TABLE IV. Values of Eqs. (3.1) and (3.7) under the restriction $\lambda^2 \lessdot \lambda_{max}^2$, where $\lambda_{max}=\mu/2=70$ Mev/c.

p_1 (Bev/c)	θ_1 (deg)	I_4[a] 10^{-6} (mb/sr)	I_1[b] 10^{-6} (mb/sr)	I_2[b] 10^{-6} (mb/sr)	I_3[b] 10^{-6} (mb/sr)
3	$\frac{1}{2}$	1.66	38.8	13.70	1.46
2.7		1.09	31.9	10.98	1.11
2.4		0.67	24.5	6.16	0.59
3	1	5.47	26.7	2.90	0.71
2.7		3.72	23.4	2.98	0.60
2.4		2.36	19.1	1.73	0.46
3	2	1.16	8.68	−0.96	0.16
2.7		0.87	8.80	−0.85	0.18
2.4		0.60	8.37	−0.69	0.17
3	5	8.73	0.39	0.00	0.00
2.7		7.82	0.52	−0.02	0.00
2.4		6.63	0.63	−0.04	0.00

[a] $I_4=\dfrac{\alpha}{8\pi^2}\dfrac{p_1p_2}{k^3}\left(\dfrac{\sin\theta_1}{1-\beta_1\cos\theta_1}\right)^2\dfrac{2.1\lambda_{max}^2}{\lambda_{max}^2+10}$.
[b] I_1, I_2, I_3 (see Table I).

IV. CONCLUSIONS

On the basis of the Mandelstam representation for the three invariant amplitudes of electromagnetic pion-pair production, we derived partial-wave dispersion relations and obtained solutions by retaining only the 1π contribution to the unphysical cut. The contribution from 2π intermediate states could be neglected because of the smallness of the amplitude for the process $\gamma+\pi\rightarrow\pi+\pi$. The amplitudes thus obtained can be used in calculating the cross section for electromagnetic pion-pair production through Eq. (3.1), which consists of purely electromagnetic terms plus a rescattering correction. The rescattering term is important only for S states of the 2π system because the average t involved is small, no matter how large the incident photon energy. The rescattering term is expressed through two functions, $\mathcal{T}_{++}{}^{0,1}(t)$ and $\mathcal{T}_{++}{}^{0,2}(t)$, for which formula (3.2) is given in terms of $N_0{}^I$ and $D_0{}^I$, the numerator and denominator functions of the π-π S-wave amplitude.[10,19]

We have estimated the rescattering correction using Desai's estimate of the π-π S wave with a virtual state (or resonance) near threshold in $I=0$. The results are that the cross section is substantially enhanced for low t, and diminished slightly for high t. Therefore, an experiment to measure the π-π effects should be performed best at low-momentum transfers which tend to emphasize low t. For purposes of illustration we have estimated the cross section with an incident photon of 6 Bev, with the result that rescattering comprises over 38% of the cross section at $\theta_1=\frac{1}{2}$ deg, and $P_1=3$ Bev/c.

We have also shown that if we observe the pion pairs produced in the forward direction by a high-energy photon, keeping the momentum transfers sufficiently small, the electromagnetic pair production dominates the nuclear production. This situation should improve as the energy of the initial photon is chosen to be higher, and if we use a large nucleus.

We conclude, therefore, that the effects of the π-π S-wave should be measurable in electromagnetic pion-pair production, if the experiment can be designed to pick out events where the momentum transfer to the nucleus is $\lesssim\mu/2$ or 70 Mev/c, with the energy of the incident photon greater than 6 Bev.

ACKNOWLEDGMENTS

I wish especially to thank Professor Geoffrey F. Chew for his suggestion of the problem explored in this paper, and for innumerable discussions during its progress.

I also wish to gratefully acknowledge the help of Mrs. Claudette G. Rugge, who assisted in numerical calculations on the IBM 709.

PHYSICAL REVIEW VOLUME 126, NUMBER 1 APRIL 1, 1962

Radiative Corrections to the Coulomb Scattering Asymmetry Function*

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(Received August 31, 1961)

An evaluation of the effect of the radiative corrections on the Coulomb scattering asymmetry function is presented. The lowest order corrections, consisting of those terms which are quadratic in the external field, are evaluated with the aid of the unitarity of the S matrix. Relative corrections to the asymmetry function are presented in tabular form for electron velocities in the range $0.6\leq\beta\leq0.9$ and electron scattering angles in the range $30°\leq\theta\leq150°$. These corrections, which are shown to be independent of the value of the energy resolution, are found to be less than 3% for the electron velocities and angles quoted above.

I. INTRODUCTION

IT is of some interest to determine to what extent radiative effects can account for the discrepancy between the experimental determination of the Mott scattering asymmetry[1] and the exact (in αZ) numerical calculations of the asymmetry function.[2] Several diffi-

* Supported in part by the U. S. Atomic Energy Commission.

[1] D. F. Nelson and R. W. Pidd, Phys. Rev. **114**, 728 (1959). References to previous experimental measurements of asymmetry are given in this paper, as well as a detailed evaluation of the earlier experiments.

[2] N. Sherman, Phys. Rev. **103**, 1601 (1956); N. Sherman and

culties are inherent in the theoretical determination of the radiative effects.

Perhaps the most formidable difficulty is an exact treatment of the Coulomb scattering wave function. We avoid this problem by expanding the S matrix in powers of the external field in the usual manner.[3] Such a procedure is unsatisfactory, since for elements of interest in Mott scattering experiments $\alpha Z \gtrsim \frac{1}{2}$; one therefore expects rather large corrections to the dominant term in such an expansion. It is hoped, however, that expressions based on this expansion will give a reasonable estimate of the radiative effects.

When the radiation field is neglected, the lowest order term in an expansion of the cross section in powers of αZ gives rise to no asymmetry. The first contribution to the asymmetry function arises from terms of relative order $\alpha^2 Z^2$ (the second Born approximation). For a Coulomb field this term results in divergent integrals. In order to extract finite results from the second-order term in the expanded S matrix, one resorts to the artifice of replacing the Coulomb field by a "screened" Coulomb field, and allowing the screening to vanish after the cross section has been computed.[4] When the radiation field is included in the S matrix, a similar situation is encountered. The first-order terms in αZ give rise to no asymmetry. To evaluate the second-order term using the expanded S matrix, screening must be included to avoid divergent integrals.

