exchanged electron, (χ_z), relative to its new nucleus, and R' is the relative coordinate between the center of masses of the two atoms in the final configuration. We must now express the coordinates $y_{z'+1}$ and R' in terms of the coordinates $\chi_1\cdots\chi_z$, $y_1\cdots y_{z'}$, R in order to perform the integration. This can be accomplished by some simple but lengthy algebra. The result is

$$y_{z'+1}=R+\chi_z-\frac{m}{M_1}\sum^{z}\chi_i+\frac{m}{M_2}\sum^{z'}y_i, \tag{A3}$$

and

$$R'=\frac{M_2}{M_{2'}}R+\frac{m^2}{M_1M'}\sum^{z}\chi_i-\frac{m}{M'}\chi_z. \tag{A4}$$

Here m is the electron mass, M_1 and M_2 are the total mass of A and B, respectively, $M_{2'}$ is the total mass of D, and M' is the reduced mass of the combination C and D.

The integral may now be performed. We note that the factors $\Phi_0^{(A)}(\chi_1\cdots\chi_z)\Phi_0^{(B)}(y_1\cdots y_{z'})$, being bound states, keep the χ and y integrations finite and limit the contribution to finite regions of χ and y. We also note that y_{z+1} depends upon R and if $\Phi_m^{(D)}$ is a bound state it will vanish exponentially for large R. Thus all integrations in M are finite and the assertion in the text is proven. It should be noted that the requirements here are somewhat weaker than those stated in the text; $\Phi_m^{(D)}$ had to be bound, but not $\Phi_n^{(C)}$.

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Possible Neutrinoless Decay Modes of the Muon*

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A phenomenological theory is given of possible "two-photon" neutrinoless muon $\rightarrow$ electron decay processes: $\mu^\pm\rightarrow e^\pm+\gamma+\gamma$; μ^-+nucleus $\rightarrow\mu^-+\gamma$(virtual Coulomb)+nucleus $\rightarrow e^-+\gamma$+nucleus; μ^-+nucleus $\rightarrow\mu^-+\gamma$(virtual Coulomb)$+\gamma$(virtual Coulomb)+nucleus $\rightarrow e^-$+nucleus. It is found that the decay, μ^-+nucleus $\rightarrow e^-+\gamma$+nucleus, is the most probable of the three under consideration and a search for this as yet experimentally uninvestigated decay is suggested.

I

The relationship between the muon ($\mu^\pm$) and the electron ($e^\pm$) is perhaps even more obscure than that between any other two elementary particle species. Thus, (a) the measured $\mu^\pm-e^\pm$ mass difference is not accompanied by any so far detected dynamical difference between the $\mu^\pm$ and $e^\pm$ (nongravitational) interactions, and (b) the neutrinoless $\mu^\pm\rightarrow e^\pm$ decays, i.e., $\mu^+\rightarrow e^++\gamma$, $\mu^+\rightarrow e^++e^++e^-$, μ^-+nucleus $\rightarrow e^-$+nucleus, $\mu^+\rightarrow e^++\gamma+\gamma$, $\mu^++e^-\rightarrow\gamma+\gamma$, etc., occur at rates smaller than 10^{-8} to 10^{-5} that of $\mu^\pm\rightarrow e^\pm+\nu+\bar{\nu}$,[1] although these neutrinoless decays are not forbidden by any well-established selection rule. It is conceivable that such an intimate connection exists between (a) and (b) that it is futile to attempt to treat any aspect of the latter without at least some comprehension of the former—on the other hand, it is equally conceivable that all the neutrinoless decays are rigorously forbidden by a selection rule whose general significance can be appreciated without any deep understanding of (a). However this may be, we shall here suppose that the various neutrinoless $\mu^\pm\rightarrow e^\pm$ decays occur at low but nonvanishing rates and that a meaningful phenomenological treatment of these rates can be given without concern about (a). In fact, we shall consider a possible two-photon mode of neutrinoless decay: $\mu^\pm\rightarrow e^\pm+\gamma+\gamma$ and shall assume, for the practical relevance of our discussion, that the various "one-photon" neutrinoless decays[2]: $\mu^\pm\rightarrow e^\pm+\gamma$; μ^-+nucleus $\rightarrow\mu^-+\gamma$(virtual

