

when the energy transfer is negligible.

The charge structure of the nucleus is taken into account through the square of the form factor $F_N(\lambda^2)$ as determined by electron-nucleus elastic scattering[17] to the other coefficients outside the bracket in Eq. (3.1a). It should be noted this nuclear form factor is generally more important than the pion form factor $F_\pi(\lambda^2)$ which, according to the Frazer-Fulco analysis,[1] is roughly the same as the charge-structure factor of a proton. The most recent results from Stanford[17] indicate that $F_\pi(\lambda^2)\approx 1/[1+(\lambda^2/20)\mu^2]$.

Also we remark that Coulomb corrections[18] should be made if $2\pi(Z\alpha/v_1)$, $2\pi(Z\alpha/v_2)\gg 1$. But even for heavy nuclei these corrections amount to less than 10% for all but the lowest energies.

It may be recalled that the original purpose of this investigation was to develop an approximate method for measuring the π-π S-wave amplitudes. We see now that this means a measurement of the two functions $\mathcal{T}_{++}{}^{0,I}(t)$. The remainder of this paper will be devoted to an estimate of the difficulty of such a measurement. In our estimates we shall employ the S amplitudes calculated by Desai[10] from the Chew-Mandelstam approach,[16] in terms of the π-π coupling constant λ and the position and width of the π-π p-wave resonance. If more reliable S amplitudes become available, they should of course be used.

B. Evaluation of the Cross Section

To be able to measure electromagnetic pion-pair production in competition with nuclear processes, we are forced to use incident photons of high energy to take advantage of the high peak in the cross section at forward angles where the momentum transfer is small. The minimum value of momentum transfer is $K/(2\epsilon_1\epsilon_2)$ or $t/2k$. As an illustration, we shall calculate the cross section (3.1) with $K=6$ Bev. The minimum momentum transfer in this case becomes approximately 0.05μ, and most of the pair production takes place with values of momentum transfer, or less than $\frac{1}{2}\mu$, as will be shown later. In this small range of λ^2, we may replace the form factors by some average $\langle F_N(\lambda^2)^2F_\pi(\lambda^2)^2\rangle_{\mathrm{av}}$, which will be close to 1 in our example. Then, when the energy of each pion is large compared to μ, the integration over the solid angle of one of the pions can be carried out in a

[17] R. Hofstadter and R. Herman, Phys. Rev. Letters **6**, 293 (1961).

[18] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954).

straightforward manner to give the result:

$$\frac{1}{\langle Z^2F_N{}^2F_\pi{}^2\rangle_{\mathrm{av}}}\frac{d^2\sigma_{1\pi}}{d\epsilon_1 d\Omega_1}$$
$$=\frac{\alpha^3}{8\pi^2}\frac{p_1p_2}{k^3}\int\int\frac{1}{\lambda^4}T_{1\pi}{}^2 d\cos\theta_2 d\phi$$
$$=\frac{\alpha^3}{\pi}\frac{p_1p_2}{k^3}\left\{\left(\frac{\sin^2\theta_1}{(\epsilon_1-p_1\cos\theta_1)^4}-\frac{1}{(\epsilon_1-p_1\cos\theta_1)^2}\right)\right.$$
$$+\left(\frac{-\frac{1}{2}\sin^2\theta_1}{(\epsilon_1-p_1\cos\theta_1)^4}+\frac{1}{(\epsilon_1-p_1\cos\theta_1)^2}\right)$$
$$\left.\times\ln\left(\frac{2\epsilon_1\epsilon_2}{k}\right)\right\}, \quad (3.3)$$

for the cross section without rescattering. This formula attains its maximum at an angle $\theta_1\approx 1/\epsilon_1$ and agrees with Drell's result[18] in this region. The angular integration over $d\Omega$ can be further carried out to obtain the Pauli-Weisskopf formula,[11] except for a factor of 2 already noted by Drell:

$$\frac{1}{\langle Z^2F_N{}^2F_\pi{}^2\rangle_{\mathrm{av}}}\frac{d\sigma_{1\pi}}{d\epsilon_1}=\frac{\alpha^3\epsilon_1\epsilon_2}{k^3}\frac{8}{3}\ln\left(\frac{\epsilon_1\epsilon_2}{2k}\right). \quad (3.4)$$

In our high-energy limit, the λ^2 dependence of the rescattering term (3.2) may be neglected in the calculation of Eq. (3.1), and we make use of the solutions obtained by Desai for pion-pair production by two real photons. The results of calculation for various momenta and small forward angles of a pion are given in Table I.