The difficult computations involved in a direct evaluation of the second-order terms[5] are avoided by application of unitarity of the S matrix. One is able to reduce the terms of interest to two types. The first type of term involves integrals over an intermediate electron state; these integrals are easily evaluated analytically. The second type of term involves integrals over an intermediate state with one electron and one photon. These integrals contain divergences at low photon energy which are subtracted out analytically. The resulting finite integrals are evaluated numerically.

Divergences which arise as the screening parameter vanishes cancel when the cross section is formed, whereas the infrared divergences cancel after a soft bremsstrahlung term is added to the cross section.

The dependence of the radiative corrections on experimental energy resolution is determined entirely by the bremsstrahlung cross section. Since the soft bremsstrahlung cross section is a multiple of the elastic scattering cross section, the asymmetry function (a ratio of cross sections) is independent of the magnitude of the energy resolution, provided the energy resolution $\Delta W \ll m$. It is shown that radiative effects cannot account for the discrepancy between the experimental and theoretical determinations of the asymmetry.

II. FORMULATION OF THE PROBLEM

Scattering of an electron by a static scalar potential may be described in terms of the S-matrix element,

$$\langle p_2 | S | p_1 \rangle = (8\pi^2 i/V)(\bar{u}_2 \gamma_4 T u_1)\delta(W_1 - W_2), \qquad (1)$$

where $p_i = (\mathbf{p}_i, iW_i)$, $i=1, 2$ represent the four-momenta of the initial and final electrons, V is a normalization volume, and where the 4×4 T matrix may be written in general

$$T = a + \gamma_4 b. \qquad (2)$$

a and b are scalar functions of $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$, the momentum transferred to the scattering center, and of the common energy W_1. Letting $d\sigma = \sigma(\mathbf{p}_1, \boldsymbol{\zeta}_1; \mathbf{p}_2)d\Omega$ represent the differential cross section for scattering from a state of momentum $\mathbf{p}_1$ and spin orientation $\boldsymbol{\zeta}_1$ (in the incident electron's rest system) to a state of momentum $\mathbf{p}_2$, one easily deduces that $\sigma(\mathbf{p}_1, \boldsymbol{\zeta}_1; \mathbf{p}_2) = I(\theta) - D(\theta)\mathbf{n}\cdot\boldsymbol{\zeta}_1$ where $\mathbf{n} = \mathbf{p}_1 \times \mathbf{p}_2 / |\mathbf{p}_1 \times \mathbf{p}_2|$ is the unit normal to the plane of scattering, and θ is the scattering angle. In terms of a and b one finds

$$I(\theta) = 4\{W_1^2[1-\beta^2 \sin^2(\theta/2)]|a|^2 + W_1^2[1-\beta^2\cos^2(\theta/2)]|b|^2 + 2W_1 m \operatorname{Re} a^* b\}, \qquad (3a)$$

$$D(\theta) = -4p_1^2 \sin\theta \operatorname{Im} a^* b. \qquad (3b)$$

In Eqs. (3), $p_1 = |\mathbf{p}_1| = |\mathbf{p}_2|$, $\beta = p_1/W_1$, and m denotes the electron mass. $I(\theta)$ is the cross section for unpolarized electrons and $P(\theta) = -D(\theta)/I(\theta)$ measures the scattering asymmetry. We will describe the effect of the radiative corrections on P by writing

$$P = P_0(1+\delta), \qquad (4)$$

where P_0 gives the value of P in the absence of the radiation field and $\delta = \delta(\beta, \theta)$ represents the relative effect of the radiative corrections at a scattering angle θ, for electrons of velocity β.

Let us denote by $S_{(j)}^{(i)}$ that term in the expanded S matrix which consists of i vertices, j of which represent interactions with the external field. Let $a_{(j)}^{(i)}$ and $b_{(j)}^{(i)}$ be the contribution of $S_{(j)}^{(i)}$ to a and b. Write $I(\theta) = I_0 + I_{\mathrm{rc}} + I_{\mathrm{br}}$ and $D(\theta) = D_0 + D_{\mathrm{rc}} + D_{\mathrm{br}}$, where I_0 and D_0 represent the lowest nonvanishing Born approximation values, I_{rc} and D_{rc} represent the radiative corrections to these values, and I_{br} and D_{br} represent the contribution from a soft bremsstrahlung term. One then finds in terms of $a_{(j)}^{(i)}$ and $b_{(j)}^{(i)}$:

$$I_{\mathrm{rc}}(\theta) = 8W_1^2 a_{(1)}^{(1)}\{[1-\beta^2\sin^2(\theta/2)] \times a_{(1)}^{(3)} + (m/W_1) b_{(1)}^{(3)}\}, \qquad (5a)$$

$$D_{\mathrm{rc}}(\theta) = -4p_1^2 \sin\theta [a_{(1)}^{(1)} \operatorname{Im} b_{(2)}^{(4)} + a_{(1)}^{(3)} \operatorname{Im} b_{(2)}^{(2)} - b_{(1)}^{(3)} \operatorname{Im} a_{(2)}^{(2)}]. \qquad (5b)$$

Introducing a screened Coulomb potential $[a_4(q) = -2\pi i \delta(W_1 - W_2) Ze/(q^2+\lambda^2)]$, and allowing $\lambda \to 0$

D. F. Nelson, *ibid.* **114**, 1541 (1959). Analytical results corresponding to the first and second Born approximations, respectively, are given by F. Gürsey, Phys. Rev. **107**, 1734 (1957), and W. R. Johnson, T. A. Weber, and C. J. Mullin, *ibid.* **121**, 933 (1961).

[3] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Cambridge, 1955), p. 320.

[4] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).

[5] R. G. Newton, Phys. Rev. **97**, 1162 (1955); **98**, 1514 (1955).

when this limit exists, one finds that the Born approximation terms in Eqs. (5) are[6]

$$a_{(1)}^{(1)}(q)=\alpha Z/q^2, \tag{6a}$$

$$b_{(1)}^{(1)}(q)=0, \tag{6b}$$

$$\mathrm{Im}a_{(2)}^{(2)}(q)=-(W_1/m)\,\mathrm{Im}b_{(2)}^{(2)}+(2\alpha Z/\beta)\ln(q/\lambda), \tag{6c}$$

$$\mathrm{Im}b_{(2)}^{(2)}(q)=(\alpha^2Z^2m/p_1K^2)\ln(2p_1/q), \tag{6d}$$

where $\mathbf{K}=\mathbf{p}_1+\mathbf{p}_2$. Similarly, one finds that those terms involving the radiation field and one interaction with the external field may be written after mass and charge renormalization as[7]

$$a_{(1)}^{(3)}(q)=-\frac{\alpha}{2\pi}a_{(1)}^{(1)}\{[(q^2+2m^2)A(q)-2]\times[\ln(m/\lambda_0)-1]+\tfrac{1}{4}(q^2-4m^2)A(q)+\tfrac{1}{2}(q^2+2m^2)B(q)-4C(q)\}, \tag{7a}$$

$$b_{(1)}^{(3)}(q)=-\frac{\alpha}{2\pi}a_{(1)}^{(1)}W_1mA(q). \tag{7b}$$

The functions $A(q)$, $B(q)$, and $C(q)$ occurring in Eqs. (7), together with several other functions occurring in the sequel are defined in the Appendix. The parameter λ_0 represents the photon mass, which will be set equal to zero after the infrared divergences have been eliminated.