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[1] *For* $\mu^+\rightarrow e^++\gamma$: S. Frankel, J. Halpern, L. Holloway, W. Wales, M. Yearian, O. Chamberlain, A. Lemonick, and F. Pipkin, Phys. Rev. Letters **8**, 123 (1962); D. Bartlett, S. Devons, and A. M. Sachs, *ibid.* 120 (1962). S. Frankel, V. Hagopian, J. Halpern, and A. L. Whetstone, Phys. Rev. **118**, 589 (1960). J. Ashkin, T. Fazzini, G. Fidecaro, N. H. Lipman, A. W. Merrison, and J. Paul, Nuovo cimento **14**, 1266 (1959). D. Berley, J. Lee, and M. Bardon, Phys. Rev. Letters **2**, 357 (1959). H. Davis, A. Roberts, and T. Zipf, Phys. Rev. Letters **2**, 221 (1959). T. O'Keefe, M. Rigby, and J. Wormald, Proc. Phys. Soc. (London) **173**, 951 (1959).

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For μ^-*+nucleus* $\rightarrow e^-$*+nucleus:* M. Conversi, L. diLella, G. Penso, M. Toller, and C. Rubbia, Phys. Rev. Letters **8**, 125 (1962); R. D. Sard, K. M. Crowe, and H. Kruger, Phys. Rev. **121**, 619 (1961). M. Conversi, L. diLella, A. Egidi, C. Rubbia, and M. Toller, Phys. Rev. **122**, 687 (1961); Nuovo cimento **18**, 1283 (1960).

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[2] Thorough phenomenological treatments of $\mu^\pm\rightarrow e^\pm+\gamma$; μ^-+nucleus $\rightarrow\mu^-+\gamma$(virtual Coulomb)+nucleus $\rightarrow e^-$+nucleus, and, of $\mu^\pm\rightarrow e^\pm+\gamma$(virtual Dalitz) $\rightarrow e^\pm+e^++e^-$, have been given by S. Weinberg and G. Feinberg, Phys. Rev. Letters **3**, 111, 244 (1959), and by M. Bander and G. Feinberg, Phys. Rev. **119**, 1427 (1960), respectively.

Coulomb)+nucleus $\rightarrow e^-$+nucleus; and $\mu^\pm \rightarrow e^\pm + \gamma$(virtual Dalitz) $\rightarrow e^\pm + e^+ + e^-$ are appreciably less probable than $\mu^\pm \rightarrow e^\pm + \gamma + \gamma$. We shall not speculate on the origin of this relative inhibition of the "one-photon" neutrinoless $\mu^\pm \rightarrow e^\pm$ processes but merely remark that if such a relative inhibition is actually operative, its presence is presumably ultimately due to the same physical factors which are responsible for the large value of $m^{(\mu)}/m^{(e)}$ [i.e., (a)] and for the generally small probability of the neutrinoless $\mu^\pm \rightarrow e^\pm$ decays [i.e., (b)]. More specifically, it may, however, be mentioned that in one form of the intermediate boson theory of weak interactions such electromagnetic properties can be ascribed to the boson that the rates of $\mu^\pm \rightarrow e^\pm + \gamma$; μ^-+nucleus $\rightarrow \mu^- + \gamma$(virtual Coulomb) +nucleus $\rightarrow e^-$+nucleus; and $\mu^\pm \rightarrow e^\pm + \gamma$(virtual Dalitz) $\rightarrow e^\pm + e^+ + e^-$ are all effectively suppressed[3] with the result that the decay $\mu^\pm \rightarrow e^\pm + \gamma + \gamma$ is expected to be more probable than any of these "one-photon" neutrinoless $\mu^\pm \rightarrow e^\pm$ decays.

II

We begin by setting down the expression for a Lorentz-invariant and gauge-invariant matrix element associated with the two-photon neutrinoless decay: $\mu^\pm \rightarrow e^\pm + \gamma + \gamma$. Such a matrix element is conveniently expressed in terms of a corresponding Lorentz-invariant and gauge-invariant effective Lagrangian, $\mathcal{L}(x)$, as ($\hbar = 1$, $c = 1$)

$$\left\langle e^\pm \gamma^{(1)} \gamma^{(2)} \middle| \int d^4x \, \mathcal{L}(x) \middle| \mu^\pm \right\rangle = (2\pi)^4 \delta^{(4)}(p^{(1)} + p^{(2)} + p^{(e)} - p^{(\mu)}) \times \langle e^\pm \gamma^{(1)} \gamma^{(2)} | \mathcal{L}(0) | \mu^\pm \rangle, \quad (1)$$

