whence, using also Eq. (9),

$$\frac{\Gamma(\mu^- + \text{nucleus} \to e^- + \text{nucleus})}{\Gamma(\mu^- + \text{nucleus} \to e^- + \gamma + \text{nucleus})} \approx \frac{27\pi}{(137)}\left(\frac{g_2^2}{g_1^2+g_2^2}\right). \quad (12)$$

Equation (12) shows that the ratio of the "double conversion coefficient" to the "single conversion coefficient" is never greater than about 0.6 for the various Z's of interest.[8] If, in addition, $g_2 \ll g_1$, as for example will be the case if the $\mathcal{L}(x)$ of Eq. (2) is associated with a nonvanishing matrix element for $\mu^\pm \to e^\pm + \pi^0 \to e^\pm + \gamma + \gamma$, the "double conversion coefficient" is negligible compared to the "single conversion coefficient."[9] Since, according to Eq. (10), this "single conversion coefficient" is significantly greater than unity, it is clearly of considerable interest to conduct an experimental search for $\mu^- + \text{nucleus} \to e^- + \gamma + \text{nucleus}$. Such a search is particularly appropriate since, of all the neutrinoless $\mu^\pm \to e^\pm$ decays, the only one so far without experimental investigation is the one which from the present point of view is expected to be the most probable, viz., $\mu^- + \text{nucleus} \to e^- + \gamma + \text{nucleus}$.

[8] Physically speaking, the "double conversion coefficient" is not even bigger than the "single conversion coefficient" because the nucleus necessarily takes up a relatively large recoil momentum ($\cong m^{(\mu)}$) in $\mu^- + \text{nucleus} \to e^- + \text{nucleus}$ while, as noted above, the nucleus' recoil momentum in $\mu^- + \text{nucleus} \to e^- + \gamma + \text{nucleus}$ is predominantly small.

[9] It should, of course, be remembered that if a nonvanishing matrix element $M(\mu^- \to e^- + \pi^0)$ exists for $\mu^- \to e^- + \pi^0$, the strong-interaction nuclear absorption of the π^0 induces $\mu^- + \text{nucleus} \to e^- + \text{nucleus}$ via I: $\mu^- + \text{nucleus} \to e^- + \pi^0 + \text{nucleus} \to e^- + \text{nucleus}$ at a rate $\sim (Z_{\text{eff}}^3 A)$. Indeed, with $M(\mu^- \to e^- + \pi) \neq 0$, the process I might be more probable than the electroma⁰netic-"single conversion" process [Eqs. (8)–(10)] which in thig case would proceed via II: $\mu^- + \text{nucleus} \to e^- + \pi^0 + \text{nucleus} \to s^- + \pi^0 + \gamma$(virtual Coulomb)$+\text{nucleus} \to e^- + \gamma + \text{nucleus}$ while, eas al ready indicated, the electromagnetic "double conversion" process, III: $\mu^- + \text{nucleus} \to e^- + \pi^0 + \text{nucleus} \to e^- + \pi^0 + \gamma$ (virtual Coulomb)$+\gamma$(virtual Coulomb)$+\text{nucleus} \to e^- + \text{nucleus}$ would in this case be forbidden [Eqs. (11) and (12) with $g_2 = 0$]. Under these circumstances, any actual nonvanishing rate for $\mu^- + \text{nucleus} \to e^- + \text{nucleus}$ would have to be ascribed to process I.

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Quantization and the Classical Hamilton-Jacobi Equation

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In this paper, a formalism for quantization is developed which starts out from the Hamilton-Jacobi expression, $\partial S/\partial t + H(\partial S/\partial q, q)$, and which leads to its usual quantum-mechanical operator equivalent by means of straightforward algebra. The quantum-mechanical operator equivalents of H and p are then seen to be the consequence of assigning a number of equally probable classical paths to a dynamical system.