It is apparent that D_{rc} is of order α^3Z^2, while the correction to I is of order α^2Z. The lowest order term in D_{rc} vanishes due to the vanishing of $b_{(1)}^{(1)}$ together with the reality of $a_{(1)}^{(1)}$ and $b_{(1)}^{(3)}$.

The cross section for emission of one soft photon, when integrated over photon angles and photon energies in the interval $[\lambda_0,\Delta W]$ may be written[8] $d\sigma_{\mathrm{br}}=Md\sigma_0$, where $d\sigma_0$ is the elastic scattering cross section and

$$M=\frac{\alpha}{\pi}\left\{[(q^2+2m^2)A(q)-2]\ln(2\Delta W/\lambda_0)+(W_1/p_1)\ln\left(\frac{W_1+p_1}{W_1-p_1}\right)-\tfrac{1}{2}(q^2+2m^2)E(q)\right\}, \tag{8}$$

where $E(q)$ is defined in the Appendix.

Writing $\Delta I=I_{\mathrm{rc}}+I_{\mathrm{br}}$ and $\Delta D=D_{\mathrm{rc}}+D_{\mathrm{br}}$, one finds:

$$\Delta I=\frac{\alpha}{\pi}4W_1^2(a_{(1)}^{(1)})^2\left\{\left[[(q^2+2m^2)A(q)-2]\times\left[\ln\left(\frac{2\Delta W}{m}\right)+1\right]+(W_1/p_1)\ln\left(\frac{W_1+p_1}{W_1-p_1}\right)-\tfrac{1}{2}(q^2+2m^2)E(q)-\tfrac{1}{4}(q^2-4m^2)A(q)-\tfrac{1}{2}(q^2+2m^2)B(q)+4C(q)\right]\times[1-\beta^2\sin^2(\theta/2)]-m^2A(q)\right\}, \tag{9}$$

which is the result originally due to Schwinger,[9] and $\Delta D=\Delta D_1+\Delta D_2$ with

$$\Delta D_1=\frac{\alpha}{2\pi}D_0\left\{[(q^2+2m^2)A(q)-2]\left[\ln\left(\frac{2\Delta W}{m}\right)+1\right]+(W_1/p_1)\ln\left(\frac{W_1+p_1}{W_1-p_1}\right)-\tfrac{1}{2}(q^2+2m^2)E(q)-\tfrac{1}{4}(q^2-4m^2)A(q)-\tfrac{1}{2}(q^2+2m^2)B(q)+4C(q)-W_1^2A(q)\right\}, \tag{10a}$$

$$\Delta D_2=-4p_1^2\sin\theta a_{(1)}^{(1)}\left\{\mathrm{Im}b_{(2)}^{(4)}-(2\alpha Z/\beta)b_{(1)}^{(3)}\ln(q/\lambda)+\frac{\alpha}{2\pi}\mathrm{Im}b_{(2)}^{(2)}\left[[(q^2+2m^2)A(q)-2]\times\ln\left(\frac{2\Delta W}{\lambda_0}\right)+\frac{W_1}{p_1}\ln\left(\frac{W_1+p_1}{W_1-p_1}\right)-\tfrac{1}{2}(q^2+2m^2)E(q)\right]\right\}. \tag{10b}$$

ΔD_1 is entirely finite, whereas ΔD_2 contains, along with the one unknown factor $\mathrm{Im}b_{(2)}^{(4)}$ both screening and infrared divergences. $\mathrm{Im}b_{(2)}^{(4)}$ will be computed in the following sections and the cancellation of all divergences in ΔD_2 will be demonstrated explicitly. From Eq. (4) it is evident that

$$\delta(\beta,\theta)=\Delta D/D_0-\Delta I/I_0. \tag{11}$$

III. DECOMPOSITION OF $\mathrm{Im}b_{(2)}^{(4)}$

Combining Eqs. (1) and (2), one sees that

$$\langle p_2|S|p_1\rangle=\frac{8\pi^2i}{V}(\bar{u}_2(\gamma_4a+b)u_1)\delta(W_1-W_2), \tag{12}$$

from which it follows that

$$\langle p_2|S+S^*|p_1\rangle=-\frac{16\pi^2}{V}(\bar{u}_2[\gamma_4\,\mathrm{Im}a+\mathrm{Im}b]u_1)\delta(W_1-W_2). \tag{13}$$

Comparing those terms of fourth order in e and second order in eZ on both sides of the equation $S^*S=I$, one finds

$$S_{(2)}^{(4)}+S_{(2)}^{(4)*}=-S_{(1)}^{(2)*}S_{(1)}^{(2)}-S_{(0)}^{(2)*}S_{(2)}^{(2)}-S_{(2)}^{(2)*}S_{(0)}^{(2)}-S_{(0)}^{(1)*}S_{(2)}^{(3)}-S_{(2)}^{(3)*}S_{(0)}^{(1)}-S_{(1)}^{(1)*}S_{(1)}^{(3)}-S_{(1)}^{(3)*}S_{(1)}^{(1)}. \tag{14}$$

By forming the matrix element of Eq. (14) with $\langle p_1|$ and $|p_2\rangle$, and omitting terms which trivially vanish, one

[6] See, for example, reference 4.
[7] J. M. Jauch and F. Rohrlich, reference 3, p. 333.
[8] J. Schwinger, Phys. Rev. **76**, 790 (1949).
[9] See, for example, reference 8, p. 812.

concludes that

$$\frac{16\pi^2}{V}(\bar{u}_2[\gamma_4 \operatorname{Im}a_{(2)}{}^{(4)}+\operatorname{Im}b_{(2)}{}^{(4)}]u_1)\delta(W_2-W_1)$$
$$=\frac{V^2}{(2\pi)^6}\int\int d\mathbf{p}d\mathbf{k}\sum_{s,\epsilon}\langle pk|S_{(1)}{}^{(2)}|p_2\rangle^*\langle pk|S_{(1)}{}^{(2)}|p_1\rangle$$
$$+\frac{V}{(2\pi)^3}\int d\mathbf{p}\sum_{s}\{\langle p|S_{(1)}{}^{(1)}|p_2\rangle^*\langle p|S_{(1)}{}^{(3)}|p_1\rangle$$
$$+\langle p|S_{(1)}{}^{(3)}|p_2\rangle^*\langle p|S_{(1)}{}^{(1)}|p_1\rangle\}. \quad (15)$$

The summations in Eq. (15) extend over s, the intermediate electron spin, and ϵ, the photon polarization.

