PHYSICAL REVIEW VOLUME 126, NUMBER 2 APRIL 15, 1962

Radiation in a Plasma. II. Equivalent Sources*

MARSHALL H. COHEN
Center for Radiophysics and Space Research, Cornell University, Ithaca, New York
(Received June 15, 1961; revised manuscript received September 29, 1961)

We postulate that the fields in a plasma satisfy the linearized plasma equations containing electric and acoustic source terms. The fields and discontinuities produced by surface layers of the sources are discussed. It is shown that the fields in a homogeneous source-free volume are given uniquely by their values in V at time zero, plus boundary values for $t>0$, either $\mathbf{n}\times\mathbf{E}$ or $\mathbf{n}\times\mathbf{H}$, and either n_1 or $\mathbf{n}\cdot\mathbf{v}$. The fields in the volume are the same as those radiated by a certain set of equivalent surface sources on the boundary.

Integral equations for the scattering by a plasma bubble are derived. The differential scattering cross sections are evaluated for a small bubble, by using the Born approximation.

I. INTRODUCTION

THE aim of this series of papers[1,2] is to develop a continuum theory of waves and radiation in a plasma. For simplicity, we use the simple picture of a plasma as a continuous loss-free electron fluid containing an isotropic pressure, with stationary ions that neutralize the electrons, on the average. There is no external magnetic field. The linearized equations are the same as those used by many other workers.[3]

In Part I of this series[1] (hereafter referred to as [I]) we added an electric current source to the equations, and derived the spectrum of Čerenkov radiation from a fast charged particle. The total radiated Čerenkov energy agreed with that given by Pines and Bohm,[4] who used a more sophisticated analysis.

In this paper we add acoustic as well as electric source terms to the linearized equations. These sources allow us to develop "equivalent sources" for the wave fields which can exist in the plasma. By "equivalent sources" we mean a set of sources which is mathematically equivalent to the actual source, in its ability to reproduce the field, or a portion of it, in some region. Such sources are useful when dealing with boundary value problems in electromagnetic theory and acoustics. Their utility lies in the conceptual simplicity which they provide in the formulation of the problems and also in the means they give for visualizing approximation procedures. It seems to us that similar concepts will also be of use in plasma boundary-value problems. In Part III of this series,[2] for example, we shall discuss the radiation from a dipole antenna in a plasma, in terms of equivalent sources.

Some preliminary results have to be obtained before any boundary value problems can be analyzed. In Sec. III we consider a homogeneous plasma, and show that the fields can be separated into two modes. The P mode is a longitudinal (radial) plasma wave at great distances from the source, and the EM mode is a transverse electromagnetic wave at great distances from the source. This mode separation is the same as in [I], but now certain of the sources generate the P mode, and the others generate the EM mode.

In Secs. IV and V we discuss surface distributions of sources, and the fields and discontinuities they produce. In Sec. VI we derive a uniqueness theorem and then show that the fields in a volume can be expressed as the radiation from equivalent sources on the boundary. These results are extensions of the well-known analogous results in electromagnetic theory and acoustics.

We then discuss two boundary value problems. In Sec. VII we consider the problem of reflection and refraction at a density discontinuity. This problem has been worked out in detail by others,[5,6] and so we discuss only the boundary conditions themselves.

In Sec. VIII we solve the problem of scattering by a bubble in a plasma, and give the scattering cross sections for the case of a small bubble. There are four cross sections, for incident P or EM waves scattering into both P and EM waves.

The $P\rightarrow EM$ scattering has recently been discussed by Tidman and Weiss,[7] using a more sophisticated analysis. Our result essentially agrees with theirs. The bubble scattering problem is of interest for solar radio astronomy. One theory for the origin of some of the solar radio bursts is that they originate as P waves by a Čerenkov process and are converted to EM waves by scattering on inhomogeneities.[8]

In this paper we use the same symbols and units (MKS rationalized) as in [I]. The time variation for harmonic sources is $e^{-i\omega t}$.

II. BASIC EQUATIONS WITH SOURCE TERMS

We postulate that the fields in the plasma satisfy the linearized inhomogeneous Maxwell and Euler equations. The source terms are interpreted as electric current and charge $(\mathbf{J},\rho)$, magnetic current and charge $(\mathbf{K},\rho^m)$, a

* This work was done at the Observatoire de Meudon, Seine-et-Oise, France, while the author was a Guggenheim Memorial Foundation Fellow.

[1] M. H. Cohen, Phys. Rev. **123**, 711 (1961).
[2] M. H. Cohen, following paper [Phys. Rev. **125**, 398 (1962)].
[3] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), Chap. 4; J. F. Denisse and J. L. Delcroix, *Théorie des Ondes dans les Plasmas* (Dunod, Paris, 1961), Chap. I.
[4] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).
[5] D. A. Tidman, Phys. Rev. **117**, 366 (1960).
[6] A. H. Kritz and D. Mintzer, Phys. Rev. **117**, 382 (1960).
[7] D. A. Tidman and G. H. Weiss, Phys. Fluids **4**, 703 (1961).
[8] V. L. Ginzburg and V. V. Zhelezniakov, Soviet Astron.—AJ **2**, 653 (1958).

fluid flux source Q, and a mechanical body source $\mathbf{F}$. The flux source generates an electric fluid (electrons) and must be connected to $(\mathbf{J},\rho)$ by a continuity equation:

$$\nabla\cdot\mathbf{J}+\partial\rho/\partial t-eQ=0. \tag{2.1}$$

The magnetic current and charge obey a separate continuity equation:

$$\nabla\cdot\mathbf{K}+\partial\rho^m/\partial t=0. \tag{2.2}$$

The field equations are

$$\nabla\times\mathbf{E}+\mu_0\partial\mathbf{H}/\partial t=-\mathbf{K}, \tag{2.3}$$

$$\nabla\times\mathbf{H}-\epsilon_0\partial\mathbf{E}/\partial t+en_0\mathbf{v}=\mathbf{J}, \tag{2.4}$$

$$\mu_0\nabla\cdot\mathbf{H}=\rho^m, \tag{2.5}$$

$$\epsilon_0\nabla\cdot\mathbf{E}+en_1=\rho, \tag{2.6}$$

$$\nabla\cdot(n_0\mathbf{v})+\partial n_1/\partial t=Q, \tag{2.7}$$

$$m(\partial/\partial t)(n_0\mathbf{v})+n_0e\mathbf{E}+\nabla p=\mathbf{F}, \tag{2.8}$$

where n_0 and v_0 are the mean density and rms velocity of the electrons; m and $-e$ are the mass and charge of an electron; $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields; $\mathbf{v}$ is the systematic velocity field imparted to the electrons by the sources; and n_1 is the field of systematic variation in density imparted to the electrons by the sources. In Eq. (2.8), p is the pressure; we assume it is given approximately to zero and first orders by

$$p=p_0+p_1=\tfrac{1}{3}n_0mv_0^2+n_1mv_0^2. \tag{2.9}$$

Equation (2.9) is obtained from the adiabatic gas law, $T\propto n^{\gamma-1}$, with $\gamma=3$. Equations (2.1) to (2.8) are redundant; Eqs. (2.5) and (2.7) may be derived from the others.