INTRODUCTION

MANY different formalisms have been developed for passing from a classical Hamiltonian or Lagrangian to a quantum-mechanical operator, but there still remains some mystery about why the differential operators which replace the momenta and energy should have the form they are given. Dirac[1] in his discussion of the action principle in his book and in subsequent papers clarifies things considerably by showing just how classical action is related to the arbitrary phase S in the time-dependent wave function $\psi \exp[(iS/\hbar)]$.

Feynman[2] in his Lagrangian formulation of the quantum mechanics goes a step further to show how the probability amplitude for a particular space-time path of a system is related to the classical Lagrangian $L(X(t), X(t))$ for this path. The total probability amplitude for a system's going from an initial state A to a final state B is the sum of all the probability amplitudes for all the possible paths from A to B. Each such path contributes equally in magnitude to the probability amplitude, but the phase for each path is different and equal to the classical action (in units of $\hbar$) for each path.

As Feynman points out, the contribution from a given path is "proportional to $\exp[iS(X(t))/\hbar]$, where the action $S(X(t)) = \int L(\dot{X}(t), X(t))dt$ is the time integral of the classical Lagrangian taken along the path in question." He shows that there is a close analogy between $\langle X'|X\rangle_\epsilon$ (the probability amplitude for finding the system at X' at time $t+\epsilon$ if it was at X at a time t) and $\exp[iS(X,X')/\hbar]$, and says: "In fact we now see that to a sufficient approximation the two quantities may be taken proportional to each other."

In view of the intimate relationship between S which, as Dirac[1] has pointed out, is the classical Hamilton-Jacobi function of the problem, and the

[1] P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, New York, 1957), 4th ed., p. 125.

[2] R. Feynman, Revs. Modern Phys. 20, 367 (1948).

quantum-mechanical probability amplitude, one may inquire whether it is possible to derive the quantum-mechanical operator for a given dynamical problem from the Hamilton-Jacobi differential equation by means of simple algebraic manipulations. It is the purpose of this note to show that this can indeed be done. One is then led automatically to a Lagrangian formulation of quantum mechanics since the algebraic expression one obtains becomes precisely the Lagrangian of the problem on passing over to the quantum-mechanical domain.

The result of this procedure is to show more clearly than has been done in the past how intimately the classical action is related to the phase of the wave function. Moreover, it is evident from this approach that the Hamilton-Jacobi function, and hence also the phase of the wave function, must be proportional to the function that generates a gauge transformation.

THE HAMILTON-JACOBI FORM AND ITS QUANTUM-MECHANICAL OPERATOR EQUIVALENT

If $H(p,q)$ is the classical Hamiltonian of a dynamical system whose generalized momenta and coordinates are p and q, respectively, then we know that it is always possible to generate a contact transformation by means of a function S which satisfies the differential equation:

$$\partial S/\partial t+H(p,q)=0, \tag{1}$$

and from which the momenta can be obtained by means of the equation

$$p=\partial S/\partial q. \tag{2}$$

We may combine (1) and (2) into the single equation,

$$\partial S/\partial t+H(\partial S/\partial q,q)=0. \tag{3}$$

The solution of Eq. (3) is the classical action function for the time interval t_0 to t—that is, it is the time integral of the classical Lagrangian:

$$S=\int_{t_0}^{t} L(\dot{q}(t),q(t))dt; \tag{4}$$

and it is this function which remains stationary for small variations of the classical orbit if those variations vanish at t_0 and at t.