with

$$\mathcal{L}(x) = (m^{(\mu)})^{-3} [g_1 \mathbf{E}(x) \cdot \mathbf{B}(x) + g_2 \tfrac{1}{2} (\mathbf{E}^2(x) - \mathbf{B}^2(x))] \cdot [\bar{\psi}_e(x) \psi_\mu(x)] + \text{H.c.}, \quad (2)$$

where $\mathbf{E}(x)$, $\mathbf{B}(x)$, $\psi_e(x)$, and $\psi_\mu(x)$ are the photon electric vector, photon magnetic vector, electron spinor, and muon spinor quantized field amplitudes and g_1, g_2 are dimensionless coupling constants.[4] The $\mathcal{L}(x)$ of Eq. (2) is not the most general Lorentz-invariant and gauge-invariant effective Lagrangian for two-photon neutrinoless decay since the field amplitudes entering into it are all taken at the same space-time point x but this $\mathcal{L}(x)$ is nevertheless adequate to demonstrate the various interesting physical features of the "two-photon" neutrinoless decays: $\mu^\pm \rightarrow e^\pm + \gamma + \gamma$; μ^-+nucleus $\rightarrow \mu^- + \gamma$(virtual Coulomb)+nucleus $\rightarrow e^- + \gamma$ +nucleus; μ^-+nucleus $\rightarrow \mu^- + \gamma$(virtual Coulomb) $+\gamma$(virtual Coulomb)+nucleus $\rightarrow e^-$+nucleus.[5]

We proceed to calculate the decay rate: $\Gamma(\mu^\pm \rightarrow e^\pm + \gamma + \gamma)$. A straightforward evaluation gives

$$\begin{aligned}
\Gamma(\mu^\pm \rightarrow e^\pm + \gamma + \gamma) &= 2\pi \int \{\delta(|\mathbf{p}^{(1)}| + |\mathbf{p}^{(2)}| + |\mathbf{p}^{(1)} + \mathbf{p}^{(2)}| - m^{(\mu)}) \\
&\quad \times \tfrac{1}{2} \sum_{\substack{\text{spins of} \\ \gamma^{(1)},\, \gamma^{(2)}, \\ e^\pm,\, \mu^\pm}} |\langle e^\pm \gamma^{(1)} \gamma^{(2)} | \mathcal{L}(0) | \mu^\pm \rangle|^2 \} \frac{1}{2} \frac{d\mathbf{p}^{(1)}}{(2\pi)^3} \frac{d\mathbf{p}^{(2)}}{(2\pi)^3} \\
&= 2\pi \int \Bigg\{ \delta(|\mathbf{p}^{(1)}| + |\mathbf{p}^{(2)}| + |\mathbf{p}^{(1)} + \mathbf{p}^{(2)}| - m^{(\mu)}) \\
&\quad \times \left[\frac{(g_1^2 + g_2^2)}{[m^{(\mu)}]^6} (1 - \hat{p}^{(1)} \cdot \hat{p}^{(2)})^2 \left(\frac{|\mathbf{p}^{(1)}|\,|\mathbf{p}^{(2)}|}{4} \right) \right] \Bigg\} \\
&\quad \times \frac{1}{2} \frac{d\mathbf{p}^{(1)}}{(2\pi)^3} \frac{d\mathbf{p}^{(2)}}{(2\pi)^3} \\
&= (128 \times 15\pi)^{-1} (g_1^2 + g_2^2) m^{(\mu)}, \quad (3)
\end{aligned}$$

with a photon-photon angular correlation $\sim (1 - \hat{p}^{(1)} \cdot \hat{p}^{(2)})^2$, a photon momentum distribution $\sim x^3[1 - \frac{3}{4}x]$ ($xm^{(\mu)}/2 \equiv |\mathbf{p}^{(1)}|$ or $|\mathbf{p}^{(2)}|$; $0 \leqq x \leqq 1$), and an electron momentum distribution $\sim y^2[1-y]^2$ ($ym^{(\mu)}/2 \equiv |\mathbf{p}^{(e)}| = |\mathbf{p}^{(1)} + \mathbf{p}^{(2)}|$; $0 \leqq y \leqq 1$). Thus, if the experimental discrimination for the energies and angles of the $\gamma^{(1)}$, $\gamma^{(2)}$ is not too sharp and if no attempt is made to detect the e^+, a fairly high proportion of any $\mu^+ \rightarrow e^+ + \gamma^{(1)} + \gamma^{(2)}$ events kinematically simulate the two-photon neutrinoless muonium annihilations: $\mu^+ + e^- \rightarrow \gamma^{(1)} + \gamma^{(2)}$, and a limit