To pick out that term on the right-hand side of Eq. (15) which corresponds to $\operatorname{Im}b_{(2)}{}^{(4)}$ the following lemma is useful.

Lemma. If $\bar{u}_2(\operatorname{Im}b+\gamma_4\operatorname{Im}a)u_1=\bar{u}_2Ou_1$, then $\operatorname{Im}b=(1/q^2K^2)\operatorname{Tr}[(-ip_2+m)O(-ip_1+m)\boldsymbol{\tau}\gamma_4\gamma_5]$ with $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2$, $\mathbf{K}=\mathbf{p}_1+\mathbf{p}_2$, and $\tau=(\mathbf{p}_1\times\mathbf{p}_2, i0)$.

The term $b_{(2)}{}^{(4)}$ thus decomposes into two parts $c+d$, where c is the contribution from the one-electron integral and d is the contribution from the one-electron, one-photon integral.

IV. EVALUATION OF Imc

Let us introduce the expressions

$$\langle p|S_{(1)}{}^{(1)}|p_1\rangle=i\frac{8\pi^2\alpha Z\delta(W_1-W)}{V(q_{10}{}^2+\lambda^2)}(\bar{u}\gamma_4u_1), \quad (16a)$$

and[10]

$$\langle p|S_{(1)}{}^{(3)}|p_1\rangle=i\frac{8\pi^2\alpha Z\delta(W_1-W)}{V(q_{10}{}^2+\lambda^2)}\times(\bar{u}[\Lambda_{4f}(p_1,p)-q_{10}{}^2\pi_f(q_{10})\gamma_4]u_1), \quad (16b)$$

with $\mathbf{q}_{10}=\mathbf{p}_1-\mathbf{p}$, where Λ_{4f} and π_f are the renormalized vertex and vacuum polarization terms, which are written out in detail in the Appendix. Substituting Eqs. (16) into the second term on the right of Eq. (15) and making use of the lemma, one finds

$$\operatorname{Im}c=\frac{\alpha^2Z^2}{2\pi}\int\frac{d\mathbf{p}\delta(W_1-W)}{2W(q_{10}{}^2+\lambda^2)(q_{20}{}^2+\lambda^2)}\times\frac{1}{q^2K^2}\operatorname{Tr}[(-ip+m)O(-ip_1+m)\boldsymbol{\tau}\gamma_4\gamma_5], \quad (17)$$

with

$$O=\gamma_4(-ip+m)[\Lambda_{4f}(p_1,p)-q_{10}{}^2\pi_f(q_{10})\gamma_4]+[\bar{\Lambda}_{4f}(p_2,p)-q_{20}{}^2\pi_f(q_{20})\gamma_4](-ip+m)\gamma_4$$
$$=-\frac{\alpha}{2\pi}\{(ip+m)F(q_{10})-i\gamma_4pW_1mA(q_{10})+W_1[2F(q_{10})+m^2A(q_{10})]\gamma_4+(ip+m)F(q_{20})-ip\gamma_4W_1mA(q_{20})+W_1[2F(q_{20})+m^2A(q_{20})]\gamma_4\}, \quad (18)$$

where $F(q)$ is defined in the Appendix. Upon evaluating the trace, Eq. (17) reduces to

$$\operatorname{Im}c=-\frac{\alpha^3Z^2mp_1}{4\pi^2}\left[\frac{1}{K^2}J_1+\frac{W_1{}^2}{q^2}J_2\right], \quad (19)$$

with

$$J_1=\int\frac{d\Omega_p(K^2-2\mathbf{p}\cdot\mathbf{K})F(q_{10})}{(q_{10}{}^2+\lambda^2)(q_{20}{}^2+\lambda^2)}, \quad (20a)$$

$$J_2=\int\frac{d\Omega_p(q^2-2\mathbf{p}\cdot\mathbf{q})A(q_{10})}{(q_{10}{}^2+\lambda^2)(q_{20}{}^2+\lambda^2)}. \quad (20b)$$

If $\mathbf{p}_1$ is chosen as the polar axis, and $\mathbf{p}_2$ is placed in the $x-z$ plane, the azimuthal integrations in Eqs. (20) may be carried out to give

$$J_1=\frac{2\pi}{p_1{}^2}\int_{-1}^{\cos\theta}d\mu\frac{F(q_{10})}{1-\mu}, \quad (21)$$

and

$$J_2=\tilde{J}_2+\Delta J_2,$$

with

$$\tilde{J}_2=\frac{2\pi}{p_1{}^2}A(q)\ln(4p_1{}^2/\lambda^2), \quad (22a)$$

$$\Delta J_2=\frac{\pi}{p_1{}^2}\int_{-1}^{1}d\mu\left\{\frac{2-\cos\theta-\mu-|\cos\theta-\mu|}{1-\mu}\times A(q_{10})-2A(q)\right\}\frac{1}{|\cos\theta-\mu|}. \quad (22b)$$