The homogeneous version of Eqs. (2.3) through (2.8) has been used by many other workers in discussing the propagation of plane waves in a homogeneous plasma.[3] It is a linear approximation, and results from neglecting ion motions and electron-ion collisions, and from assuming that electron-electron interactions can be represented by the fluid pressure term.

The sources introduced in Eqs. (2.3) to (2.8) are all well-known from electromagnetic theory[9] and acoustics.[10] They are externally prescribed and generate the fields in the plasma. We could logically introduce further kinds of sources, for example, electric double layer sources, or heat sources, but the ones we have are sufficient for our purposes. With them we can define and utilize equivalent sources for the various kinds of waves which can exist in the plasma.

When the medium is homogeneous we may rewrite the power flow theorem [I] to include the effect of all the sources working. From Eqs. (2.3) to (2.8), setting n_0 and v_0^2 constant, we have

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}+mv_0^2n_1\mathbf{v})+\frac{\partial}{\partial t}\times\left[\frac{\epsilon_0}{2}E^2+\frac{\mu_0}{2}H^2+\tfrac{1}{2}n_0mv^2+\tfrac{1}{2}n_0mv_0^2\left(\frac{n_1}{n_0}\right)^2\right]$$
$$=-\mathbf{E}\cdot\mathbf{J}-\mathbf{H}\cdot\mathbf{K}+\mathbf{v}\cdot\mathbf{F}+mv_0^2\left(\frac{n_1}{n_0}\right)Q. \tag{2.10}$$

III. MODE SEPARATION IN A HOMOGENEOUS MEDIUM

A. Definitions

In our subsequent work we shall be dealing mainly with homogeneous media, or with media where the inhomogeneities are confined to surfaces where n_0 and v_0^2 are discontinuous. In each homogeneous region we can proceed as in [I] and separate the fields into plasma (P) and electromagnetic (EM) modes. The characteristic features of these modes are that the P mode contains no magnetic field, and the EM mode contains no charge accumulation. The P mode is essentially an acoustic field, and, at great distances from the source, it is a longitudinal (radial) wave. The EM mode is the ordinary electromagnetic field which would exist in a dispersive medium of relative dielectric constant $\epsilon_r=1-\omega_p^2/\omega^2$, and, at great distances from the source, it is a transverse wave.

In order to see how the source terms in Eqs. (2.3) to (2.8) are to be distributed between the two modes, we give first the differential equations for n_1 and $\mathbf{H}$. From the field equations (2.3) to (2.8), putting n_0 and v_0^2 constant, we derive

$$\left(\nabla^2-\frac{1}{v_0^2}\frac{\partial^2}{\partial t^2}-\frac{1}{D^2}\right)n_1=-\frac{\rho}{eD^2}-\frac{1}{v_0^2}\frac{\partial Q}{\partial t}+\frac{1}{mv_0^2}\nabla\cdot\mathbf{F}, \tag{3.1}$$

$$\left(\nabla\times\nabla\times+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\omega_p^2}{c^2}\right)\frac{\partial\mathbf{H}}{\partial t}=-\epsilon_0\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\mathbf{K}+\nabla\times\frac{\partial\mathbf{J}}{\partial t}-\frac{e}{m}\nabla\times\mathbf{F}, \tag{3.2}$$

where $D=v_0/\omega_p$. The source terms are not independent, for $\mathbf{J}$, ρ and Q are connected by Eq. (2.1). Apart from this, however, we may say that ρ, Q, and the irrotational part of $\mathbf{F}$ generate n_1; $\mathbf{K}$, $\mathbf{J}$, and the solenoidal part of $\mathbf{F}$ generate $\mathbf{H}$. But n_1 is connected only with the P mode, and $\mathbf{H}$ is connected only with the EM mode. Thus ρ, Q, and $\mathbf{F}_p$ generate the P mode, and $\mathbf{K}$, $\mathbf{J}$, and $\mathbf{F}_e$ generate the EM mode. The components of $\mathbf{F}$ are defined by

[9] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 464.

[10] P. M. Morse and K. U. Ingard, *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1961), Vol. XI/1, p. 4.

$\mathbf{F}=\mathbf{F}_p+\mathbf{F}_e$; $\nabla\times\mathbf{F}_p=\nabla\cdot\mathbf{F}_e=0$. From Eq. (3.1), $\mathbf{F}_p$ is equivalent to a dipole source distribution for the P mode, and from Eq. (3.2) $\mathbf{F}_e$ is equivalent to an electric current source distribution for the EM mode.

In this manner we are led to define the EM mode (the set of fields $\mathbf{E}_e$, $\mathbf{H}$, $\mathbf{v}_e$) by the following set of equations [I]:

$$\nabla\times\mathbf{E}_e+\mu_0\partial\mathbf{H}/\partial t=-\mathbf{K}, \tag{3.3}$$

$$\nabla\times\mathbf{H}-\epsilon_0\partial\mathbf{E}_e/\partial t+en_0\mathbf{v}_e=\mathbf{J}, \tag{3.4}$$

$$\mu_0\nabla\cdot\mathbf{H}=\rho^m, \tag{3.5}$$

$$\epsilon_0(\partial^2/\partial t^2+\omega_p^2)\nabla\cdot\mathbf{E}_e=-(\partial/\partial t)(\nabla\cdot\mathbf{J}), \tag{3.6}$$

$$n_0(\partial^2/\partial t^2+\omega_p^2)\nabla\cdot\mathbf{v}_e=(\omega_p^2/e)\nabla\cdot\mathbf{J}, \tag{3.7}$$

$$n_0m\partial\mathbf{v}_e/\partial t+n_0e\mathbf{E}_e=\mathbf{F}_e. \tag{3.8}$$

These equations are redundant; (3.3), (3.4), and (3.8) are independent, and the others may be derived from them.

The P mode (the set of fields $\mathbf{E}_p$, $\mathbf{v}_p$, n_1) is defined by

$$\nabla\times\mathbf{E}_p=0, \tag{3.9}$$

$$\epsilon_0\partial\mathbf{E}_p/\partial t-en_0\mathbf{v}_p=0, \tag{3.10}$$

$$\epsilon_0\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\nabla\cdot\mathbf{E}_p+e\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)n_1=\omega_p^2\rho+e\frac{\partial Q}{\partial t}, \tag{3.11}$$

$$n_0\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\nabla\cdot\mathbf{v}_p+\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\frac{\partial n_1}{\partial t}=\frac{\omega_p^2}{e}\frac{\partial\rho}{\partial t}+\frac{\partial^2Q}{\partial t^2}, \tag{3.12}$$

$$n_0m\partial\mathbf{v}_p/\partial t+n_0e\mathbf{E}_p+mv_0^2\nabla n_1=\mathbf{F}_p. \tag{3.13}$$

Equations (3.10), (3.11) and (3.13) are independent, and (3.9) and (3.12) may be derived from them.

It may be verified that the sum of the two modes is the total field which satisfies Eqs. (2.3) to (2.8), except at the plasma frequency, $\omega=\omega_p$. At this frequency the plasma can support an undamped standing wave which is independent of the sources.