We shall now show that we can obtain the Lagrangian operator for the quantum-mechanical analog of the classical path by a simple algebraic transformation of the expression,

$$\partial S/\partial t+H(\partial S/\partial q,q). \tag{5}$$

Of course the final result will not be the quantum-mechanical Lagrangian, since the operator will not be applied to a wave function. But we shall see in the concluding section of this paper how we are to pass over to the quantum-mechanical case. We begin by noting that we can write

$$\frac{\partial S}{\partial t}=e^{-iS/\hbar}\left(\frac{\hbar}{i}\frac{\partial}{\partial t}\right)e^{iS/\hbar}=e^{iS/\hbar}\left(\frac{-\hbar}{i}\frac{\partial}{\partial t}\right)e^{-iS/\hbar},$$

and

$$\frac{\partial S}{\partial q}=e^{-iS/\hbar}\left(\frac{\hbar}{i}\frac{\partial}{\partial q}\right)e^{iS/\hbar}=e^{iS/\hbar}\left(\frac{-\hbar}{i}\frac{\partial}{\partial q}\right)e^{-iS/\hbar}. \tag{6}$$

We shall also require an expression for $(\partial S/\partial q)^2$ in terms of the operator $[(\hbar/i)\partial/\partial q]$. We have

$$e^{-iS/\hbar}\left(\frac{\hbar}{i}\right)^2\left(\frac{\partial^2}{\partial q^2}\right)e^{-iS/\hbar}=e^{-iS/\hbar}\left(\frac{\hbar}{i}\right)^2\left[\left(\frac{i}{\hbar}\frac{\partial^2 S}{\partial q^2}\right)e^{iS/\hbar}+\left(\frac{i}{\hbar}\right)^2\left(\frac{\partial S}{\partial q}\right)^2e^{iS/\hbar}\right],$$

and

$$e^{iS/\hbar}\left(\frac{-\hbar}{i}\right)^2\left(\frac{\partial^2}{\partial q^2}\right)e^{-iS/\hbar}=e^{iS/\hbar}\left(\frac{-\hbar}{i}\right)^2\left[\left(\frac{-i}{\hbar}\frac{\partial^2 S}{\partial q^2}\right)e^{-iS/\hbar}+\left(\frac{-i}{\hbar}\right)^2\left(\frac{\partial S}{\partial q}\right)^2e^{-iS/\hbar}\right]. \tag{7}$$

We can combine these two equations to obtain

$$e^{-iS/\hbar}\left(\frac{\hbar}{i}\frac{\partial}{\partial q}\right)^2e^{iS/\hbar}+e^{iS/\hbar}\left(\frac{-\hbar}{i}\frac{\partial}{\partial q}\right)^2e^{-iS/\hbar}=2\left(\frac{\partial S}{\partial q}\right)^2. \tag{8}$$

If $H(\partial S/\partial q,q)$ is the Hamiltonian of a dynamical system, it is quadratic in $\partial S/\partial q$ and we may therefore replace $\partial S/\partial q$ in this expression by its operator equivalent (8).

We thus have

$$H\left(\frac{\partial S}{\partial q},q\right)=\tfrac{1}{2}\left[e^{-iS/\hbar}H\left(\frac{\hbar}{i}\frac{\partial}{\partial q},q\right)e^{iS/\hbar}+e^{iS/\hbar}H\left(\frac{-\hbar}{i}\frac{\partial}{\partial q},q\right)e^{-iS/\hbar}\right]. \tag{9}$$

We have thus obtained an exact expression for the classical Hamiltonian of the system in terms of its quantum-mechanical operator and its complex conjugate by a purely forward mathematical procedure. As it stands, the operator does not of course operate on a wave function but instead on the function $e^{+iS/\hbar}$ (and the complex conjugate operator operates on the complex conjugate function $e^{-iS/\hbar}$). We may, in a manner of speaking, refer to $e^{iS/\hbar}$ as the classical wave function that must be assigned to a classical orbit of the system.

Equation (9) is exact as long as H is a linear or a quadratic function of the momenta. But even if H

contained p to powers higher than 2, (9) would still hold since the additional terms would contain as factors derivatives of S of the form $\partial^n S/\partial q^n$ with $n\geq 2$. These, however, can all be made to vanish since S can always be chosen so that $\partial S/\partial q$ is not an explicit function of q. Indeed, we know from the usual definition of S that $\partial S/\partial q$ is a function only of p so that $\partial^2 S/\partial q^2=0$.