In Eqs. (21) and (22) $\mu=\cos(\sphericalangle\mathbf{p}_1,\mathbf{p})$; the limit $\lambda\to0$ has been taken except in Eq. (22a). The μ integrations in Eqs. (21) and (22) are now carried out to give

$$J_1=-\frac{4\pi}{p_1{}^2}\left\{-\left[\ln\left(\frac{m}{\lambda_0}\right)-1\right]\left[\Gamma\left(\frac{4p_1{}^2}{m^2}\right)-\Gamma\left(\frac{q^2}{m^2}\right)\right]+\tfrac{1}{2}\ln\left(\frac{4p_1{}^2}{q^2}\right)-\frac{1}{8}\left[\Lambda\left(\frac{4p_1{}^2}{m^2}\right)-\Lambda\left(\frac{q^2}{m^2}\right)\right]+2\left[\Delta\left(\frac{4p_1{}^2}{m^2}\right)-\Delta\left(\frac{q^2}{m^2}\right)\right]-\frac{1}{4}\left[\Sigma\left(\frac{4p_1{}^2}{m^2}\right)-\Sigma\left(\frac{q^2}{m^2}\right)\right]-\frac{1}{2}\left[\Xi\left(\frac{4p_1{}^2}{m^2}\right)-\Xi\left(\frac{q^2}{m^2}\right)\right]\right\}, \quad (23a)$$

$$J_2=\frac{4\pi}{p_1{}^2}A(q)\ln(q/\lambda)+\frac{2\pi}{m^2p_1{}^2}[m^2A(q)-1]\ln(4p_1{}^2/q^2)-\frac{2\pi}{m^2p_1{}^2}[\Theta(q^2/m^2,4p_1{}^2/m^2)-m^2B(q)]. \quad (23b)$$

The functions Γ, Λ, Δ, Σ, Ξ, and Θ occurring in Eqs. (23) are listed in the Appendix. Equations (23) are substi-

[10] J. M. Jauch and F. Rohrlich, reference 3, p. 333. One should note that the use of the renormalized functions is unnecessary since the ultraviolet divergences cancel on the right-hand side of Eq. (15).

tuted into Eq. (19), which then becomes

$$\operatorname{Im}c=(2\alpha Z/\beta)b_{(1)}{}^{(3)}\ln(q/\lambda)-\frac{\alpha}{\pi}\frac{\alpha^2Z^2m}{p_1K^2}\times\left[\Gamma\left(\frac{4p_1^2}{m^2}\right)-\Gamma\left(\frac{q^2}{m^2}\right)\right]\ln\left(\frac{m}{\lambda_0}\right)+\frac{\alpha}{\pi}\frac{\alpha^2Z^2m}{p_1K^2}\Phi(\beta,\theta),\quad(24)$$

where $\Phi(\beta,\theta)$ is defined in the Appendix.

Comparing Eq. (24) with Eq. (10b), one sees that the screening divergences in ΔD_2 cancel. Since the infrared divergence in Eq. (24) does not cancel that occurring in Eq. (10b), one must expect further infrared divergences in Imd.

V. EVALUATION OF Imd

Introduce

$$\langle kp|S_{(1)}{}^{(2)}|p_1\rangle=-i\frac{e^2}{V}\frac{1}{(2\omega V)^{\frac{1}{2}}}a_4(q_1)\left[\frac{\bar{u}e_\mu[-i(k+p)+m]\gamma_4u_1}{(k+p)^2+m^2}+\frac{\bar{u}\gamma_4[-i(p_1-k)+m]e_\mu u_1}{(p_1-k)^2+m^2}\right],\quad(25a)$$

$$\langle kp|S_{(1)}{}^{(2)}|p_2\rangle^*=-i\frac{e^2}{V}\frac{1}{(2\omega V)^{\frac{1}{2}}}a_4{}^*(q_2)\left[\frac{\bar{u}_2\gamma_4[-i(k+p)+m]e_\mu u}{(k+p)^2+m^2}+\frac{\bar{u}_2e_\mu[-i(p_2-k)+m]\gamma_4u}{(p_2-k)^2+m^2}\right],\quad(25b)$$

with

$$\mathbf{q}_i=\mathbf{p}_i-\mathbf{p}-\mathbf{k},\quad i=1,2;$$

then

$$\langle kp|S_{(1)}{}^{(2)}|p_2\rangle^*\langle kp|S_{(1)}{}^{(2)}|p_1\rangle=-\frac{e^4}{V^3}\frac{1}{2\omega}\frac{4\pi^2Z^2e^2}{(q_1^2+\lambda^2)(q_2^2+\lambda^2)}\cdot(\bar{u}_2\bar{N}_{2\mu}u)(\bar{u}N_{1\mu}u_1)\delta(W_1-W_2)\delta(W_1-W-\omega),\quad(26)$$

where

$$\bar{N}_{2\mu}=\frac{\gamma_4[-i(p+k)+m]\gamma_\mu}{(p+k)^2+m^2}+\frac{\gamma_\mu[-i(p_2-k)+m]\gamma_4}{(p_2-k)^2+m^2},\quad(27a)$$

$$N_{1\mu}=\frac{\gamma_\mu[-i(p+k)+m]\gamma_4}{(p+k)^2+m^2}+\frac{\gamma_4[-i(p_1-k)+m]\gamma_\mu}{(p_1-k)^2+m^2}.\quad(27b)$$

Substituting Eqs. (26) into the first term on the right of Eq. (15) and applying the lemma it follows that

$$\operatorname{Im}d=-\frac{2\alpha^3Z^2m}{(2\pi)^3}\times\int\int\frac{d\mathbf{p}d\mathbf{k}\,\delta(W_1-W-\omega)}{2W2\omega(q_1^2+\lambda^2)(q_2^2+\lambda^2)}J(\mathbf{p},\mathbf{k}),\quad(28)$$

where

$$mJ(\mathbf{p},\mathbf{k})=\frac{1}{q^2K^2}\operatorname{Tr}[(-ip_2+m)O(-ip_1+m)\boldsymbol{\tau}\gamma_4\gamma_5]\quad(29)$$

with $O=O_1+O_2+O_3+O_4$, where

$$D^2O_1=[m^2(ip+m)-mp\cdot k+(m^2+p\cdot k)ik+2(m^2W_1+p\cdot k\omega)\gamma_4],\quad(30a)$$

$$DD_1O_2=[(p\cdot p_1-p\cdot k)(ip+m)+mp\cdot k+ikp\cdot p_1+2mW\gamma_4ik-m\omega\gamma_4ip+(W_1+W)\gamma_4ikip+(m^2\omega+2W_1p\cdot p_1)\gamma_4],\quad(30b)$$

$$DD_2O_3=[(p\cdot p_2-p\cdot k)(ip+m)+mp\cdot k+ikp\cdot p_2+2mWik\gamma_4-m\omega ip\gamma_4+(W_1+W)ipik\gamma_4+(m^2\omega+2W_1p\cdot p_2)\gamma_4],\quad(30c)$$

$$D_1D_2O_4=\{(p_1\cdot k+p_2\cdot k-p_1\cdot p_2)(ip+m)-m(p\cdot k+2W\omega)+[p\cdot k-p\cdot p_1-p\cdot p_2-2W(W_1+W)-m^2]ik-2W(p_1\cdot p_2-p_1\cdot k-p_2\cdot k)\gamma_4\}.\quad(30d)$$

In Eqs. (30), $D=p\cdot k$, $D_1=p_1\cdot k$, and $D_2=p_2\cdot k$; the fact that plane-wave projection operators bracket O has been used to simplify the O_i.