TABLE I. Results of calculation of Eq. (3.1) for $K=6$ Bev $=43\ \mu$.

p_1/μ (Bev/c)	θ_1 (deg)	I_1[a] 10^{-6} (mb/sr)	I_2[b] 10^{-6} (mb/sr)	I_3[c] 10^{-6} (mb/sr)
3	$\frac{1}{2}$	49.4	14.80	1.87
2.7		40.2	11.89	1.33
2.4		31.0	6.68	0.87
3	1	37.4	4.05	1.13
2.7		32.0	3.84	0.96
2.4		25.6	2.18	0.72
3	2	17.1	−0.92	0.40
2.7		16.1	−0.75	0.39
2.4		14.3	−0.74	0.35
3	5	2.2	−0.11	0.043
2.7		2.3	−0.18	0.045
2.4		2.4	−0.25	0.043

[a] $I_1=\frac{\alpha^3}{8\pi}\frac{p_1p_2}{k^3}\int\int\frac{1}{\lambda^4}T_{1\pi}{}^2 d\Omega_2\ (\mu^{-2}\ \mathrm{mb})$

[b] $I_2=\frac{\alpha^3}{8\pi^2}\frac{p_1p_2}{k^3}\int\int\frac{1}{\lambda^4}2T_{1\pi}\ \mathrm{Re}T' d\Omega_2$

[c] $I_3=\frac{\alpha^3}{8\pi^2}\frac{p_1p_2}{k^3}\int\int\frac{1}{\lambda^4}|T'|^2 d\Omega_2.$

[19] S. D. Drell, Phys. Rev. Letters **5**, 278 (1960).

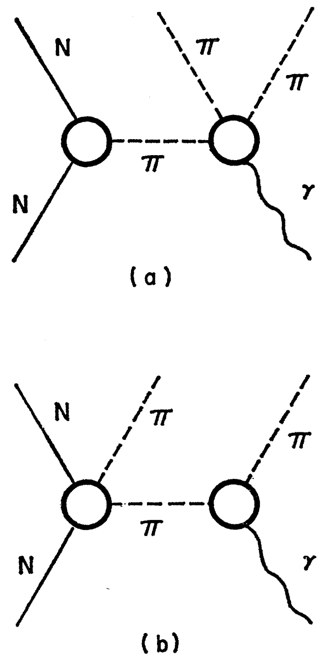

FIG. 3. One-pion exchange diagram for $\gamma+N\rightarrow 2\pi+N$.

The results in Table I show that the cross section becomes larger at smaller angles and at larger momentum of pions. As expected from simple kinematical considerations, small λ in the denominator of Eq. (3.1) favors pion production at forward angles and consequently smaller values of t on the average. Thus, the rescattering is also a maximum at the smallest angles. At larger angles θ_1 it may be noted that the rescattering term changes sign. Observe that rescattering in the most favorable regions increases the total cross section by more than 30%.

C. Nuclear Pion-Pair Production

Our attention is now turned to pion-pair production by the competing nuclear process. Since we are concerned with production in the narrow forward angular cone, the most likely production mechanism is that of a peripheral collision with exchange of a single pion. At small angles and high energies, as discussed by Drell,[19] the importance of this mechanism should be enhanced relative to others that correspond to an exchange of two or more pions. In this connection, we have to consider two types of one-pion exchange diagram—one involving the process $\gamma+\pi\rightarrow\pi+\pi$ at a vertex, and one involving the reaction $\pi+N\rightarrow\pi+N$—as shown in Fig. 3.

Using the extrapolation formula given by Chew and Low,[20] we can estimate the cross sections for the above mechanisms. The single-nucleon cross section for

[20] G. F. Chew and F. E. Low, Phys. Rev. **118**, 1640 (1959).

the mechanism shown in Fig. 3(a) is

$$\sigma=\frac{f^2}{2\pi}\int\int dpdt\frac{p[(p^2+m^2)^{\frac{1}{2}}-m](t-1)/2}{k^2(p^2+\mu)^2}\sigma_{\pi\gamma\to\pi\pi}(t). \quad (3.5)$$

If we use $\sigma_{\pi\gamma\to\pi\pi}$ as given by Wong,[4] a numerical calculation of the cross section (3.5) for various values of initial photon energy leads to the results shown in Table II. As is evident from the above comparison, this mechanism accounts for only about 1% of the observed cross section and so may be ignored.

Presumably, therefore, the Drell mechanism shown in Fig. 3(b) is the dominant nuclear process, and in fact this mechanism has been shown to account, at least qualitatively, for the experimental results obtained by Kilner *et al.*[21] at a photon energy of 1.23 Bev. The dominance of this mechanism should grow with increasing energy for the incident photon. The single-nucleon cross section has been given by Drell[19] as

$$\frac{d^2\sigma}{d\epsilon_1 d\Omega_1}=\frac{\alpha}{8\pi^2}\left\{\frac{\epsilon_1(k-\epsilon_1)}{k^3}\frac{\sin^2\theta_1}{(1-\beta_1\cos\theta_1)^2}\right\}\sigma_{\pi p}(k-\epsilon_1), \quad (3.6)$$

where $\sigma_{\pi p}(k-\epsilon_1)$ is the total elastic cross section for pion-nucleon scattering at a pion energy $k-\epsilon_1$.