If we combine (6) and (9) we can write (5) in the form

$$\frac{\partial S}{\partial t}+H\left(\frac{\partial S}{\partial q},q\right)=\tfrac{1}{2}\left\{e^{-iS/\hbar}\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial t}\right)+H\left(\frac{\hbar}{i}\frac{\partial}{\partial q},q\right)\right]e^{iS/\hbar}\right.$$
$$\left.+e^{iS/\hbar}\left[\left(\frac{-\hbar}{i}\frac{\partial}{\partial t}\right)+H\left(\frac{-\hbar}{i}\frac{\partial}{\partial q},q\right)\right]e^{-iS/\hbar}\right\}. \quad (10)$$

THE HAMILTON-JACOBI FORM AND THE CLASSICAL LAGRANGIAN

To see what physical significance the expression on the right-hand side of (10) is to have in quantum mechanics, we shall first consider its classical counterpart, the Hamilton-Jacobi form (5). As we have noted, S is just the function that generates a contact transformation from one set of dynamical variables, q, p to another set Q, P such that

$$p=\partial S/\partial q, \quad -P=\partial S/\partial Q,$$
and
$$K-H=\partial S/\partial t, \quad (11)$$

where $K(Q,P)$ ultimately must be determined on the basis of one's choice of S. We also have, resulting from such a transformation,

$$(\sum p\dot{q}-H)dt\rightarrow[\sum P\dot{Q}-K(Q,P)+dS/dt]dt. \quad (12)$$

If one chooses $S(p,q,t,P,Q)$ to satisfy the Hamilton-Jacobi equation, then one must have

$$K=0, \quad \text{and} \quad \dot{P}=\dot{Q}=0, \quad (13)$$

so that P and Q are both constants of the motion.

For the time being we shall not impose the condition (13) on K but instead leave it as an arbitrary function to be determined later. Furthermore, we shall not introduce Q and P as constants of the motion at this stage, but allow them to vary when we apply the variational principle to the motion of our system. Keeping this in mind we may now treat the Hamilton-Jacobi form $\partial S/\partial t+H$ as some function of p, q, P, Q whose behavior we want to investigate by means of a variational principle.

At this point we shall make no use of the fact that $\partial S/\partial t+H$ is to vanish along the classical path, but we shall see that this is a consequence of our variational principle for the correct Lagrangian. We shall now show that if we choose K properly, the Hamilton-Jacobi form itself is the correct Lagrangian of the system as far as arbitrary variations are concerned; or differs from the correct Lagrangian by, at most, a perfect differential. This variational principle will then also give P and Q as constants of the motion.

Since we can obtain the Hamilton-Jacobi form from the expression

$$\partial S/\partial t+H-K \quad (14)$$

by setting K equal to zero, we shall first see what our variational principle leads to if we take this expression for our Lagrangian. We shall transform it by using the relations

$$\frac{dS}{dt}=\sum\frac{\partial S}{\partial q}\dot{q}+\sum\frac{\partial S}{\partial p}\dot{p}+\sum\frac{\partial S}{\partial Q}\dot{Q}+\sum\frac{\partial S}{\partial P}\dot{P}+\frac{\partial S}{\partial t}. \quad (15)$$

On substituting this into (14) and noting that

$$\sum\frac{\partial S}{\partial p}\dot{p}+\sum\frac{\partial S}{\partial P}\dot{P}=0,$$

which is one of the properties of a contact transformation, we obtain the expression

$$dS/dt-\sum p\dot{q}+\sum P\dot{Q}+H-K, \quad (16)$$

where we have used the first two equations of (11).