B. Klein-Gordon Equations

The fields and their potential functions satisfy the inhomogeneous Klein-Gordon equation, as in [I]. The equation for n_1 was given in Eq. (3.1). The field n_1 of the plasma mode acts as a potential function for $\mathbf{E}_p$:

$$\epsilon_0(\partial^2/\partial t^2+\omega_p^2)\mathbf{E}_p=-ev_0^2\nabla n_1+(e/m)\mathbf{F}_p. \tag{3.14}$$

The EM mode is conveniently discussed by using two sets of potentials. One involves $\mathbf{J}$ and $\mathbf{F}_e$ and the other only $\mathbf{K}$, so that by superposition the total field is the sum of the partial fields due to the two modes. For the first component, let

$$\mu_0\mathbf{H}=\nabla\times\mathbf{A},\quad \mathbf{E}_e=-\partial\mathbf{A}/\partial t-\nabla\phi. \tag{3.15}$$

If we follow the derivation in [I], and let

$$\nabla\cdot\frac{\partial\mathbf{A}}{\partial t}+\frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\phi=0, \tag{3.16}$$

then the potentials $\mathbf{A}$ and ϕ are given by

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\omega_p^2}{c^2}\right)\frac{\partial\mathbf{A}}{\partial t}=-\mu_0\frac{\partial\mathbf{J}}{\partial t}+\mu_0\frac{e}{m}\mathbf{F}_e, \tag{3.17}$$

$$\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\omega_p^2}{c^2}\right)\phi=\frac{1}{\epsilon_0}\nabla\cdot\frac{\partial\mathbf{J}}{\partial t}. \tag{3.18}$$

For the component due to $\mathbf{K}$, let

$$\epsilon_0\mathbf{E}_e=\nabla\times\mathbf{A}_m,\quad \frac{\partial\mathbf{H}}{\partial t}=\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\mathbf{A}_m+\nabla\frac{\partial\phi_m}{\partial t}. \tag{3.19}$$

If we let

$$c^2\nabla\cdot\mathbf{A}_m+\partial\phi_m/\partial t=0, \tag{3.20}$$

then

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\omega_p^2}{c^2}\right)\mathbf{A}_m=\epsilon_0\mathbf{K}, \tag{3.21}$$

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\omega_p^2}{c^2}\right)\frac{\partial\phi_m}{\partial t}=\frac{1}{\mu_0}\frac{\partial\rho^m}{\partial t}. \tag{3.22}$$

IV. SURFACE DISTRIBUTIONS OF SOURCES AND FIELD DISCONTINUITIES

A. General Case

We now consider surface distributions of sources and the discontinuities in the fields that they produce. Surface distributions are defined as the limits of thin dense volume distributions, as the thickness goes to zero. We shall use the same symbols for the two-dimensional surface distributions as we used in Secs. II and III for the three-dimensional volume distributions; whether we are discussing two- or three-dimensional distributions will be clear from the particular context.

Let the surface S (Fig. 1) separate two homogeneous regions of the plasma. The normal $\mathbf{n}$ points into region 1. The average properties of the plasma, n_0 and v_0^2, may be different on the two sides of S.

A distribution of electric current $\mathbf{J}$ on S produces a discontinuity in tangential magnetic field, $\mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2)=\mathbf{J}$. This follows by integrating Eq. (2.4) around a contour such as that shown in Fig. 1, and assuming that $\mathbf{E}$ and $\mathbf{v}$ are finite as the contour shrinks to zero. A distribution of magnetic current similarly produces a discontinuity in tangential electric field, $\mathbf{n}\times(\mathbf{E}_2-\mathbf{E}_1)=\mathbf{K}$. The normal

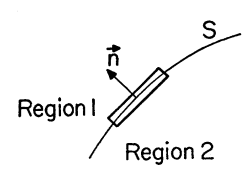


FIG. 1. The surface S separates two homogeneous regions of the plasma.

components of $\mathbf{H}$ and $\mathbf{E}$ have discontinuities given by the surface charge distributions: $\mathbf{n}\cdot(\mathbf{H}_1-\mathbf{H}_2)=\rho^m/\mu_0$, and $\mathbf{n}\cdot(\mathbf{E}_1-\mathbf{E}_2)=\rho/\epsilon_0$. These are obtained by integrating Eqs. (2.5) and (2.6) over the volume whose cross section is shown in Fig. 1, and assuming that n_1 remains finite as the volume shrinks to zero.

A surface distribution of Q produces a discontinuity in the normal component of fluid flux: $\mathbf{n}\cdot(n_{01}\mathbf{v}_1-n_{02}\mathbf{v}_2)=Q$, from Eq. (2.7) and the requirement that n_1 be finite. If the plasma has the same density on the two sides of S, then $n_{01}=n_{02}=n_0$, and Q simply produces a discontinuity in the normal component of velocity.

A surface distribution of $\mathbf{F}$ produces a discontinuity in pressure. Let F be the force per unit area, directed along the normal. By integrating Eq. (2.8) over the infinitesimal volume whose cross section is shown in Fig. 1, we have $F=p_{11}-p_{12}=m(v_{01}^2n_{11}-v_{02}^2n_{12})$, assuming that $p_{01}=p_{02}$, and that $\mathbf{v}$ and $\mathbf{E}$ are finite.

The discontinuity in $\mathbf{n}\times(n_0\mathbf{v})$ is obtained by taking the curl of Eq. (2.8) and integrating over the infinitesimal rectangular surface of Fig. 1. The result[11] is shown below in Eq. (4.7).

The following list summarizes the sources and the discontinuities.

$$J=\mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2), \tag{4.1}$$

$$\rho=\epsilon_0\mathbf{n}\cdot(\mathbf{E}_1-\mathbf{E}_2), \tag{4.2}$$

$$Q=\mathbf{n}\cdot(n_{01}\mathbf{v}_1-n_{02}\mathbf{v}_2), \tag{4.3}$$

$$\mathbf{K}=\mathbf{n}\times(\mathbf{E}_2-\mathbf{E}_1), \tag{4.4}$$

$$\rho^m=\mu_0\mathbf{n}\cdot(\mathbf{H}_1-\mathbf{H}_2), \tag{4.5}$$

$$F=m(v_{01}^2n_{11}-v_{02}^2n_{12}), \tag{4.6}$$

$$\mathbf{n}\times\nabla F=m\mathbf{n}\times(n_{02}\partial\mathbf{v}_2/\partial t-n_{01}\partial\mathbf{v}_1/\partial t)+e\mathbf{n}\times(n_{02}\mathbf{E}_2-n_{01}\mathbf{E}_1). \tag{4.7}$$

Except for Eq. (4.7), these are the same relations that exist separately for electromagnetic theory and for acoustics.

B. Homogeneous Case

When the medium is the same on the two sides of S we can find simple expressions for the discontinuities in the P and EM modes individually. Assume the time dependence $e^{-i\omega t}$, and let $X=\omega_p^2/\omega^2$. We have the requirement that the *total* field $\mathbf{E}$, $\mathbf{H}$, $\mathbf{v}$, n_1 remain finite everywhere, but we shall see that the *component* fields $\mathbf{E}_p$, $\mathbf{E}_e$, etc., have singularities at the surface.