$$(g_1^2 + g_2^2) < 1 \times 10^{-19}, \quad (4)$$

can be obtained from Eq. (3) and the experimental upper limit[1]

$$\left[\frac{\Gamma(\mu^+ \rightarrow e^+ + \gamma + \gamma) + \Gamma(\mu^+ + e^- \rightarrow \gamma + \gamma)}{\Gamma(\mu^+ \rightarrow e^+ + \nu + \bar{\nu})} \right]_{\text{exper}} < 5 \times 10^{-6}. \quad (5)$$

We also calculate, on the basis of the $\mathcal{L}(x)$ of Eq. (2), the muonium annihilation rate:

$$\Gamma(\mu^+ + e^- \rightarrow \gamma + \gamma) = (\tfrac{1}{4}) \left(\frac{m^{(e)}}{m^{(\mu)}} \times \frac{1}{137} \right)^3 (g_1^2 + g_2^2) m^{(\mu)}, \quad (6)$$

so that

$$\frac{\Gamma(\mu^+ + e^- \rightarrow \gamma + \gamma)}{\Gamma(\mu^+ \rightarrow e^+ + \gamma + \gamma)} = (32 \times 15\pi) \left(\frac{m^{(e)}}{m^{(\mu)}} \times \frac{1}{137} \right)^3 \ll 1, \quad (7)$$

[3] S. A. Bludman and J. A. Young, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), p. 564.

[4] If both g_1 and g_2 are nonvanishing, the corresponding $\mathcal{L}(x)$ violates parity conservation. Also, no essential generalization is obtained if one replaces $[\bar{\psi}_e \psi_\mu]$ by $[\bar{\psi}_e (1 + \delta\gamma_5) \psi_\mu]/(1+\delta^2)^{\frac{1}{2}}$ with $0 < \delta \leqq 1$.

[5] Any actual nonlocality in $\mathcal{L}(x)$ is expected to extend over a linear dimension no greater than $[m^{(\mu)}]^{-1}$ and hence can have only a relatively small effect on our various conclusions.

whence it is clear that any approximately 50 Mev more or less antiparallel time-coincident $\gamma^{(1)}$, $\gamma^{(2)}$, emitted in neutrinoless decay by stopped μ^+ in matter, arise predominantly from $\mu^+ \to e^+ + \gamma^{(1)} + \gamma^{(2)}$ rather than from $\mu^+ + e^- \to \gamma^{(1)} + \gamma^{(2)}$. This predominance is a general consequence of the small $\mu^+ - e^-$ overlap in muonium and of our assumption of a local $\mu^+ - e^- - \gamma\gamma$ effective interaction.[5]

III

We now recall that negative muons always decay from lowest Bohr orbits in μ^--mesic atoms so that one or both of the photons in the two-photon neutrinoless $\mu^- \to e^-$ decay may be provided by the atom's nuclear Coulomb field—thus, and as we have already indicated, we anticipate "two-photon" processes of the type: $\mu^- +$nucleus $\to \mu^- + \gamma$(virtual Coulomb)$+$nucleus $\to e^- + \gamma +$nucleus, and $\mu^- +$nucleus $\to \mu^- + \gamma$(virtual Coulomb)$+\gamma$(virtual Coulomb)$+$nucleus $\to e^- +$nucleus. We proceed to discuss the rates of these processes remembering that we must now take within the $\mathfrak{L}(x)$ of Eq. (2):

$$\mathbf{E}(x) = Ze(-\nabla_{\mathbf{x}})(-4\pi/\nabla_{\mathbf{x}}^2)\rho(|\mathbf{x}|) + \{\mathbf{E}(x)\}_{\text{quantized photon field}},$$
$$\mathbf{B}(x) = \{\mathbf{B}(x)\}_{\text{quantized photon field}},$$