We shall require the time integral of this expression to be stationary for small variations of S. Since dS/dt is a total derivative, this vanishes automatically because δS must vanish at the upper and lower limits of the integral. We thus have for our stationary condition

$$\int_{t_0}^{t}\left[-\sum p\delta\dot{q}-\sum\dot{q}\delta p+\sum P\delta\dot{Q}+\sum\dot{Q}\delta P+\sum\frac{\partial H}{\partial p}\delta p\right.$$
$$\left.+\sum\frac{\partial H}{\partial q}\delta q-\sum\frac{\partial K}{\partial Q}\delta Q-\sum\frac{\partial K}{\partial P}\delta P\right]dt=0.$$

After some partial integrations and the discarding of total time derivatives, we obtain the Hamiltonian equations:

$$\begin{aligned}\dot{q}&=\partial H/\partial p, & \dot{Q}&=\partial K/\partial P,\\ \dot{p}&=-\partial H/\partial q, & \dot{P}&=-\partial K/\partial Q.\end{aligned} \quad (17)$$

If we now place the arbitrary function of K equal to zero, we obtain [from (17)] the canonical equations of motion with Q and P as constants of the motion. At the same time (14) goes over into the Hamilton-Jacobi form (5) and we see that a variational principle applied to this expression as the Lagrangian leads to the correct equations of motion.

Since (16) must vanish along the correct path, and since $Q=0$ for $K=0$, we obtain

$$dS/dt=\sum p\dot{q}-H=\mathfrak{L},$$

so that

$$S=\int\mathfrak{L}dt.$$

It is instructive to see that the Hamilton-Jacobi form differs from the usual expression for the Lagrangian by a perfect differential so that either expression may be used in a variational principle. We have

$$\begin{aligned}\mathfrak{L} &= \sum p\dot{q} - H \\ &= \sum p\dot{q} + \partial S/\partial t - (\partial S/\partial t + H) \\ &= \sum (\partial S/\partial q)\dot{q} + \partial S/\partial t - (\partial S/\partial t + H).\end{aligned}$$

If we now make use of the fact that Q is a constant of the motion, we see that

$$\sum (\partial S/\partial q)\dot{q} + \partial S/\partial t = dS/dt,$$

so that

$$\mathfrak{L} = dS/dt - (\partial S/\partial t + H).$$

Since in the usual variational principle the integral

$$\int_{t_0}^{t} \mathfrak{L} dt = \int \left[\frac{dS}{dt} - \left(\frac{\partial S}{\partial t} + H\right)\right] dt$$

is varied, we see that we may equally well vary the integral

$$\int_{t_0}^{t} \left(\frac{\partial S}{\partial t} + H\right) dt.$$

In carrying out this variation we must keep in mind that although the integrand in the above expression vanishes along the actual path, it does not vanish along the varied path.

If we vary the above integral directly, we have (not taking Q as a constant of the motion)

$$\begin{aligned}\delta \int_{t_0}^{t} &\left[\frac{\partial S(q,Q,p,P)}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right)\right] dt \\ &= \int_{t_0}^{t} \left[\frac{\partial}{\partial q}\left(\frac{\partial S}{\partial t} + H\right)\delta q + \frac{\partial}{\partial p}\left(\frac{\partial S}{\partial t} + H\right)\delta p + \frac{\partial}{\partial t}\left(\frac{\partial S}{\partial Q}\right)\delta Q \right. \\ &\qquad \left. + \frac{\partial}{\partial t}\left(\frac{\partial S}{\partial P}\right)\delta P\right] dt = 0.\end{aligned}$$

On setting the coefficients of δq and δp separately equal to zero, we see that $(\partial S/\partial t + H)$ can be a function neither of p nor of q and hence must be a constant which we can choose equal to zero. We thus get the Hamilton-Jacobi equation as a consequence of the variational principle.

On setting the coefficient of δQ equal to zero and using the fact that $\partial S/\partial Q$ is equal to P, we see that the variational principle also gives us P as a constant of the motion. Along the varied path, of course, neither $(\partial S/\partial t + H)$ nor dP/dt vanishes.

Since (5) behaves like a Lagrangian for a classical path, the operator in (10) when applied to an appropriate complex function should give us the quantum-mechanical Lagrangian.

