Formal Theory of Rearrangement Collisions. II*

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A recent derivation of a new formulation of the rearrangement collision problem is extended by utilizing an extension of the theory of the optical potential.

I. INTRODUCTION

In a recent note[1] a new formulation of the transition matrix for a rearrangement collision was derived. In this form the difficulty arising from the fact that the initial and final states in the matrix element are not orthogonal was eliminated. A by-product was the result that the effective interaction in the matrix element was not the potential in the initial (final) state, but only that part of the potential which changes the initial (final) state.

In another recent note[2] the idea of the optical potential was extended to include the description by a set of coupled optical potential equations. The ideas of this note are used here to motivate an extension of I. With this new result the effective potential is now only that part of the potential which changes a whole set of initial (final) states.

An alternative derivation of the result of I is presented in the next section. This derivation is then extended in Sec. III to give the new result. The result is then discussed and the problems associated with it pointed out.

II. DERIVATION OF THE PREVIOUS RESULT

In this section we shall simply quote the usually accepted result for the general rearrangement collision. We shall then show in what sense the results of I are equivalent to this result. In the process we shall introduce the notation which will be necessary for the proof of the new result of this note in Sec. III.

The general rearrangement collision can be denoted by

$$A_0+B_0\rightarrow C_0+D_0, \tag{1}$$

where A, B, C, D are each systems with internal degrees of freedom and the subscripts are used to denote that each is in its ground state (this assumption can easily be removed). The total Hamiltonian H can be broken up into the Hamiltonian, H_i, of the initial non-interacting particles, A and B and their interaction V_i. Or it may be broken up into the Hamiltonian, H_f, of the final non-interacting particles, C and D and their interaction V_f. That is,

$$H=H_i+V_i=H_f+V_f. \tag{2}$$

We may define a set, $\chi_i(p,n)$ of initial states which represents A and B in the internal state (n) and relative motion in the plane wave state p. This satisfies

$$[E_i(p,n)-H_i]\chi_i(p,n)=0. \tag{3}$$

Similarly the final non-interacting states $\chi_f(p,n)$ satisfy

$$[E_f(p,n)-H_i]\chi_f(p,n)=0. \tag{4}$$

The total wave function for the system can be written

$$\Psi_i^{(+)}(p,0)=\chi_i(p,0)+(1/a_i)V_i\Psi_i^{(+)}(p,0), \tag{5}$$

where

$$a_i=E-H_i+i\eta, \tag{6}$$

and η is a positive *infinitesimal*, which is the mathematical device used to insure that $\Psi^{(+)}$ has only outgoing waves asymptotically. We may also define the time reversed wave function which will also be useful

$$\Psi_f^{(-)}(p,0)=\chi_f(p,0)+(1/a_f^\dagger)V_f\Psi_f^{(-)}(p,0), \tag{7}$$

where

$$a_f^\dagger=E-H_f-i\eta. \tag{8}$$

We now quote the usual[3] forms for the transition matrix for (1)

$$T=\langle\chi_f(p',0)\,|\,V_f\,|\,\Psi_i^{(+)}(p,0)\rangle =\langle\Psi_f^{(-)}(p',0)\,|\,V_i\,|\,\chi_i(p,0)\rangle. \tag{9}$$

We shall write these in another form by noting that

$$a=a_f-V_f=a_i-V_i \tag{10}$$

and

$$a\Psi_i^{(+)}=i\eta\chi_i, \tag{11a}$$

$$a^\dagger\Psi_f^{(-)}=-i\eta\chi_f. \tag{11b}$$

Thus (suppressing the arguments)

$$T=i\eta\{\langle\chi_f|\Psi_i^{(+)}\rangle-\langle\chi_f|\chi_i\rangle\} =i\eta\{\langle\Psi_f^{(-)}|\chi_i\rangle-\langle\chi_f|\chi_i\rangle\}. \tag{12}$$

It is a simple matter (see Sec. III) to show that $\langle\chi_f|\chi_i\rangle$ is a finite number so that its contribution to T vanishes in the physical limit, $\eta=0$.

In order to discuss I, it is necessary to define a pro-

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[1] M. H. Mittleman, Phys. Rev. **122**, 1930 (1961). (Hereafter referred to as I.)

[2] M. H. Mittleman and R. Pu, preceding paper [Phys. Rev. **125**, 370 (1962)].

[3] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

jection operator in the initial (final) state. This projects onto the ground states of A and B (C and D) and leaves their relative motion unchanged.