Let us now compare formula (3.6) with Table I to see how serious the nuclear competition is. At a photon energy of 6 Bev, the energy $k-\epsilon_1$ of the pion scattered by the nucleon is 3 Bev, when the momentum ϵ_1 of the other pion is 3 Bev/c. At 3 Bev the total pion-nucleon cross section is about 30 mb, and under these conditions, at small forward angles formula (3.6) predicts the values as given in Table III.

These cross sections are larger than the electromagnetic cross section by factors of 20 to 2000, so it appears hopeless to observe the purely electromagnetic pair production if only one of the pions is measured. The situation may be improved by going to a heavy nucleus to take advantage of the Z^2 factor in Eq. (3.1a), but the Drell mechanism is also largely coherent and will increase by approx $A^{\frac{2}{3}}$.

If one can measure the differential cross section for observing *both* pions in a small forward cone with the momentum transfer to the nucleus limited to a small value, then the electromagnetic process has a chance to be observed in comparison with the nuclear process, essentially because of the high power of momentum transfer in the denominator of the electromagnetic production cross section. For this purpose we have again formula (3.1) under various conditions as in Table I, but with an additional restriction that $\lambda\leqslant\lambda_{\max}$, where for illustration we choose $\lambda_{\max}=\mu/2$ (70 Mev/c). Formula (3.5) is modified to impose this restriction on λ such that $\sigma_{\pi N}$ is replaced by

$$\int_0^{\lambda_{\max}^2}\frac{d\sigma_{\pi N}}{d\lambda^2}d\lambda^2,$$

where $\lambda^2=(p'-p)^2$ is the invariant square of momentum transfer to the nucleon. In the region of energy and angles under consideration, diffraction behavior dominates elastic π-N scattering, and we assume a phenomenological formula,

$$d\sigma_{\pi N}/d\lambda^2=a^2/(t_0+\lambda^2)^2, \quad (3.7)$$

to represent the forward diffraction peak. Here, the parameter t_0 determines the width of the peak, while a^2 is a normalization factor. This formula has been fitted with various experimental data[22,23] to give $t_0=10\,\mu^2$ and $a^2=2.1$ mb.

The results of calculation for the nuclear as well as the electromagnetic pion pair production with limited momentum transfer are given in Table IV. We may add here that the interference between the electromagnetic production amplitude and the nuclear production amplitude by peripheral collision should be negligible, since the former is mainly real while the latter is presumed to be mostly imaginary (the π-N forward diffraction amplitude is imaginary).

We note in Table IV that the effect of the π-π final-state interactions, as measured in terms of I_2+I_3, is larger than the nuclear production I_4 at $\theta_1=\frac{1}{2}$ deg. We note also that an even lower limit on λ will be beneficial, since the cross sections of Eq. (3.1) have not changed much under the restriction of $\lambda<\mu/2$.

TABLE II. Numerical values of formula (3.5) in comparison with the experimental cross section for pion-pair production by photons incident on protons.

k (Mev)	$\sigma(k)$ of (3.5) (μb)	Experimental $\sigma(k)$ (μb)
500	0.80	~70[a]
700	0.58	
1000	0.38	~50[b]
2000	0.16	

[a] See reference 22.
[b] See reference 23.

TABLE III. Cross sections of Eq. (3.6) for $k=6$ Bev and $p_1=3$ Bev/c.

θ (deg)	$d^2\sigma/d\epsilon_1 d\Omega_1$ of Eq. (3.5) (10^{-4} mb/sr)
$\frac{1}{2}$	9.68
1	31.8
2	68.1
5	50.7

[21] J. R. Kilner, R. E. Diebold, and R. L. Walker, Phys. Rev. Letters **5**, 518 (1960).

[22] J. M. Sellen, G. Cocconi, V. T. Cocconi, and E. L. Hart, Phys. Rev. **113**, 1323 (1959).

[23] R. G. Thomas, Lawrence Radiation Laboratory Report UCRL-8965 (unpublished); K. Wang, T. Wang, T. Ting, V. G. Ivanov, Yu. Katyshev, E. N. Kladnitskaya, L. A. Kuluyukina, N. D. Ty, A. V. Nikitin, S. Z. Otvinovskii, M. I. Solovev, R. Sosnovskii, and M. D. Shafranov, Soviet Phys.—JETP **11**, 313 (1960).