By integrating Eq. (3.8) along a line perpendicular to S, we obtain

$$\int(n_0e\mathbf{E}_e-i\omega n_0m\mathbf{v}_e)\cdot d\mathbf{l}=F_e,$$

where F_e is now a two-dimensional scalar source. It is the solenoidal force per unit area, directed along the normal. As the thickness of the surface layer shrinks to zero, we must have $\mathbf{E}_e$ and $\mathbf{v}_e$ become infinite, in order for the integral to yield the finite value, F_e. Since $\mathbf{E}$ and $\mathbf{v}$ are finite, we must have the opposite singularities in $\mathbf{E}_p$ and $\mathbf{v}_p$:

$$\int(n_0e\mathbf{E}_p-i\omega n_0m\mathbf{v}_p)\cdot d\mathbf{l}=-F_e.$$

Substitution from (3.10) gives the singularity in $\mathbf{E}_p$:

$$n_0e(1-X)/X\int\mathbf{E}_p\cdot d\mathbf{l}=F_e.$$

Integration of Eq. (3.9) now gives the discontinuity in $\mathbf{n}\times\mathbf{E}_p$:

$$\mathbf{n}\times(\mathbf{E}_{p1}-\mathbf{E}_{p2})=\frac{X}{n_0e(1-X)}\mathbf{n}\times\nabla F. \tag{4.8}$$

In Eq. (4.8) we have replaced F_e with F, because $\mathbf{n}\times\nabla F_e=\mathbf{n}\times\nabla F$.

The discontinuity in $\mathbf{n}\cdot\mathbf{E}_p$ is obtained by integrating Eq. (3.11)

$$\mathbf{n}\cdot(\mathbf{E}_{p1}-\mathbf{E}_{p2})=-\frac{\rho X}{\epsilon_0(1-X)}+\frac{ieQ}{\epsilon_0\omega(1-X)}. \tag{4.9}$$

This discontinuity in n_1 is given in Eq. (4.6), and the discontinuity in $\mathbf{v}_p$ follows from Eq. (3.10) and the discontinuities in $\mathbf{E}_p$.

The discontinuities in the EM mode are

$$\mathbf{n}\times(\mathbf{E}_{e2}-\mathbf{E}_{e1})=\mathbf{K}+\frac{X}{n_0e(1-X)}\mathbf{n}\times\nabla F, \tag{4.10}$$

$$\mathbf{n}\cdot(\mathbf{E}_{e1}-\mathbf{E}_{e2})=\frac{\nabla\cdot\mathbf{J}}{i\omega\epsilon_0(1-X)}, \tag{4.11}$$

$$(\mathbf{v}_{e1}-\mathbf{v}_{e2})=\frac{e}{i\omega m}(\mathbf{E}_{e1}-\mathbf{E}_{e2}). \tag{4.12}$$

The discontinuity in $\mathbf{H}$ is given in Eqs. (4.1) and (4.5). It may be verified that the discontinuities in the EM and P modes combine to give the discontinuities in the total fields given by Eqs. (4.1) to (4.7).

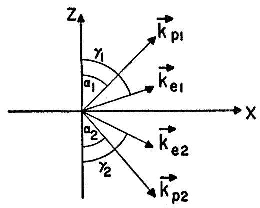
A static distribution of ρ and F will give rise to a static P mode, but not an EM mode. The discontinuities in the fields of the P mode are properly obtained from Eqs. (4.8) and (4.9) by taking the limit as ω goes to zero. A static distribution of Q and $\mathbf{J}$ will generate a static EM mode, with discontinuities given by Eqs. (4.1) and (4.3).

V. FIELDS OF PLANE SURFACE DISTRIBUTIONS

As an illustration of the use of the discontinuity formulas developed above, and for later use, we compute the fields due to some plane surface distributions. Let

[11] J. A. Stratton, see reference 9, p. 191.

FIG. 2. Plane waves generated by sources on the plane $z=0$.



the coordinate system be defined as in Fig. 2. The sources are on the plane $z=0$, and $\mathbf{n}=\mathbf{z}$.

A distribution which is uniform on the plane except for a constant phase progression, $\exp[i(kx-\omega t)]$, will in general radiate six plane waves, as indicated by the arrows in Fig. 2. There are two *EM* waves, for the two polarizations, in directions $\mathbf{k}_{e1}$ and $\mathbf{k}_{e2}$, and there are longitudinal *P* waves in the directions $\mathbf{k}_{p1}$ and $\mathbf{k}_{p2}$. The lengths of these vectors are the appropriate propagation constants for plane *EM* and *P* waves, $k_e=\omega(1-X)^{\frac{1}{2}}/c$, and $k_p=\omega(1-X)^{\frac{1}{2}}/v_0$. The vectors are in the (x,z) plane and in such a direction that their projections on the x axis are equal to k.

Six boundary conditions suffice to determine the six waves; we may use Eqs. (4.1), (4.2), (4.4), and (4.6). These correspond to the independent sources, since Q and ρ^m are given by Eqs. (2.1) and (2.2).

When the medium is the same in the two regions, the calculation of the radiated fields becomes particularly simple. In this case the corresponding components on the two sides must have equal amplitudes, by symmetry, and so the required number of boundary conditions is reduced. We now give the solutions for several cases.

A. Distribution of J, ϱ, Q

Let the sources on the plane $z=0$ consist of the following (the time factor $\exp(-i\omega t)$ is suppressed):

$$\mathbf{J}=\mathbf{x}J_0e^{ikx},\quad \rho=\rho_0e^{ikx},\quad Q=-iQ_0e^{ikx},$$

where $kJ_0-\omega\rho_0+eQ_0=0$. Since the medium is the same on the two sides of S, the field consists of two equal-amplitude plane *EM* waves at angles $\gamma=\gamma_1=\gamma_2=\sin^{-1}(k/k_e)$, and two equal-amplitude plane *P* waves at angles $\alpha=\alpha_1=\alpha_2=\sin^{-1}(k/k_p)$. These angles are shown in Fig. 2.

The *EM* mode is polarized with $\mathbf{H}$ having only a y component. By Eq. (4.1) the $\mathbf{H}$ waves are

$$\mathbf{H}_1=-\tfrac{1}{2}\mathbf{y}J_0\exp(i\mathbf{k}_{e1}\cdot\mathbf{r}),\quad \mathbf{H}_2=\tfrac{1}{2}\mathbf{y}J_0\exp(i\mathbf{k}_{e2}\cdot\mathbf{r}).\quad(5.1)$$

Associated with these waves of $\mathbf{H}$ are plane waves of $\mathbf{E}_e$ and $\mathbf{v}_e$.

The *P* mode consists of longitudinal plane waves. By Eq. (4.9),

$$\mathbf{E}_{p1}=A\mathbf{k}_{p1}\exp(i\mathbf{k}_{p1}\cdot\mathbf{r}),\quad \mathbf{E}_{p2}=A\mathbf{k}_{p2}\exp(i\mathbf{k}_{p2}\cdot\mathbf{r}),\quad(5.2)$$

where

$$A=\frac{1}{2k_p\epsilon_0\cos\alpha}\left[-\frac{\rho_0X}{1-X}+\frac{eQ_0}{\omega(1-X)}\right].$$

Associated with these waves of $\mathbf{E}_p$ are plane waves of $\mathbf{v}_p$ and n_1.