These operators, $\pi_i(0)$ and $\pi_f(0)$, have the properties

$$[\pi_i,a_i]=[\pi_f,a_f]=0. \tag{13}$$

With these we may also define the optical potential wave functions $\Psi_{c_i}^{(+)}(p,0)$ and $\Psi_{c_f}^{(-)}(p,0)$ by

$$\Psi_{c_i}^{(+)}(p,0)=\pi_i(0)\Psi_i^{(+)}(p,0) \tag{14a}$$

and

$$\Psi_{c_f}^{(-)}(p,0)=\Psi_f^{(-)}(p,0)\pi_f(0). \tag{14b}$$

Using (5) and (7), these satisfy

$$a_i\Psi_{c_i}^{(+)}(p,0)-\pi_i(0)V_i\Psi_i^{(+)}(p,0)=i\eta\chi_i(p,0) \tag{15a}$$

and

$$\Psi_{c_f}^{(-)}(p,0)a_f-\Psi_f^{(-)}(p,0)V_f\pi_f(0)=i\eta\chi_f(p,0). \tag{15b}$$

We may now turn to the results of I

$$T=\langle\Psi_f^{(-)}(p,0)|[V_i,\pi_i(0)]|\Psi_i^{(+)}(p,0)\rangle \tag{16a}$$

and

$$T=\langle\Psi_f^{(-)}(p',0)|[\pi_f(0),V_f]|\Psi_i^{(+)}(p,0)\rangle. \tag{16b}$$

Now substitution of (14), (15), and (11) in (16) yields

$$T=i\eta\{\langle\Psi_f^{(-)}|\chi_i\rangle-\langle\chi_f|\Psi_{c_i}^{(+)}\rangle\} \tag{17a}$$

and

$$T=i\eta\{\langle\chi_f|\Psi_i^{(+)}\rangle-\langle\Psi_{c_f}^{(-)}|\chi_i\rangle\}. \tag{17b}$$

We assert here that the last term in each of the brackets is a finite number in the physical limit, $\eta=0$, so that these are equivalent to (12). This will be demonstrated in the appendix.

III. THE EXTENDED RESULT

The idea behind the derivation of (16) given in I was that elastic scattering contains no rearrangement. That is: although the elastic scattering wave function, $\Psi_c(0)$, is not formally orthogonal to the rearranged state the physically observable part contains no rearranged state. Another way of saying this is that the initial state (A_0+B_0) is orthogonal to the final state (C_0+D_0) when they (A and B) or (C and D) are well separated. (One might think here of the exchange of an electron in a collision between two atoms.) Thus all the elastic scattering may be removed from the wave function before the rearranged amplitude is extracted. This makes the effective interaction only that part which changes the ground state.

We now remark that there is a large class of in-elastic scatterings which satisfy the same conditions. That is, the states $(A_n+B_{n'})$ have the same property provided only that n and n' are bound states. Therefore, we now conjecture that the following are also valid expressions for the T matrix:

$$T=\langle\Psi_f^{(-)}(p',0)|[\pi_f,V_f]|\Psi_i^{(+)}(p,0)\rangle \tag{18a}$$

and

$$T=\langle\Psi_f^{(-)}(p',0)|[V_i,\pi_i]|\Psi_i^{(+)}(p,0)\rangle. \tag{18b}$$

Here π_i and π_f are projection operators onto *any* sum of *bound* states

$$\pi_i=\sum_{n=0}\pi_i(n),\quad \pi_f=\sum_{n=0}\pi_f(n). \tag{19}$$

In order to demonstrate this, we use the results of the preceding paper which indicates that we need merely replace $\pi(0)\rightarrow\pi$ and carry over the results.

Equation (18) can now be put into the form of (17), except that $\Psi_{c_i}^{(+)}$ and $\Psi_{c_f}^{(-)}$ must now be interpreted as a sum in the light of the new definition of π. Again we assert and leave for the appendix the demonstration that the last term in (17) gives no contribution (even with the new definition of Ψ_c). Thus, our extended result is

$$T=\langle\Psi_f^{(-)}(p',0)|[V_i,\pi_i]|\Psi_i^{(+)}(p,0)\rangle \tag{20a}$$

and

$$T=\langle\Psi_f^{(-)}(p',0)|[\pi_f,V_f]|\Psi_i^{(+)}(p,0)\rangle. \tag{20b}$$

We reiterate, the π's are sums over *any* group of *bound* states containing the initial (final) state.

We have shown that (20) and (9) are equivalent and that (9) is simpler if the exact solutions are available. The question is, which is the best form to start with when approximations are necessary. If one is limited to the first Born approximations for $\Psi^{(+)}$, then the two forms of (9) give the same result. This is not the case for (20). On the other hand, there is no reason to expect that approximation on the two forms should give identical results. We are expanding in different potentials (V_i or V_f) depending on which form we use. Arguments concerning the lack of orthogonality between the initial and final states indicate that (20) is superior to (9), but that does not specify which form of π we should use in (20).