Computing the traces of the O_i in Eqs. (30) according to Eq. (29), one discovers that $J=J_1+J_2$, where

$$J_1=\frac{1}{K^2}\left\{\frac{m^2}{2D^2}+\frac{p\cdot p_1}{DD_1}-\frac{p_1\cdot p_2}{2D_1D_2}\right\}\times(K^2-2\mathbf{p}\cdot\mathbf{K})+(\mathbf{p}_1\leftrightarrow\mathbf{p}_2),\quad(31a)$$

$$J_2=\frac{1}{K^2}\left\{-\frac{K^2+2\mathbf{k}\cdot\mathbf{K}}{2D}+\frac{K^2+2\mathbf{p}\cdot\mathbf{K}}{D_1}-\frac{m^2\mathbf{k}\cdot\mathbf{K}}{D^2}-\frac{p\cdot p_1(2\mathbf{k}\cdot\mathbf{K})}{DD_1}+\frac{2(W_1+W)(\omega\mathbf{p}\cdot\mathbf{K}-W\mathbf{k}\cdot\mathbf{K})}{DD_1}-\frac{K^2(p\cdot k+2W\omega)}{2D_1D_2}+\frac{2W_1\omega(2\mathbf{p}\cdot\mathbf{K})+(2W^2+m^2-p\cdot k)(2\mathbf{k}\cdot\mathbf{K})}{2D_1D_2}\right\}+\frac{1}{q^2}\left\{\frac{4W_1W\mathbf{k}\cdot\mathbf{q}-2W_1\omega\mathbf{p}\cdot\mathbf{q}-W\omega q^2}{DD_1}\right\}+\frac{4}{q^2K^2}\left\{\frac{2W_1(W_1+W)(\mathbf{p}\times\mathbf{k})\cdot\boldsymbol{\tau}}{DD_1}\right\}+(\mathbf{p}_1\leftrightarrow\mathbf{p}_2).\quad(31b)$$

J_1 behaves like $1/k^2$ for small k, from which it follows that the contribution of J_1 to $\mathrm{Im}d$ gives rise to logarithmic divergences. Let us decompose $\mathrm{Im}d=\mathrm{Im}d_1+\mathrm{Im}d_2$ corresponding to the decomposition of J into J_1 and J_2.

To evaluate the divergent part of $\mathrm{Im}d_1$, one introduces the photon mass λ_0 and divides the ω integration range into two parts $[\lambda_0,x]$ and $[x, W_1-m]$, where $\lambda_0\ll x\ll W_1-m$. In particular $q_1\rightarrow q_{10}$ and $q_2\rightarrow q_{20}$ in the first region. Write $\mathrm{Im}d_1=\mathrm{Im}\tilde{d}_1+\Delta\,\mathrm{Im}d_1$; then

$$\mathrm{Im}\tilde{d}_1=-\frac{\alpha^3Z^2mp_1}{(2\pi)^3K^2}\int\frac{d\Omega_p(K^2-2\mathbf{p}\cdot\mathbf{K})}{(q_{10}^2+\lambda^2)(q_{20}^2+\lambda^2)}\times\int_{\lambda_0}^{x}kd\omega\int d\Omega_k\left\{\frac{m^2}{2(\mathbf{p}\cdot\mathbf{k}-W_1\omega)^2}+\frac{p\cdot p_1}{(\mathbf{p}\cdot\mathbf{k}-W_1\omega)(\mathbf{p}_1\cdot\mathbf{k}-W_1\omega)}-\frac{p_1\cdot p_2}{2(\mathbf{p}_1\cdot\mathbf{k}-W_1\omega)(\mathbf{p}_2\cdot\mathbf{k}-W_1\omega)}\right\}. \quad (32)$$

The integrals in Eq. (32) may be carried out to give

$$\mathrm{Im}\tilde{d}_1=-\frac{\alpha}{\pi}\frac{\alpha^2Z^2m}{p_1K^2}\left[\Gamma\left(\frac{4p_1^2}{m^2}\right)-\Gamma\left(\frac{q^2}{m^2}\right)\right]\ln\left(\frac{2x}{\lambda_0}\right)-\frac{\alpha}{2\pi}[(q^2+2m^2)A(q)-2]\,\mathrm{Im}b_{(2)}{}^{(2)}\ln\left(\frac{2x}{\lambda_0}\right)+\frac{\alpha}{\pi}\frac{\alpha^2Z^2m}{p_1K^2}\Psi(\beta,\theta), \quad (33)$$

where $\Psi(\beta,\theta)$ is given in the Appendix.

In the second region one has

$$\Delta\,\mathrm{Im}d_1=-\frac{\alpha^3Z^2m}{(2\pi)^3K^2}\int_x^{W_1-m}pkdk\times\int\frac{d\Omega_pd\Omega_k[K^2-2\mathbf{p}\cdot\mathbf{K}]}{q_1^2q_2^2}\times\left\{\frac{m^2}{2D^2}+\frac{p\cdot p_1}{DD_1}-\frac{p_1\cdot p_2}{2D_1D_2}\right\}. \quad (34)$$