TRANSITION TO THE QUANTUM-MECHANICAL LAGRANGIAN

To see how we are to pass over to the quantum-mechanical Lagrangian we shall start out by considering (10) again for a classical path. We have seen that the classical path is given by that particular choice of S which makes

$$\int_{t_0}^{t} \left\{ e^{-iS/\hbar}\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial t}\right) + H\left(\frac{\hbar}{i}\frac{\partial}{\partial q}, q\right)\right] e^{iS/\hbar} + e^{iS/\hbar}\left[\left(\frac{-\hbar}{i}\frac{\partial}{\partial t}\right) + H\left(-\frac{\hbar}{i}\frac{\partial}{\partial q}, q\right)\right] e^{-iS/\hbar}\right\} dt \quad (18)$$

a minimum. This leads to a single well-defined S which is a functional of the entire path. It is important to note here that S is determined by the entire path along which the system moves and hence must be treated differently from the way in which any function of the coordinates q appearing in the operator is treated.

As long as we are dealing with a classical system, the path is completely determined, and only one S (which is the total action of the system along the path) can be assigned to the system. This is equivalent to saying that there is a unit probability for finding the system on that particular path which makes (18) a minimum, and that we may assign to this path the probability amplitude $e^{iS/\hbar}$ so that the probability of finding the system somewhere along the path is precisely

$$e^{-iS/\hbar} e^{iS/\hbar} = 1.$$

This in itself is a rather remarkable way of considering the classical motion of a particle, for it appears that even without going over to quantum mechanics there exists an operator (of the same formal structure as the Schrödinger operator) which when operating on a complex function of unit absolute value gives the classical equations of motion. Just as in quantum mechanics, we may in classical mechanics assign to the motion of the particle a complex probability amplitude which is just this complex function. Since this function is of unit amplitude, the probability of finding the particle anywhere but on the classically determined path, is zero.

Although we have introduced a classical probability amplitude, the interpretation of the quantity $e^{-iS/\hbar}e^{iS/\hbar}$ is not quite the same as the Born interpretation of $\psi^*\psi$. Whereas the latter is related to the probability of finding the particle in a small neighborhood, of x, y, z, the former relates to the probability of finding the particle in a classically permissible orbit. In other words, assigning to a particle the classical probability amplitude $e^{iS/\hbar}$ does not mean that there is an equal probability of finding the particle at each point of the classical orbit, but rather that the unit probability applies to the entire path.

To pass over to the quantum-mechanical Lagrangian

density we first note that we may no longer speak of a precisely determined path for our system, but must instead assign to it a group of permissible classical paths in the manner described by Feynman in reference 2. The introduction of a set of permissible paths for our system is in itself a departure from classical mechanics since the solution of the Hamilton-Jacobi equation gives a single well-defined path for the system if the force field in which the system is moving, and the initial conditions are given. However, it is precisely because of the unpredictable fluctuations in the interaction of the system with the force field that a group of equally probable paths must be considered. This unpredictability arises because the system interacts with the force field through the absorption and emission of individual quanta by a process that cannot be analyzed in detail. If no force field were present, the system (particle) would move along a well-determined classical path and the classical and quantum-mechanical descriptions would be equivalent. Indeed, as long as a particle is in free flight, it is meaningless to differentiate between its wave and particle properties, and one is then justified in treating it as a particle.

Considering now a particle moving in a force field, we must introduce a group of paths each of which will have associated with it a well-defined action S. This action will of course vary from path to path, but the probability amplitude associated with a given path will no longer be $e^{iS/\hbar}$. Instead we shall assign to the ith path a probability amplitude $A^{\frac{1}{2}}e^{iS_i/\hbar}$ where A is a constant normalizing factor and is the same for each of the paths.