where $\rho(|\mathbf{x}|)$ is the (normalized-to-unity) charge distribution within the nucleus. We then have:

$$\begin{aligned}
\Gamma(\mu^- + \text{nucleus} &\to e^- + \gamma + \text{nucleus}) \\
&= 2\pi \int \Bigg\{ \delta(|\mathbf{p}^{(1)}| + |\mathbf{p}^{(e)}| - m^{(\mu)}) \\
&\quad \times \tfrac{1}{2} \sum_{\substack{\text{spins of} \\ \gamma^{(1)},\, e^-,\, \mu^-}} \left| \left\langle e^-\gamma^{(1)} \left| \int d\mathbf{x}\; \mathfrak{L}(\mathbf{x},0) \right| \mu^- \right\rangle \right|^2 \Bigg\} \frac{d\mathbf{p}^{(1)}}{(2\pi)^3} \frac{d\mathbf{p}^{(e)}}{(2\pi)^3} \\
&\cong 2\pi \int \Bigg\{ \delta(|\mathbf{p}^{(1)}| + |\mathbf{p}^{(e)}| - m^{(\mu)}) \Bigg[\frac{(g_1^2+g_2^2)}{(m^{(\mu)})^6} (4\pi)^3 (Ze)^2 \\
&\quad \times \left(\frac{\tfrac{1}{2}(\mathbf{p}^{(1)} \times \mathbf{p}^{(e)})^2}{|\mathbf{p}^{(1)} + \mathbf{p}^{(e)}|^4} \right) (2\mathbf{p}^{(1)})^{-1} \Bigg] \\
&\quad \times \left| \int \exp[-i(\mathbf{p}^{(1)} + \mathbf{p}^{(e)}) \cdot \mathbf{x}] \rho(|\mathbf{x}|) \varphi(|\mathbf{x}|) d\mathbf{x} \right|^2 \Bigg\} \\
&\quad \times \frac{d\mathbf{p}^{(1)}}{(2\pi)^3} \frac{d\mathbf{p}^{(e)}}{(2\pi)^3}, \qquad (8)
\end{aligned}$$

where $\varphi(|\mathbf{x}|)$ is the lowest Bohr orbit space wave function of the μ^-. Approximate evaluation of the integrals over $\mathbf{x}$ and over $\mathbf{p}^{(e)}$ yields, with Z_{eff} defined as in the theory of muon capture[6]:

$$\Gamma(\mu^- + \text{nucleus} \to e^- + \gamma + \text{nucleus}) \approx (2/9\pi) \frac{Z_{\text{eff}}^4 Z}{(137)^4} (g_1^2 + g_2^2) m^{(\mu)}, \qquad (9)$$

so that, using also Eq. (3),

$$\frac{\Gamma(\mu^- + \text{nucleus} \to e^- + \gamma + \text{nucleus})}{\Gamma(\mu^\pm \to e^\pm + \gamma + \gamma)} \approx \left(\frac{256 \times 15}{9} \right) \frac{Z_{\text{eff}}^4 Z}{(137)^4} = \begin{cases} 7: \text{Cu} - Z_{\text{eff}} = 21.1 \\ 130: \text{Pb} - Z_{\text{eff}} = 34.2. \end{cases} \qquad (10)$$

Equation (10) indicates that the "single conversion coefficient" is significantly greater than unity if $Z \geqq 25$. We should also mention that the dependence of

$$\begin{aligned}
&\left\{ \tfrac{1}{2} \sum_{\substack{\text{spins of} \\ \gamma^{(1)},\, e^-,\, \mu^-}} \left| \left\langle e^-\gamma^{(1)} \left| \int d\mathbf{x}\; \mathfrak{L}(\mathbf{x},0) \right| \mu^- \right\rangle \right|^2 \right\} \\
&= \Bigg\{ \frac{\text{const}}{|\mathbf{p}^{(1)}|} \frac{(\mathbf{p}^{(1)} \times \mathbf{p}^{(e)})^2}{|\mathbf{p}^{(1)} + \mathbf{p}^{(e)}|^4} \\
&\qquad \times \left| \int \exp[-i(\mathbf{p}^{(1)} + \mathbf{p}^{(e)}) \cdot \mathbf{x}] \rho(|\mathbf{x}|) \varphi(|\mathbf{x}|) d\mathbf{x} \right|^2 \Bigg\}
\end{aligned}$$

[Eq. (8)] on $\mathbf{p}^{(1)}$, $\mathbf{p}^{(e)}$ is such that the most probable kinematic configurations of the emitted photon and electron are specified by $\mathbf{p}^{(1)} \approx -\mathbf{p}^{(e)} \approx \tfrac{1}{2} m^{(\mu)} \hat{p}^{(1)}$, so that on the average the nucleus takes up relatively little recoil momentum. Physically, this is a consequence of the relatively large abundance of low-momentum virtual photons in the Coulomb field of the nucleus.