To integrate $\Delta\,\mathrm{Im}d_1$ numerically it is convenient to subtract from the integrand a function which is easily integrable analytically, and which agrees with the integrand of Eq. (34) as $k\rightarrow 0$. The resulting numerical procedure will then be independent of x. The function which is subtracted is

$$\Delta'\,\mathrm{Im}d_1=-\frac{\alpha^3Z^2m}{(2\pi)^3K^2}\int_x^{W_1-m}p_1kdk\times\int\frac{d\Omega_pd\Omega_k[K^2-2\mathbf{p}_0\cdot\mathbf{K}]}{q_{10}^2q_{20}^2}\times\left[\frac{m^2}{2D_0^2}+\frac{p_0\cdot p_1}{D_0D_1}-\frac{p_1\cdot p_2}{2D_1D_2}\right]$$
$$=-\frac{\alpha}{\pi}\frac{\alpha^2Z^2m}{p_1K^2}\left[\Gamma\left(\frac{4p_1^2}{m^2}\right)-\Gamma\left(\frac{q^2}{m^2}\right)\right]\ln\left(\frac{W_1-m}{x}\right)-\frac{\alpha}{2\pi}[(q^2+2m^2)A(q)-2]\times\mathrm{Im}b_{(2)}{}^{(2)}\ln\left(\frac{W_1-m}{x}\right), \quad (35)$$

where $p_0=(\mathbf{p}_0,iW_1)$ with $\mathbf{p}_0=\mathbf{p}p_1/p$ and $D_0=p_0\cdot k$. Writing $\Delta^2\,\mathrm{Im}d_1=\Delta\,\mathrm{Im}d_1-\Delta'\,\mathrm{Im}d_1$, one concludes that

$$\mathrm{Im}d=-\frac{\alpha}{\pi}\frac{\alpha^2Z^2m}{p_1K^2}\left[\Gamma\left(\frac{4p_1^2}{m^2}\right)-\Gamma\left(\frac{q^2}{m^2}\right)\right]\ln\frac{2(W_1-m)}{\lambda_0}-\frac{\alpha}{2\pi}[(q^2+2m^2)A(q)-2]\,\mathrm{Im}b_{(2)}{}^{(2)}\times\ln\frac{2(W_1-m)}{\lambda_0}+\frac{\alpha}{\pi}\frac{\alpha^2Z^2m}{p_1K^2}[\Psi(\beta,\theta)-\Omega(\beta,\theta)], \quad (36)$$

with

$$\frac{\alpha}{\pi}\frac{\alpha^2Z^2m}{p_1K^2}\Omega(\beta,\theta)=-\Delta^2\,\mathrm{Im}d_1-\mathrm{Im}d_2. \quad (37)$$

The function $\Omega(\beta,\theta)$, which is a finite five-dimensional integral, is evaluated numerically to give the results summarized in Table I.

Combining Eqs. (24) and (36), one concludes that

$$\mathrm{Im}b_{(2)}{}^{(4)}=\frac{2\alpha Z}{\beta}b_{(1)}{}^{(3)}\ln\left(\frac{q}{\lambda}\right)-\frac{\alpha}{2\pi}[(q^2+2m^2)A(q)-2]\,\mathrm{Im}b_{(2)}{}^{(2)}\times\ln\frac{2(W_1-m)}{\lambda_0}+\frac{\alpha}{\pi}\frac{\alpha^2Z^2m}{p_1K^2}\times\left\{\Phi(\beta,\theta)+\Psi(\beta,\theta)-\Omega(\beta,\theta)+\left[\Gamma\left(\frac{4p_1^2}{m^2}\right)-\Gamma\left(\frac{q^2}{m^2}\right)\right]\ln\frac{2(W_1-m)}{m}\right\}, \quad (38)$$

TABLE I. Numerical values of $\Omega(\beta,\theta)$.

$\beta \backslash \theta$	30°	60°	90°	120°	150°
0.6	−0.812	−0.280	−0.005	0.000	0.000
0.7	−1.42	−0.543	−0.134	−0.002	0.000
0.8	−3.03	−1.22	−0.342	−0.009	−0.002
0.9	−11.8	−3.94	−1.05	−0.350	−0.010

and

$$\Delta D_2=-4p_1^2\sin\theta a_{(1)}^{(1)}\Bigg\{\frac{\alpha}{\pi}\frac{\alpha^2Z^2m}{p_1K^2}\Bigg(\Phi+\Psi-\Omega+\left[\Gamma\left(\frac{4p_1^2}{m^2}\right)-\Gamma\left(\frac{q^2}{m^2}\right)\right]\ln\frac{2(W_1-m)}{m}\Bigg)+\frac{\alpha}{2\pi}\operatorname{Im}b_{(2)}^{(2)}\Bigg([(q^2+2m^2)A(q)-2]\times\ln\left(\frac{\Delta W}{W_1-m}\right)+\frac{W_1}{p_1}\ln\frac{W_1+p_1}{W_1-p_1}-\tfrac{1}{2}(q^2+2m^2)E(q)\Bigg)\Bigg\}, \quad (39)$$

from which one deduces

$$\delta=\frac{\alpha}{2\pi}\Bigg\{\left[\frac{q^2-4m^2}{4}-W_1^2+\frac{8W_1^2m^2}{(4W_1^2-q^2)}\right]A(q)-[(q^2+2m^2)A(q)-2]\left[1+\ln\frac{2(W_1-m)}{m}\right]+\tfrac{1}{2}(q^2+2m^2)B(q)-4C(q)+2\left[\ln\left(\frac{2p_1}{q}\right)\right]^{-1}\times\Bigg[\Phi(\beta,\theta)+\Psi(\beta,\theta)-\Omega(\beta,\theta)+\left(\Gamma\left(\frac{4p_1^2}{m^2}\right)-\Gamma\left(\frac{q^2}{m^2}\right)\right)\ln\frac{2(W_1-m)}{m}\Bigg]\Bigg\}. \quad (40)$$

Equation (40) is evaluated numerically for several values of β and θ to give the results for $\delta(\beta,\theta)$ which are presented in Table II.[11]

TABLE II. Numerical values of $(2\pi/\alpha)\delta(\beta,\theta)$.

$\beta \backslash \theta$	30°	60°	90°	120°	150°
0.6	6.78	5.75	4.92	4.47	4.38
0.7	8.00	6.44	5.17	4.60	4.61
0.8	11.3	8.31	6.06	5.36	5.70
0.9	26.6	15.2	9.15	8.44	10.2

[11] A program for the computation of $\delta(\beta,\theta)$ for arbitrary values of β and θ is available.

VI. DISCUSSION

In the preceding sections we have shown that the dominant term (in a series expansion in powers of αZ) of the Coulomb scattering asymmetry function can be written in the form

$$P=P_0(1+\delta),$$

where P_0 is the asymmetry function in the absence of radiative effects, and $\delta(\beta,\theta)$, which is defined in Eq. (40), gives the radiative corrections. We have noted that $\delta(\beta,\theta)$ is independent of the energy resolution ΔW provided that $\Delta W \ll m$. From the values of $\delta(\beta,\theta)$ given in Table II it is apparent that for a given value of θ, δ increases with increasing β. Similarly, for a given value of β, the largest radiative effects occur for small scattering angles. This latter result is not surprising in view of the optical theorem which relates the forward scattering amplitude to the integrated bremsstrahlung cross section.