B. Distribution of K, ϱ^m

Let the sources consist of the following:

$$\mathbf{K}=\mathbf{x}K_0e^{ikx},\quad \rho^m=\rho_0{}^me^{ikx},$$

where $kK_0-\omega\rho_0{}^m=0$. This particular distribution of magnetic current will radiate *EM* waves polarized with $\mathbf{E}_e$ having only a y-component. By Eq. (4.10) these fields are

$$\mathbf{E}_{e1}=\tfrac{1}{2}\mathbf{y}K_0\exp(i\mathbf{k}_{e1}\cdot\mathbf{r}),\quad \mathbf{E}_{e2}=-\tfrac{1}{2}\mathbf{y}K_0\exp(i\mathbf{k}_{e2}\cdot\mathbf{r}).\quad(5.3)$$

The magnetic current does not produce a *P* mode.

C. Distribution of *F*

Let the source on S consist of a distribution of F:

$$F=F_0e^{ikx}.$$

The *P* mode is obtained from Eq. (4.6):

$$n_{11}=(F_0/2mv_0{}^2)\exp(i\mathbf{k}_{p1}\cdot\mathbf{r}),$$
$$n_{12}=-(F_0/2mv_0{}^2)\exp(i\mathbf{k}_{p2}\cdot\mathbf{r}).\quad(5.4)$$

The *EM* mode in general consists of four waves. By using the conditions (4.1) and (4.10), however, we find that there are only two waves, polarized with $\mathbf{E}_e$ in the (x,z) plane. The electric fields are

$$\mathbf{E}_{e1}=B\mathbf{y}\times\mathbf{k}_{e1}\exp(i\mathbf{k}_{e1}\cdot\mathbf{r}),$$
$$\mathbf{E}_{e2}=B\mathbf{y}\times\mathbf{k}_{e2}\exp(i\mathbf{k}_{e2}\cdot\mathbf{r}),\quad(5.5)$$

where

$$B=-(iXF_0\tan\gamma)/2n_0e(1-X).$$

D. Distribution of Static ϱ, *F*

The fields of a uniform static sheet of charge, ρ_0 coul/m^2, are found from Eqs. (3.1), (3.14), and (4.9), by letting $\omega\rightarrow 0$. The results are

$$\begin{aligned}\mathbf{E}_1&=\mathbf{z}(\rho_0/2\epsilon_0)e^{-z/D},\quad \mathbf{E}_2=-\mathbf{z}(\rho_0/2\epsilon_0)e^{z/D},\\ n_{11}&=(\rho_0/2eD)e^{-z/D},\quad n_{12}=(\rho_0/2eD)e^{z/D}.\end{aligned}\quad(5.6)$$

The fields of a uniform static distribution of F evidently must have the same form as in Eq. (5.6). From Eq. (4.6) we derive

$$\begin{aligned}\mathbf{E}_1&=\mathbf{z}(F/2en_0D)e^{-z/D},\quad \mathbf{E}_2=\mathbf{z}(F/2en_0D)e^{z/D},\\ n_{11}&=(F/2mv_0{}^2)e^{-z/D},\quad n_{12}=-(F/2mv_0{}^2)e^{z/D}.\end{aligned}\quad(5.7)$$

VI. UNIQUENESS THEOREM

We derive a uniqueness theorem by a simple extension of the procedure used by Stratton.[12] Let V be a

[12] J. A. Stratton, see reference 9, p. 486.

homogeneous source-free volume bounded by the surface S with outward normal $\mathbf{n}$. Let $\mathbf{\Gamma}_1 \equiv (\mathbf{E}_1,\mathbf{H}_1,\mathbf{v}_1,n_{11})$ and $\mathbf{\Gamma}_2$ be two solutions of Eqs. (2.3) to (2.8) which are identical at time $t=0$ at all points of V. The difference field, $\mathbf{\Gamma}=\mathbf{\Gamma}_1-\mathbf{\Gamma}_2$ also satisfies Eqs. (2.3) to (2.8) by linearity, and from Eq. (2.10) we have

$$\iint_S (\mathbf{E}\times\mathbf{H}+mv_0^2n_1\mathbf{v})\cdot\mathbf{n}ds$$
$$=-\frac{\partial}{\partial t}\iiint_V [\tfrac{1}{2}\epsilon_0E^2+\tfrac{1}{2}\mu_0H^2 + \tfrac{1}{2}n_0mv^2+\tfrac{1}{2}n_0mv_0^2(n_1/n_0)^2]dv.$$

If the values on S of either $\mathbf{E}\times\mathbf{n}$ or $H\times\mathbf{n}$, and also either n_1 or $\mathbf{v}\cdot\mathbf{n}$, are zero for $t\geq 0$, then the left-hand side is zero. In this case the right-hand side is also zero, so that $\mathbf{\Gamma}=0$, for $t\geq 0$. The fields in V are therefore uniquely determined by their values in V at $t=0$, plus the boundary values for $t\geq 0$: either $\mathbf{E}\times\mathbf{n}$ or $\mathbf{H}\times\mathbf{n}$, and either n_1 or $\mathbf{v}\cdot\mathbf{n}$.

It is permissible to let a portion of S go to infinity. In this case one assumes that the medium has a slight loss, so that the integrals of $\mathbf{E}\times\mathbf{H}$ and $mv_0^2n_1\mathbf{v}$ vanish.[12] If the sources oscillate harmonically, then the steady-state components of the fields in V are given uniquely by the boundary values: either $\mathbf{E}\times\mathbf{n}$ or $\mathbf{H}\times\mathbf{n}$, and either n_1 or $\mathbf{v}\cdot\mathbf{n}$.

By using the ideas of surface sources and field discontinuities developed above, we can state explicitely how the interior field is determined by the boundary values. The procedure is analogous to that used by Schelkunoff for the electromagnetic equivalence theorem.[13] Consider a source-free volume V bounded by the surface S with outward normal $\mathbf{n}$, and let there be an external harmonic source. Assume that the medium is continuous across S. (If we are interested in a surface S where the plasma is discontinuous, we may work with a surface S' a little inside of S, and let $S'\rightarrow S$.)

We now seek the total field radiated by the external source together with the following set of surface sources on S: $\mathbf{J}=\mathbf{n}\times\mathbf{H}$, $Q=n_0\mathbf{n}\cdot\mathbf{v}$, $\mathbf{K}=\mathbf{E}\times\mathbf{n}$, $F=mv_0^2n_1$. Consider the field which is equal to zero in V, and equal to the original field outside V. It has the desired singularities or discontinuities at all sources, and satisfies all boundary conditions. It, therefore, is the field we seek. By linearity the surface sources by themselves must produce the negative of the original fields inside V, and zero outside. We thus have the theorem: the set of surface sources on S,$\mathbf{J}=\mathbf{H}\times\mathbf{n}$, $Q=-n_0\mathbf{n}\cdot\mathbf{v}$, $\mathbf{K}=\mathbf{n}\times\mathbf{E}$, $F=-mv_0^2n_1$, radiating in the given plasma, produce identically the original field in V, and zero outside of V. In the event that S is a surface of discontinuity, we must use the fields $(\mathbf{E},\mathbf{H},\mathbf{v},n_1)$ and the medium (n_0,v_0^2) appropriate to the *interior* of V.