APPENDIX

We shall consider here a rearrangement in which only one particle "changes place." For definiteness the reaction (1) will be considered in which A and B are two atoms and C is A less one electron and D is B plus one electron. We shall only consider the matrix element

$$M=\langle\Psi_{c_f}^{(-)}(n),\chi_i\rangle. \tag{A1}$$

This can be written explicitly as

$$M=\int\Phi_n^{(c)}(\chi_1\cdots\chi_{z-1})\Phi_m^{(d)}(y_1\cdots y_{z'},y_{z'+1})\Psi_{n,m}^{(-)}(R')\times\Phi_0^{(A)}(\chi_1\cdots\chi_z)\Phi_0^{(B)}(y_1\cdots y_{z'})\times e^{i\mathbf{p}\cdot\mathbf{R}}d\chi_1\cdots d\chi_z dy_1\cdots dy_{z'}dR. \tag{A2}$$

We have assumed that A has z and B has z' electrons. The coordinates $\chi_1\cdots\chi_z$, and $y_1\cdots y_{z'}$ are the coordinates of the electrons relative to their respective nuclei. R is the relative coordinate between the center of mass of the two atoms (we do not set the ratio of electron to nuclear mass equal to zero). $y_{z'+1}$ is the coordinate of the

exchanged electron, (χ_z), relative to its new nucleus, and R' is the relative coordinate between the center of masses of the two atoms in the final configuration. We must now express the coordinates $y_{z'+1}$ and R' in terms of the coordinates $\chi_1 \cdots \chi_z$, $y_1 \cdots y_{z'}$, R in order to perform the integration. This can be accomplished by some simple but lengthy algebra. The result is

$$y_{z'+1}=R+\chi_z-\frac{m}{M_1}\sum^{z}\chi_i+\frac{m}{M_2}\sum^{z'}y_i, \qquad \text{(A3)}$$

and

$$R'=\frac{M_2}{M_{2'}}R+\frac{m^2}{M_1M'}\sum^{z}\chi_i-\frac{m}{M'}\chi_z. \qquad \text{(A4)}$$

Here m is the electron mass, M_1 and M_2 are the total mass of A and B, respectively, $M_{2'}$ is the total mass of D, and M' is the reduced mass of the combination C and D.

The integral may now be performed. We note that the factors $\Phi_0^{(A)}(\chi_1 \cdots \chi_z)\Phi_0^{(B)}(y_1 \cdots y_{z'})$, being bound states, keep the χ and y integrations finite and limit the contribution to finite regions of χ and y. We also note that y_{z+1} depends upon R and if $\Phi_m^{(D)}$ is a bound state it will vanish exponentially for large R. Thus all integrations in M are finite and the assertion in the text is proven. It should be noted that the requirements here are somewhat weaker than those stated in the text; $\Phi_m^{(D)}$ had to be bound, but not $\Phi_n^{(C)}$.

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Possible Neutrinoless Decay Modes of the Muon*

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A phenomenological theory is given of possible "two-photon" neutrinoless muon → electron decay processes: $\mu^\pm \to e^\pm+\gamma+\gamma$; μ^-+nucleus → $\mu^-+\gamma$(virtual Coulomb)+nucleus → $e^-+\gamma$+nucleus; μ^-+nucleus → $\mu^-+\gamma$(virtual Coulomb)+γ(virtual Coulomb)+nucleus → e^-+nucleus. It is found that the decay, μ^-+nucleus → $e^-+\gamma$+nucleus, is the most probable of the three under consideration and a search for this as yet experimentally uninvestigated decay is suggested.

I

THE relationship between the muon ($\mu^\pm$) and the electron ($e^\pm$) is perhaps even more obscure than that between any other two elementary particle species. Thus, (a) the measured $\mu^\pm - e^\pm$ mass difference is not accompanied by any so far detected dynamical difference between the $\mu^\pm$ and $e^\pm$ (nongravitational) interactions, and (b) the neutrinoless $\mu^\pm \to e^\pm$ decays, i.e., $\mu^+ \to e^+ + \gamma$, $\mu^+ \to e^+ + e^+ + e^-$, μ^-+nucleus → e^-+nucleus, $\mu^+ \to e^+ + \gamma + \gamma$, $\mu^+ + e^- \to \gamma + \gamma$, etc., occur at rates smaller than 10^{-8} to 10^{-5} that of $\mu^\pm \to e^\pm + \nu + \bar{\nu}$,[1] although these neutrinoless decays are not forbidden by any well-established selection rule. It is conceivable that such an intimate connection exists between (a) and (b) that it is futile to attempt to treat any aspect of the latter without at least some comprehension of the former—on the other hand, it is equally conceivable that all the neutrinoless decays are rigorously forbidden by a selection rule whose general significance can be appreciated without any deep understanding of (a). However this may be, we shall here suppose that the various neutrinoless $\mu^\pm \to e^\pm$ decays occur at low but nonvanishing rates and that a meaningful phenomenological treatment of these rates can be given without concern about (a). In fact, we shall consider a possible two-photon mode of neutrinoless decay: $\mu^\pm \to e^\pm + \gamma + \gamma$ and shall assume, for the practical relevance of our discussion, that the various "one-photon" neutrinoless decays[2]: $\mu^\pm \to e^\pm + \gamma$; μ^-+nucleus → $\mu^-+\gamma$(virtual

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