TABLE IV. Values of Eqs. (3.1) and (3.7) under the restriction $\lambda^2 \lessdot \lambda_{max}^2$, where $\lambda_{max} = \mu/2 = 70$ Mev/c.

p_1 (Bev/c)	θ_1 (deg)	I_4[a] 10^{-6} (mb/sr)	I_1[b] 10^{-6} (mb/sr)	I_2[b] 10^{-6} (mb/sr)	I_3[b] 10^{-6} (mb/sr)
3	$\frac{1}{2}$	1.66	38.8	13.70	1.46
2.7		1.09	31.9	10.98	1.11
2.4		0.67	24.5	6.16	0.59
3	1	5.47	26.7	2.90	0.71
2.7		3.72	23.4	2.98	0.60
2.4		2.36	19.1	1.73	0.46
3	2	1.16	8.68	−0.96	0.16
2.7		0.87	8.80	−0.85	0.18
2.4		0.60	8.37	−0.69	0.17
3	5	8.73	0.39	0.00	0.00
2.7		7.82	0.52	−0.02	0.00
2.4		6.63	0.63	−0.04	0.00

[a] $I_4 = \dfrac{\alpha}{8\pi^2}\dfrac{p_1 p_2}{k^3}\left(\dfrac{\sin\theta_1}{1-\beta_1\cos\theta_1}\right)^2 \dfrac{2.1\lambda_{max}^2}{\lambda_{max}^2+10}$.

[b] I_1, I_2, I_3 (see Table I).

IV. CONCLUSIONS

On the basis of the Mandelstam representation for the three invariant amplitudes of electromagnetic pion-pair production, we derived partial-wave dispersion relations and obtained solutions by retaining only the 1π contribution to the unphysical cut. The contribution from 2π intermediate states could be neglected because of the smallness of the amplitude for the process $\gamma+\pi\rightarrow\pi+\pi$. The amplitudes thus obtained can be used in calculating the cross section for electromagnetic pion-pair production through Eq. (3.1), which consists of purely electromagnetic terms plus a rescattering correction. The rescattering term is important only for S states of the 2π system because the average t involved is small, no matter how large the incident photon energy. The rescattering term is expressed through two functions, $\mathcal{T}_{++}^{0,1}(t)$ and $\mathcal{T}_{++}^{0,2}(t)$, for which formula (3.2) is given in terms of N_0^I and D_0^I, the numerator and denominator functions of the π-π S-wave amplitude.[10,19]

We have estimated the rescattering correction using Desai's estimate of the π-π S wave with a virtual state (or resonance) near threshold in $I=0$. The results are that the cross section is substantially enhanced for low t, and diminished slightly for high t. Therefore, an experiment to measure the π-π effects should be performed best at low-momentum transfers which tend to emphasize low t. For purposes of illustration we have estimated the cross section with an incident photon of 6 Bev, with the result that rescattering comprises over 38% of the cross section at $\theta_1=\frac{1}{2}$ deg, and $P_1=3$ Bev/c.

We have also shown that if we observe the pion pairs produced in the forward direction by a high-energy photon, keeping the momentum transfers sufficiently small, the electromagnetic pair production dominates the nuclear production. This situation should improve as the energy of the initial photon is chosen to be higher, and if we use a large nucleus.

We conclude, therefore, that the effects of the π-π S-wave should be measurable in electromagnetic pion-pair production, if the experiment can be designed to pick out events where the momentum transfer to the nucleus is $\lesssim\mu/2$ or 70 Mev/c, with the energy of the incident photon greater than 6 Bev.

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Radiative Corrections to the Coulomb Scattering Asymmetry Function*

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An evaluation of the effect of the radiative corrections on the Coulomb scattering asymmetry function is presented. The lowest order corrections, consisting of those terms which are quadratic in the external field, are evaluated with the aid of the unitarity of the S matrix. Relative corrections to the asymmetry function are presented in tabular form for electron velocities in the range $0.6\leq\beta\leq0.9$ and electron scattering angles in the range $30°\leq\theta\leq150°$. These corrections, which are shown to be independent of the value of the energy resolution, are found to be less than 3% for the electron velocities and angles quoted above.

I. INTRODUCTION

It is of some interest to determine to what extent radiative effects can account for the discrepancy between the experimental determination of the Mott scattering asymmetry[1] and the exact (in αZ) numerical calculations of the asymmetry function.[2] Several diffi-

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[1] D. F. Nelson and R. W. Pidd, Phys. Rev. **114**, 728 (1959). References to previous experimental measurements of asymmetry are given in this paper, as well as a detailed evaluation of the earlier experiments.

[2] N. Sherman, Phys. Rev. **103**, 1601 (1956); N. Sherman and