[13] S. A. Schelkunoff, Phys. Rev. **56**, 308 (1939).

We have derived this theorem in such a way that the four surface sources radiate into the given plasma, i.e., they radiate in the presence of whatever boundaries and inhomogeneities happen to be present, both inside and outside V. When the interior is homogeneous, however, the calculation may be greatly simplified, by noting that the four sources together give zero outside V. We may thus replace the exterior medium with whatever we wish and not affect the interior fields. We replace the exterior medium with a plasma identical to the one inside V, and then the calculation is made with all four sources radiating in a homogeneous unbounded plasma.

According to the uniqueness theorem, all four sources are not necessary. Either $\mathbf{J}$ or $\mathbf{K}$, plus either Q or F, should suffice. We can see how this comes about by using the concept of replacing the exterior medium, as we have just discussed. For example, replace the exterior medium with a rigid impenetrable boundary which is also a perfect electric conductor. Let this boundary shrink onto S. It will "short circuit" both $\mathbf{J}$ and Q, leaving the interior fields produced solely by the distributions of $\mathbf{K}$ and F radiating against a rigid impenetrable electric conductor. Similar remarks have been made for the electromagnetic field by Schelkunoff,[13] and by Harrington.[14]

VII. REFLECTION AT A DENSITY DISCONTINUITY

In this section we discuss the boundary conditions for the problem of reflection and refraction of a plane wave incident on a plasma density discontinuity. The general problem itself has already been discussed by Kritz and Mintzer.[6] Field[15] has discussed the special case of a plasma-vacuum boundary.

There are six reflected or refracted waves of unknown amplitude. Evidently, we must have six boundary conditions, and so we need physical arguments concerning six independent source components from the group in Eqs. (4.1) to (4.7).

The plasma contains no magnetic current, so that Eq. (4.4) gives the usual electrical condition of continuity of $\mathbf{n}\times\mathbf{E}$. We also assume that the boundary is free, so that $F=0$, and Eq. (4.6) gives the usual acoustic condition of continuity of pressure.

The second usual acoustic condition is the continuity of $\mathbf{n}\cdot\mathbf{v}$. This must also be imposed in the plasma, since it ensures that the fluid remain continuous. The normal flux $\mathbf{n}\cdot(n_0\mathbf{v})$ then is discontinuous, and Eq. (4.3) implies the existence of a fluid flux source, of strength $Q=(n_{01}-n_{02})\mathbf{n}\cdot\mathbf{v}$. By the continuity equation (2.1) this Q is connected to a distribution of ρ and $\mathbf{J}$. The current $\mathbf{J}$, however, is of second order and can be ignored; thus there is a sheet of charge on the boundary: $\rho=ieQ/\omega$.

Our description has now become rather unrealistic, for it contains a singular sheet of charge on the

[14] R. F. Harrington, *Time-Harmonic Electromagnetic Fields* (McGraw-Hill Book Company, Inc., New York, 1961), Chap. 3.
[15] G. B. Field, Astrophys. J. **124**, 555 (1956).

boundary. Evidently, this charge is spread through some region of finite thickness, and the boundary is not as sharp as we have implied. Kritz and Mintzer[6] have discussed this layer of charge and have shown that the thickness is $\mathbf{n}\cdot\mathbf{v}/\omega$. The details of the charge distribution near the boundary cannot be obtained from a simplified linear analysis, but, presumably, $\mathbf{n}\cdot\mathbf{v}/\omega$ is the correct order of magnitude for the thickness of the region where n_1 differs markedly from the simple sinusoidal waveform.

The last boundary condition is obtained by setting $\mathbf{J}=0$; and Eq. (4.1) gives the continuity of $\mathbf{n}\times\mathbf{H}$. The six conditions thus are the continuity of $\mathbf{n}\times\mathbf{H}$, $\mathbf{n}\times\mathbf{E}$, $\mathbf{n}\cdot\mathbf{v}$, and $v_0^2 n_1$. Detailed calculations for reflection coefficients and radiation efficiencies are given by Tidman[5] and by Kritz and Mintzer.[6]

VIII. SCATTERING BY A PLASMA BUBBLE

A. Equivalent Sources

Consider a plasma with mean electron density n_0 and mean square thermal velocity v_0^2. Let the plasma contain a bubble, i.e., a region where the mean density is $(n_0+\Delta n)$ and the mean square velocity is $(v_0^2+\Delta v^2)$. We shall consider only bubbles for which $\Delta n/n_0 \ll 1$ and $\Delta v^2/v_0^2 \ll 1$ and whose linear dimensions are much smaller than the P mode wavelength. Assume that the static pressure inside is equal to that outside, so that, to a first-order approximation,

$$\Delta n/n_0 = -\Delta v^2/v_0^2. \tag{8.1}$$

A weak harmonic source is outside the bubble. Define the set of incident fields, $\mathbf{\Gamma}^i \equiv (\mathbf{E}^i,\mathbf{H}^i,\mathbf{v}^i,n_1^i)$, as those that would exist if Δn and Δv^2 were zero; the set of total fields, $\mathbf{\Gamma}^t$, as those actually existing in the presence of the bubble; and the set of scattered fields as the difference: $\mathbf{\Gamma}^s = \mathbf{\Gamma}^t - \mathbf{\Gamma}^i$.

We now derive equations for $\mathbf{\Gamma}^s$ from the field equations (2.3) to (2.8). These equations apply equally well to $\mathbf{\Gamma}^i$ and $\mathbf{\Gamma}^t$. By subtracting the set applicable to $\mathbf{\Gamma}^i$ from those for $\mathbf{\Gamma}^t$, we obtain

$$\nabla\times\mathbf{E}^s + \mu_0 \partial\mathbf{H}^s/\partial t = 0, \tag{8.2}$$

$$\nabla\times\mathbf{H}^s - \epsilon_0\partial\mathbf{E}^s/\partial t + en_0\mathbf{v}^s = -e\Delta n\mathbf{v}^t, \tag{8.3}$$

$$\nabla\cdot\mathbf{H}^s = 0, \tag{8.4}$$

$$\epsilon_0\nabla\cdot\mathbf{E}^s + en_1^s = 0, \tag{8.5}$$

$$n_0\nabla\cdot\mathbf{v}^s + \partial n_1^s/\partial t = -\nabla\cdot(\Delta n\mathbf{v}^t), \tag{8.6}$$

$$m\frac{\partial}{\partial t}(n_0\mathbf{v}^s) + n_0 e\mathbf{E}^s + mv_0^2\nabla n_1^s = -m\frac{\partial}{\partial t}(\Delta n\mathbf{v}^t) - \Delta n e\mathbf{E}^t - m\nabla(\Delta v^2 n_1^t). \tag{8.7}$$

Thus, $\mathbf{\Gamma}^s$ satisfies the field equations and is generated by the sources

$$\begin{aligned}\mathbf{J} &= -e\Delta n\mathbf{v}^t,\\ Q &= -\nabla\cdot(\Delta n\mathbf{v}^t),\\ \mathbf{F} &= -m\frac{\partial}{\partial t}(\Delta n\mathbf{v}^t) - \Delta n e\mathbf{E}^t - m\nabla(\Delta v^2 n_1^t).\end{aligned} \tag{8.8}$$

Note that the equivalent sources of $\mathbf{\Gamma}^s$ do not contain an electric charge component. The current $\mathbf{J}$ and the fluid source Q together satisfy the continuity equation (2.1).