Within the range of values of β and θ listed in Table II, a maximum value of the radiative corrections of about 3% occurs at $\beta=0.9$ and $\theta=30°$. From these results it is clear that the radiative corrections cannot account for the discrepancy between the experimental and theoretical values of the asymmetry function.

ACKNOWLEDGMENTS

The authors wish to thank Professor G. W. Ford and Professor N. Sherman for valuable conversations. Thanks are also due to R. P. Brumbach, R. C. O'Connell, and J. D. Rozics for checking various parts of this calculation. We wish to thank the staffs of the Midwestern Universities Research Association and Notre Dame computing centers for assistance with the numerical program.

APPENDIX

The following symbols occur at various places in the text. Their definitions are collected here for simplicity.

$$A(q)=\int_0^1\frac{dv}{m^2+q^2v(1-v)}. \quad (A1)$$

$$B(q)=\int_0^1\frac{dv}{m^2+q^2v(1-v)}\ln\left(1+\frac{q^2}{m^2}v(1-v)\right). \quad (A2)$$

$$C(q)=\int_0^1 dv\,v(1-v)\ln\left(1+\frac{q^2}{m^2}v(1-v)\right). \quad (A3)$$

$$E(q)=\int_0^1\frac{dv\,W_1}{(m^2+q^2v(1-v))[p_1^2-q^2v(1-v)]^{\frac{1}{2}}}\times\ln\left(\frac{W_1+[p_1^2-q^2v(1-v)]^{\frac{1}{2}}}{W_1-[p_1^2-q^2v(1-v)]^{\frac{1}{2}}}\right). \quad (A4)$$

The renormalized vertex and vacuum polarization terms may be written in terms of A, B, and C as

$$\Lambda_{4f}(p_1,p_2)=-\frac{\alpha}{2\pi}\Big\{\gamma_4([(q^2+2m^2)A(q)-2][\ln(m/\lambda_0)-1]+\tfrac{1}{4}q^2A(q)+\tfrac{1}{2}(q^2+2m^2)B(q))+\frac{m}{2}q_i\sigma_{i4}A(q)\Big\}, \quad \text{(A5)}$$

$$\pi_f(q)=-\frac{2\alpha}{\pi}\frac{1}{q^2}C(q). \quad \text{(A6)}$$

Furthermore,

$$F(q)=[(q^2+2m^2)A(q)-2]\times[\ln(m/\lambda_0)-1]+\tfrac{1}{4}q^2A(q)+\tfrac{1}{2}(q^2+2m^2)B(q)-m^2A(q)-4C(q), \quad \text{(A7)}$$

$$\Gamma(a)=\int_0^1\frac{vdv}{(1-v)}\ln[1+av(1-v)], \quad \text{(A8)}$$

$$\Lambda(a)=\int_0^1\frac{dv}{v(1-v)}\ln[1+av(1-v)], \quad \text{(A9)}$$

$$\Delta(a)=\int_0^1\frac{dv(1-2v)(\frac{1}{3}v^3-\frac{1}{2}v^2)}{v(1-v)}\ln[1+av(1-v)], \quad \text{(A10)}$$

$$\Xi(a)=\int_0^1\ln[1+av(1-v)]dv, \quad \text{(A11)}$$

$$\Sigma(a)=\int_0^1\frac{dv}{v(1-v)}\{\tfrac{1}{2}\ln^2[1+av(1-v)]-v(1-v)\ln^2[1+av(1-v)]-2v(1-v)\mathfrak{L}_2[-av(1-v)]\}, \quad \text{(A12)}$$

where

$$\mathfrak{L}_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$$

is Euler's dilogarithm,

$$\Theta\left(\frac{q^2}{m^2},\frac{4p_1^2}{m^2}\right)=\int_0^1dv\left[\ln\left(\frac{1+(q^2/m^2)v(1-v)}{1+(4p_1^2/m^2)v(1-v)}\right)-\frac{1}{1+(q^2/m^2)v(1-v)}\times\ln\left(\frac{1+(q^2/m^2)v(1-v)}{1+(4p_1^2/m^2)v(1-v)}\right)\right], \quad \text{(A13)}$$

$$\Phi(\beta,\theta)=\left\{\Gamma\left(\frac{4p_1^2}{m^2}\right)-\tfrac{1}{8}\Lambda\left(\frac{4p_1^2}{m^2}\right)+2\Delta\left(\frac{4p_1^2}{m^2}\right)-\tfrac{1}{2}\Xi\left(\frac{4p_1^2}{m^2}\right)-\tfrac{1}{4}\Sigma\left(\frac{4p_1^2}{m^2}\right)+\tfrac{1}{2}\ln\left(\frac{4p_1^2}{m^2}\right)\right\}-\left\{\frac{4p_1^2}{m^2}\to\frac{q^2}{m^2}\right\}+\frac{W_1^2K^2}{2m^2q^2}\left\{2[m^2A(q)-1]\times\ln(q/2p_1)+\Theta\left(\frac{q^2}{m^2},\frac{4p_1^2}{m^2}\right)-m^2B(q)\right\}, \quad \text{(A14)}$$

$$\Psi(\beta,\theta)=\left[\frac{1}{4}\frac{W_1}{p_1}\ln\frac{W_1+p_1}{W_1-p_1}+\tfrac{1}{8}(q^2+2m^2)E(q)\right]\times\ln\left(\frac{4p_1^2}{q^2}\right)-\tfrac{1}{8}H(q^2,4p_1^2)-\tfrac{1}{8}m^2I(q^2,4p_1^2), \quad \text{(A15)}$$

with

$$H(q^2,4p_1^2)=\int_0^1\frac{dv}{v(1-v)}\left[\ln^2\left(\frac{W_1+[p_1^2-q^2v(1-v)]^{\frac{1}{2}}}{W_1-[p_1^2-q^2v(1-v)]^{\frac{1}{2}}}\right)-(q^2\to 4p_1^2)\right], \quad \text{(A16)}$$

$$I(q^2,4p_1^2)=\int_0^1\frac{dv(1-2v)}{(1-v)}\times\left[\frac{W_1}{[m^2+q^2v(1-v)][p_1^2-q^2v(1-v)]^{\frac{1}{2}}}\times\ln\left(\frac{W_1+[p_1^2-q^2v(1-v)]^{\frac{1}{2}}}{W_1-[p_1^2-q^2v(1-v)]^{\frac{1}{2}}}\right)-(q^2\to 4p_1^2)\right]. \quad \text{(A17)}$$