We proceed to calculate the rate of $\mu^- +$nucleus $\to \mu^- + \gamma$(virtual Coulomb)$+\gamma$(virtual Coulomb)$+$nucleus $\to e^- +$nucleus and so to obtain the "double conversion coefficient." We have, analogously to Eq. (8)[7]:

$$\begin{aligned}
\Gamma(\mu^- + \text{nucleus} &\to e^- + \text{nucleus}) \\
&= 2\pi \int \Bigg\{ \delta(|\mathbf{p}^{(e)}| - m^{(\mu)}) \\
&\quad \times \tfrac{1}{2} \sum_{\substack{\text{spins of} \\ e^-,\, \mu^-}} \left| \left\langle e^- \left| \int d\mathbf{x}\; \mathfrak{L}(\mathbf{x},0) \right| \mu^- \right\rangle \right|^2 \Bigg\} \frac{d\mathbf{p}^{(e)}}{(2\pi)^3} \\
&= 2\pi \int \Bigg\{ \delta(|\mathbf{p}^{(e)}| - m^{(\mu)}) \left[\frac{(g_2^2)}{4(m^{(\mu)})^6} (Ze)^4 \tfrac{1}{2} \right] \\
&\quad \times \left| \int \exp(-i\mathbf{p}^{(e)} \cdot \mathbf{x}) \left(\nabla_{\mathbf{x}} \frac{4\pi}{\nabla_{\mathbf{x}}^2} \rho(|\mathbf{x}|) \right)^2 \varphi(|\mathbf{x}|) d\mathbf{x} \right|^2 \Bigg\} \\
&\quad \times \frac{d\mathbf{p}^{(e)}}{(2\pi)^3} \approx 6 \frac{Z_{\text{eff}}^4 Z}{(137)^5} (g_2^2) m^{(\mu)}, \qquad (11)
\end{aligned}$$

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[6] See, for example, H. Primakoff, Revs. Modern Phys. **31**, 802 (1959).

[7] Equation (11) and the experimental upper limit on $\Gamma(\mu^- +$nucleus $\to e^- +$nucleus) given in reference 1, viz., $\Gamma(\mu^- +$Cu nucleus $\to e^- +$Cu nucleus$) < (2 \times 10^{-7}) \times \Gamma(\mu^-$ capture by Cu nucleus), yield the limit $g_2^2 < 1 \times 10^{-20}$, which is to be compared with the limit $(g_1^2 + g_2^2) < 1 \times 10^{-19}$ given in Eq. (4).

whence, using also Eq. (9),

$$\frac{\Gamma(\mu^-+\text{nucleus}\to e^-+\text{nucleus})}{\Gamma(\mu^-+\text{nucleus}\to e^-+\gamma+\text{nucleus})}\approx\frac{27\pi}{(137)}\left(\frac{g_2^2}{g_1^2+g_2^2}\right).\quad(12)$$

Equation (12) shows that the ratio of the "double conversion coefficient" to the "single conversion coefficient" is never greater than about 0.6 for the various Z's of interest.[8] If, in addition, $g_2\ll g_1$, as for example will be the case if the $\mathcal{L}(x)$ of Eq. (2) is associated with a nonvanishing matrix element for $\mu^\pm\to e^\pm+\pi^0\to e^\pm+\gamma+\gamma$, the "double conversion coefficient" is negligible compared to the "single conversion coefficient."[9] Since, according to Eq. (10), this "single conversion coefficient" is significantly greater than unity, it is clearly of considerable interest to conduct an experimental search for $\mu^-+\text{nucleus}\to e^-+\gamma+\text{nucleus}$. Such a search is particularly appropriate since, of all the neutrinoless $\mu^\pm\to e^\pm$ decays, the only one so far without experimental investigation is the one which from the present point of view is expected to be the most probable, viz., $\mu^-+\text{nucleus}\to e^-+\gamma+\text{nucleus}$.