The scattered P field is obtained from Eq. (3.1). For a harmonic source, the solution to this equation is

$$n_1^s = \frac{1}{4\pi}\iiint_{V+}\left(\frac{\rho}{eD^2} - \frac{i\omega Q}{v_0^2} - \frac{1}{mv_0^2}\nabla\cdot\mathbf{F}\right)\frac{e^{ik_p r}}{r}dv, \tag{8.9}$$

where V^+ includes V, the volume of the bubble, and r is the distance between an element in the scattering volume and the field point. Upon substitution for the sources from Eq. (8.8), and using Eq. (8.1), we obtain

$$n_1^s = \frac{e}{4\pi m v_0^2}\iiint_{V+}\frac{e^{ik_p r}}{r}\nabla\cdot(\Delta n\mathbf{E}^t)dv - \frac{1}{4\pi n_0} \times \iiint_{V+}\frac{e^{ik_p r}}{r}\nabla^2(\Delta n\, n_1^t)dv.$$

We can use Gauss' theorem to convert each of these integrals into a different volume integral plus a surface integral. In each case the surface integral vanishes because $\Delta n=0$ on the surface. These operations leave

$$n_1^s = -\frac{e}{4\pi m v_0^2}\iiint_{V+}\Delta n\,\mathbf{E}^t\cdot\nabla\left(\frac{e^{ik_p r}}{r}\right)dv - \frac{1}{4\pi n_0} \times \iiint_{V+}\Delta n\, n_1^t\,\nabla^2\left(\frac{e^{ik_p r}}{r}\right)dv.$$

We now replace V^+ by V, because Δn is zero outside V. Substituting also $\nabla^2(e^{ik_p r}/r) = -k_p^2(e^{ik_p r}/r)$ gives the result:

$$n_1^s = -\frac{e}{4\pi m v_0^2}\iiint_{V}\Delta n\,\mathbf{E}^t\cdot\nabla\left(\frac{e^{ik_p r}}{r}\right)dv + \frac{k_p^2}{4\pi n_0} \times \iiint_{V}\Delta n\, n_1^t\frac{e^{ik_p r}}{r}dv. \tag{8.10}$$

Equation (8.10) is one of a set of simultaneous integral equations for the scattered field. The interpretation of Eq. (8.10) is that the equivalent sources of the scat-

tered P field are the distribution of monopoles, proportional to $(\Delta n\, n_1^t)$, and the distribution of dipoles, proportional to $(\Delta n\, \mathbf{E}^t)$.

The scattered EM field is obtained from Eq. (3.17) for the vector potential, which we rewrite as

$$(\nabla^2+k_e^2)\mathbf{A}=-\mu_0\mathbf{J}+i\mu_0(e/\omega m)\mathbf{F}_e. \tag{8.11}$$

Now $\mathbf{F}_e$ is the solenoidal component of $\mathbf{F}$. In Eq. (8.8) the term $m\nabla(\Delta v^2 n_1^t)$ is irrotational and so does not contribute to the scattered EM field. The term $\Delta n(m\partial\mathbf{v}^t/\partial t+e\mathbf{E}^t)$, however, will generally have both a solenoidal and an irrotational component. If we attempt to separate these components, we become involved with surface distributions because Δn changes rapidly at S. On the other hand, it is permissible, and simpler, to use in Eq. (8.11) the total source $\mathbf{F}$ rather than $\mathbf{F}_e$; this will give a different vector potential, but $\mathbf{H}\propto\nabla\times\mathbf{A}$ will be unchanged. Thus, if we substitute from Eq. (8.8), we have

$$(\nabla^2+k_e^2)\mathbf{A}'=-i\mu_0(e^2/\omega m)\Delta n\, \mathbf{E}^t, \tag{8.12}$$

where $\mu_0\mathbf{H}=\nabla\times\mathbf{A}'$. The scattered EM field is therefore obtained from the equivalent current

$$\mathbf{J}_{eq}=i(e^2/\omega m)\Delta n\, \mathbf{E}^t, \tag{8.13}$$

radiating into a homogeneous medium of relative dielectric constant $(1-X)$.

The current (8.13) may be obtained by analogy from the usual theory of scattering by dielectrics. The source of the scattered field is the polarization current $\mathbf{J}=-i\omega\Delta\epsilon\, \mathbf{E}^t$; taking $\epsilon=\epsilon_0(1-X)$ gives (8.13).

B. Scattering Cross Sections. Incident P Wave

An incident plane P wave has the fields

$$\begin{aligned}\mathbf{E}_p^i&=\mathbf{z}E_0e^{ik_pz},\\ \mathbf{v}_p^i&=-i\mathbf{z}(\epsilon_0\omega/en_0)E_0e^{ik_pz},\\ n_1^i&=-i\epsilon_0(k_p/e)E_0e^{ik_pz},\end{aligned} \tag{8.14}$$

and the incident power density is

$$P_p^i=\epsilon_0v_0(1-X)^{\frac{1}{2}}E_0E_0^*/X.$$

We now use the Born approximation and say that the incident field is approximately equal to the total field, in the bubble. We also assume that $k_pr\gg1$ and that the linear dimensions of the bubble are much less than k_p^{-1}. From Eq. (8.10) we thus derive the scattered plasma field:

$$n_1^s=-i\langle\Delta n\rangle_{\text{av}}E_0V\frac{ek_p}{mv_0^2}\left(\cos\theta+\frac{1-X}{X}\right)\frac{e^{ik_pr}}{4\pi r},$$

where the origin of the coordinate system is in the bubble and θ is the polar angle from the z axis. The other components of the field are found from Eqs. (3.14) and (3.10), using the far-field approximation:

$$\mathbf{E}_p^s=\mathbf{r}\langle\Delta n\rangle_{\text{av}}E_0V\left(\cos\theta+\frac{1-X}{X}\right)\frac{e^{ik_pr}}{4\pi rD^2n_0},$$

$$\mathbf{v}_p^s=-i\mathbf{r}\langle\Delta n\rangle_{\text{av}}E_0V\left(\cos\theta+\frac{1-X}{X}\right)\frac{\omega\epsilon_0e^{ik_pr}}{4\pi rD^2en_0^2}.$$

The scattered power density per unit solid angle is $r^2mv_0^2n_1^sv_p^{s*}$. Dividing this quantity by P_p^i gives the differential scattering cross section:

$$\sigma_{pp}=\frac{V^2}{16\pi^2D^4}\left(\frac{\langle\Delta n\rangle_{\text{av}}}{n_0}\right)^2\left(\cos\theta+\frac{1-X}{X}\right)^2. \tag{8.15}$$