We must now replace (18) by an integral over a sum of terms each of which is similar to the integral in (18) but in which S varies from term to term. We thus have for the integral that is to be stationary the expression

$$A\int_{t_0}^{t}\left\{\sum_i\left(e^{-iS_i/\hbar}\left[\frac{\hbar}{i}\frac{\partial}{\partial t}+H\left(\frac{\hbar}{i}\frac{\partial}{\partial q},q\right)\right]e^{iS_i/\hbar}\right)\right.$$
$$\left.+\sum_i\left(e^{iS_i/\hbar}\left[\frac{-\hbar}{i}\frac{\partial}{\partial t}+H\left(\frac{-\hbar}{i}\frac{\partial}{\partial q},q\right)\right]e^{-iS_i/\hbar}\right)\right\}dt. \quad (19)$$

We must not now vary this integral with respect to the separate S_i's as independent quantities, since such a variation would lead to a set of separate classical paths. Instead, we must assign to the complete ensemble of paths a single probability amplitude and vary the integral with respect to it. To carry out this variation we shall have to transform the integrand so that this single probability amplitude (together with its complex conjugate) appears as a factor to which an operator applies just the way $e^{iS/\hbar}$ and $e^{-iS/\hbar}$ appear in the classical integrand (10). We shall now show that this can be done in a purely formal way and is a mathematical consequence of (19).

If we consider the sums in the integrand of (19), we see that because of the quantities in the brackets are linear operators we have

$$\sum_i\left(e^{iS_i/\hbar}\left[\frac{\hbar}{i}\frac{\partial}{\partial t}+H\left(\frac{\hbar}{i}\frac{\partial}{\partial q},q\right)\right]e^{iS_i/\hbar}\right)$$
$$=(\sum_i e^{-iS_i/\hbar})\left[\frac{\hbar}{i}\frac{\partial}{\partial t}+H\left(\frac{\hbar}{i}\frac{\partial}{\partial q},q\right)\right](\sum_i e^{iS_i/\hbar})$$
$$+\sum_{\substack{i,j\\ i\neq j}} e^{-i(S_i-S_j)/\hbar}\left[\frac{\partial S_j}{\partial t}+H\left(\frac{\partial S_j}{\partial q},q\right)\right]. \quad (20)$$

Since (20) is to be integrated over the entire domain and the classical action is very large compared with $\hbar$, the second term in this expression will be very small compared with the first term because $e^{-i(S_i-S_j)/\hbar}$ for $i\neq j$ is a very rapidly oscillating function. This term will vanish in any case since $\partial S_j/\partial t+H(\partial S_j/\partial q,q)$ vanishes along the jth classical trajectory.

We see from (20) that the probability amplitude that must be assigned to the ensemble of paths is precisely the sum of the individual probability amplitudes assigned to each classical path. We thus have obtained a single probability amplitude by this process which is a sum of complex terms.

We may compare this with Feynman's first postulate on page 371 of reference 2 in which he states:

"If an ideal measurement is performed to determine whether a particle has a path lying in a region of space-time then the probability that the result will be affirmative is the absolute square of a sum of complex contributions, one from each path in the region." The difference between what Feynman has done and the results obtained here, is that whereas he has introduced the probability amplitude as a postulate, we have derived it.

Since the probability amplitude for the entire domain of all possible paths available to our system is proportional to the wave function of the system, we may replace $\sum_i e^{iS_i/\hbar}$ by the wave amplitude ψ. Our variational principle then becomes

$$\delta\int_{t_0}^{t}\left\{\psi^*\left[\frac{\hbar}{i}\frac{\partial}{\partial t}+H\left(\frac{\hbar}{i}\frac{\partial}{\partial q},q\right)\right]\psi\right.$$
$$\left.+\psi\left[\frac{-\hbar}{i}\frac{\partial}{\partial t}+H\left(\frac{-\hbar}{i}\frac{\partial}{\partial q},q\right)\right]\psi^*\right\}dt=0, \quad (21)$$

where the variation is to be taken with respect to ψ and ψ^*. This leads to the usual Schrödinger equation and its complex conjugate.