[8] Physically speaking, the "double conversion coefficient" is not even bigger than the "single conversion coefficient" because the nucleus necessarily takes up a relatively large recoil momentum ($\cong m^{(\mu)}$) in $\mu^-+\text{nucleus}\to e^-+\text{nucleus}$ while, as noted above, the nucleus' recoil momentum in $\mu^-+\text{nucleus}\to e^-+\gamma+\text{nucleus}$ is predominantly small.

[9] It should, of course, be remembered that if a nonvanishing matrix element $M(\mu^-\to e^-+\pi^0)$ exists for $\mu^-\to e^-+\pi^0$, the strong-interaction nuclear absorption of the π^0 induces $\mu^-+\text{nucleus}\to e^-+\text{nucleus}$ via I: $\mu^-+\text{nucleus}\to e^-+\pi^0+\text{nucleus}\to e^-+\text{nucleus}$ at a rate $\sim(Z_{\text{eff}}^3A)$. Indeed, with $M(\mu^-\to e^-+\pi)\neq 0$, the process I might be more probable than the electroma⁰netic- "single conversion" process [Eqs. (8)–(10)] which in thig case would proceed via II: $\mu^-+\text{nucleus}\to e^-+\pi^0+\text{nucleus}\to s^-+\pi^0+\gamma(\text{virtual Coulomb})+\text{nucleus}\to e^-+\gamma+\text{nucleus}$ while, eas al ready indicated, the electromagnetic "double conversion" process, III: $\mu^-+\text{nucleus}\to e^-+\pi^0+\text{nucleus}\to e^-+\pi^0+\gamma$ (virtual Coulomb)$+\gamma$(virtual Coulomb)$+\text{nucleus}\to e^-+\text{nucleus}$ would in this case be forbidden [Eqs. (11) and (12) with $g_2=0$]. Under these circumstances, any actual nonvanishing rate for $\mu^-+\text{nu-leus}\to e^-+\text{nucleus}$ would have to be ascribed to process I.

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Quantization and the Classical Hamilton-Jacobi Equation

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In this paper, a formalism for quantization is developed which starts out from the Hamilton-Jacobi expression, $\partial S/\partial t+H(\partial S/\partial q,q)$, and which leads to its usual quantum-mechanical operator equivalent by means of straightforward algebra. The quantum-mechanical operator equivalents of H and p are then seen to be the consequence of assigning a number of equally probable classical paths to a dynamical system.

INTRODUCTION

MANY different formalisms have been developed for passing from a classical Hamiltonian or Lagrangian to a quantum-mechanical operator, but there still remains some mystery about why the differential operators which replace the momenta and energy should have the form they are given. Dirac[1] in his discussion of the action principle in his book and in subsequent papers clarifies things considerably by showing just how classical action is related to the arbitrary phase S in the time-dependent wave function $\psi\exp[(iS/\hbar)]$.

Feynman[2] in his Lagrangian formulation of the quantum mechanics goes a step further to show how the probability amplitude for a particular space-time path of a system is related to the classical Lagrangian $L(X(t),X(t))$ for this path. The total probability amplitude for a system's going from an initial state A to a final state B is the sum of all the probability amplitudes for all the possible paths from A to B. Each such path contributes equally in magnitude to the probability amplitude, but the phase for each path is different and equal to the classical action (in units of $\hbar$) for each path.

As Feynman points out, the contribution from a given path is "proportional to $\exp[iS(X(t))/\hbar]$, where the action $S(X(t))=\int L(\dot{X}(t),X(t))dt$ is the time integral of the classical Lagrangian taken along the path in question." He shows that there is a close analogy between $\langle X'|X\rangle_\epsilon$ (the probability amplitude for finding the system at X' at time $t+\epsilon$ if it was at X at a time t) and $\exp[iS(X,X')/\hbar]$, and says: "In fact we now see that to a sufficient approximation the two quantities may be taken proportional to each other."

In view of the intimate relationship between S which, as Dirac[1] has pointed out, is the classical Hamilton-Jacobi function of the problem, and the

[1] P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, New York, 1957), 4th ed., p. 125.
[2] R. Feynman, Revs. Modern Phys. 20, 367 (1948).