The source of the scattered EM field is the equivalent current (8.13). We approximate $\mathbf{E}^t$ by $\mathbf{E}_p^i$, and assume that $k_er\gg1$ and that the bubble is small. The distant field thus is the radiation from the short dipole of current moment

$$\mathbf{J}_{eq}V=i\mathbf{z}(e^2/\omega m)\langle\Delta n\rangle_{\text{av}}E_0V.$$

The distant electric field of this dipole is

$$\mathbf{E}_e^s=-\boldsymbol{\theta}\langle\Delta n\rangle_{\text{av}}E_0V\frac{\omega_p^2\sin\theta}{c^2n_0}\frac{e^{ik_er}}{4\pi r}. \tag{8.16}$$

The scattered power density, per unit solid angle, is $r^2c\epsilon_0(1-X)^{\frac{1}{2}}E_e^sE_e^{s*}$, so that the differential scattering cross section is

$$\sigma_{pe}=\frac{V^2X}{16\pi^2D^4}\left(\frac{v_0}{c}\right)^3\left(\frac{\langle\Delta n\rangle_{\text{av}}}{n_0}\right)^2\sin^2\theta. \tag{8.17}$$

This result is of importance in the theory of radio bursts from the sun. One possible step in the production of these bursts is the conversion of plasma to electromagnetic waves by scattering on irregularities of electron density, and Eq. (8.17) is the cross section for this process. The integrated form of Eq. (8.17) has already been used by Ginzburg and Zhelezniakov[8] in their discussions of solar bursts; they obtained it by applying the elementary theory of scattering by dielectrics to the plasma.

The $P\rightarrow EM$ scattering has also been considered by Tidman and Weiss,[7] who used a much more elaborate procedure. Our results essentially agree with theirs; for example, their Eq. (76) reduces to Eq. (8.16) above for the case of a small bubble, provided the incident P wave has a wavelength substantially greater than a Debye length.

C. Scattering Cross Sections. Incident EM Wave

The incident wave now consists of the following fields:

$$\begin{aligned}\mathbf{E}_e^i&=\mathbf{x}E_0e^{ik_ez},\\ \mathbf{H}^i&=\mathbf{y}c\epsilon_0(1-X)^{\frac{1}{2}}e^{ik_ez},\\ \mathbf{v}_e^i&=-i\mathbf{x}(e/\omega m)E_0e^{ik_ez},\end{aligned} \tag{8.18}$$

and the incident power density is

$$P_e{}^i = c\epsilon_0(1-X)^{\frac{1}{2}}E_0E_0{}^*.$$

The scattered P field is found from Eq. (8.10). Using the Born approximation, we have the distant field:

$$n_1{}^s = -i\langle\Delta n\rangle_{\mathrm{av}}E_0V\frac{ek_p\cos\chi}{mv_0{}^2}\frac{e^{ik_pr}}{4\pi r}, \tag{8.19}$$

where χ is the angle between the incident electric vector, and the direction from the bubble to the field point. The other components of the P field may be found from Eqs. (3.14) and (3.10). The scattering cross section is

$$\sigma_{ep} = \frac{V^2}{16\pi^2D^4X}\left(\frac{v_0}{c}\right)\left(\frac{\langle\Delta n\rangle_{\mathrm{av}}}{n_0}\right)^2\cos^2\chi. \tag{8.20}$$

The scattered EM field is the radiation from the equivalent current (8.13). With the Born approximation, the distant electric field is

$$\mathbf{E}_e{}^s = -\chi\langle\Delta n\rangle_{\mathrm{av}}E_0V\frac{\omega_p{}^2\sin\chi}{c^2n_0}\frac{e^{ik_er}}{4\pi r}. \tag{8.21}$$

The cross section is

$$\sigma_{ee} = \frac{V^2}{16\pi^2D^4}\left(\frac{v_0}{c}\right)^4\left(\frac{\langle\Delta n\rangle_{\mathrm{av}}}{n_0}\right)^2\sin^2\chi. \tag{8.22}$$

This equation is well known, and is used to discuss the scattering of radio waves on density irregularities in the ionosphere. Again, it is usually derived by the elementary theory of scattering by dielectrics.

D. Discussion

We list again the four scattering cross sections, to facilitate comparison.

$$\sigma_{pp} = \frac{V^2}{16\pi^2D^4}\left(\frac{\langle\Delta n\rangle_{\mathrm{av}}}{n_0}\right)^2\left(\cos\theta + \frac{1-X}{X}\right)^2,$$

$$\sigma_{pe} = \frac{V^2X}{16\pi^2D^4}\left(\frac{v_0}{c}\right)^3\left(\frac{\langle\Delta n\rangle_{\mathrm{av}}}{n_0}\right)^2\sin^2\theta,$$

$$\sigma_{ep} = \frac{V^2}{16\pi^2D^4X}\left(\frac{v_0}{c}\right)\left(\frac{\langle\Delta n\rangle_{\mathrm{av}}}{n_0}\right)^2\cos^2\chi,$$

$$\sigma_{ee} = \frac{V^2}{16\pi^2D^4}\left(\frac{v_0}{c}\right)^4\left(\frac{\langle\Delta n\rangle_{\mathrm{av}}}{n_0}\right)^2\sin^2\chi.$$

1. The scattered EM components (σ_{pe},σ_{ee}) have the usual polar diagram, involving the sine squared of the angle from the incident electric vector. σ_{ep} has the $\cos^2\chi$ diagram because the scattered P mode is essentially an acoustic field, with a maximum along the dipole axis. σ_{pp} contains a spherically symmetric component because its source contains a monopole component; this component is missing from σ_{ep} because its incident field has no charge accumulation.

2. The cross sections have different orders of magnitude because of the different exponents on (v_0/c). Note, however, that the ratio σ_{ee}/σ_{pp} is approximately $(v_0/c)^4 = (\lambda_p/\lambda_e)^4$, in agreement with what one might expect from Rayleigh scattering. In some problems, the "size" of the elementary scattering volume is essentially selected by the wavelength; so that, for example, in a region full of irregularities, the P-P scattering is not necessarily enormously greater than the EM-EM scattering.

3. The frequency dependence of σ_{pe} and σ_{ep} is in the opposite sense, provided the other quantities (e.g., V) remain constant as ω is varied. We have, however, assumed [I] that for P waves to propagate, $0.5 < X < 1.0$; so that in any event there cannot be much variation in σ_{pe} and σ_{ep}.

4. Since $D^4 = v_0{}^4/\omega_p{}^4$, the cross sections are strongly dependent on electron density. In particular, σ_{pe} is proportional to $n_0{}^3$. The efficiency of conversion of a P to an EM wave may thus increase strongly as the P wave travels up in the ionosphere (or down in the solar corona) towards its plasma level.

5. The form of σ_{pp} depends on Eq. (8.1), whereas, σ_{pe} and σ_{ee} are independent of Eq. (8.1), and σ_{ep} has only a second order dependence on Eq. (8.1). In a more realistic problem it will be difficult to establish a boundary condition on pressure, but in any event σ_{pe}, σ_{ee}, and σ_{ep} (to first order) will be independent of the pressure condition.