Radiation in a Plasma. III. Metal Boundaries*

MARSHALL H. COHEN
Center for Radiophysics and Space Research, Cornell University, Ithaca, New York
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The reflection of a plane wave in a plasma at a metal boundary is discussed by using the linear continuum theory. A positive sheath is the consequence of requiring the net current flow into the boundary to be zero. The dipole antenna in a plasma is considered from the scattering point of view. Integral equations for the dipole fields are derived but not solved. The fields and radiation resistances for a short linear current filament are given.

I. INTRODUCTION

IN Parts I and II of this series[1,2] (hereafter referred to as [I] and [II]) we have discussed the radiation from sources in a plasma, in terms of a linear continuum theory. In this paper we wish to extend the discussion to include metal boundaries.

We start by considering the reflection of a plane wave at a plane metal boundary. Bohm and Gross[3] have discussed this problem. They took the sheath to be the boundary of the plasma. Particles are reflected from the sheath with various phases, according to their velocities, and so energy is lost from the coherent wave motion upon reflection.

Our point of view is different from that of Bohm and Gross. Instead of starting with a sheath next to the metal wall, we assume that the plasma electron fluid extends smoothly to the wall. The sheath then arises as a consequence of the wave's hitting the wall, when we apply the condition that there can be no net current flow into the wall. In this way the linear continuum theory can account for the existence of a sheath. Some details, however, such as the distribution of potential within the sheath, are unrealistic.

We then wish to apply these concepts to a dipole antenna radiating in a plasma. A discussion in terms of scattering is very fruitful for the dipole in free space, for it provides simple approximation methods. It appears that this method may be of use for the dipole in plasma, also. In Sec. V, therefore, we derive an appropriate scattering theorem. In Sec. VI we apply the theorem to a dipole in plasma. The problem is much more complicated than the free-space dipole, principally because the equivalent sources on the wire must include the fluid sources Q and F [II] as well as the usual electric current, **J**. We derive integral equations for the fields, but make no attempt to solve them. We do, however, show the results of a calculation of the far fields and radiation resistance that would be obtained from a short sinusoidal filament of electric current in the plasma.

We make extensive use of several concepts developed in [I] and [II]. The first of these is that the total field in the plasma can be split into two modes. One, the plasma (P) mode, has no magnetic field. At great distances from the source it is a longitudinal (radial) wave with propagation constant $\omega(1-X)^{\frac{1}{2}}/v_0$, where $X=\omega_p^2/\omega^2$, $v_0=$rms thermal velocity. The electromagnetic (EM) mode has no charge accumulation and is the ordinary electromagnetic wave in a dispersive medium of relative dielectric constant $(1-X)$. At great distances from the source it is a transverse wave with propagation constant $\omega(1-X)^{\frac{1}{2}}/c$.

The second concept is that of sources in the plasma. We shall use electric current and charge, **J**, ρ; magnetic current and charge, **K**, ρ^m; a fluid flux source, Q, and a body force **F**. The continuity equations are

$$\nabla\cdot\mathbf{J}+\partial\rho/\partial t-eQ=0, \tag{1.1}$$

$$\nabla\cdot\mathbf{K}+\partial\rho^n/\partial t=0. \tag{1.2}$$

Surface distributions of these sources are connected to discontinuities in the fields in a manner entirely analogous to the surface discontinuities found in electromagnetic theory and acoustics.

The symbols and units (mks rationalized) used in this paper are the same as in [I] and [II]. The time variation for harmonic sources is $e^{-i\omega t}$.

II. BOUNDARY CONDITIONS

In Fig. 1 we show a Cartesian coordinate system with unit vectors **x**, **y**, **z**. A homogeneous plasma occupies the region $z>0$, and the plane $z=0$ is a rigid wall with infinite electrical conductivity. We seek the boundary conditions to be imposed on waves in the plasma.

The first condition is

$$\mathbf{z}\times\mathbf{E}=0, \tag{2.1}$$

since the wall is an electric conductor. It would be sufficient to specify similarly either $\mathbf{z}\cdot\mathbf{v}$ or n_1 [II]. The comparable acoustic problem would, in fact, use the condition $\mathbf{z}\cdot\mathbf{v}=0$, since the fluid could not penetrate the wall. In our case, however, the fluid consists of electrons which *are* able to penetrate the wall. The condition $\mathbf{z}\cdot\mathbf{v}=0$ is therefore too restrictive. We shall be more general and assume a surface admittance; we can always recover the case $\mathbf{z}\cdot\mathbf{v}=0$ by letting the admittance parameters go to zero.

The surface admittance is a useful concept in electro-

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[1] M. H. Cohen, Phys. Rev. **123**, 711 (1961).
[2] M. H. Cohen, preceding paper [Phys. Rev. **125**, 389 (1962)].
[3] D. Bohm and E. P. Gross, Phys. Rev. **79**, 992 (1950).

magnetic theory and in acoustics. It is not a quantity which is usually brought into the boundary conditions, since, for example, in a reflection problem its value generally depends on the angle of incidence. Nevertheless, we shall assume a surface admittance relation and shall use it as a boundary condition. Admittedly, we shall not have solved the boundary-value problem, but we shall be able to discuss in a simple manner the reflection and absorption coefficients and the formation of a sheath, in terms of the admittance coefficients.

In some other problems, the admittance in fact is merely a convenient symbol, and boundary conditions (e.g. continuity of $\mathbf{n}\times\mathbf{E}$ and $\mathbf{n}\times\mathbf{H}$) also have to be brought explicitly into the problem.[4] In many acoustic cases, however, the surface admittance is nearly independent of the incident wave, and it is then used instead of other boundary conditions.[5]

The acoustic analogy to our reflection problem would lead to a linear relation between $\mathbf{z}\cdot\mathbf{v}$ and n_1 on the plane $z=0$, since the admittance is the ratio of velocity to excess pressure. The excess pressure is the body force in acoustics; but in the plasma the electric field also contributes to the body force, and we should include it in the admittance. We assume, therefore, the bilinear admittance relation:

$$\mathbf{z}\cdot\mathbf{v}=Y_a\mathbf{z}\cdot\mathbf{E}+Y_b n_1. \tag{2.2}$$

The admittance coefficients Y_a and Y_b may be functions of the configuration of the incident wave, e.g., of the angle of incidence, or of the mode type. They may also be functions of frequency. We assume that Eq. (2.2) holds also for zero frequency, in which case Y_a and Y_b are negative real numbers.

In principle, the coefficients Y_a and Y_b can be measured, since for each incident wave there are two reflected waves (one *EM* and one *P* wave). A calculation of the coefficients themselves, however, would have to start from the opposite point of view. One would have to solve in detail the plasma-metal boundary problem, including the fields and electron motions inside the metal.

When we apply the relations (2.1) and (2.2) to the reflection problem, we shall find that in general there is a net transport of electron flux across the boundary. In principle this is permissible, since we have an infinite plasma and a boundary with an infinite sink capacity. In a laboratory situation, however, the flux transport would lead to a charge separation and so could not be allowed. Furthermore, in a region such as the ionosphere, the boundary cannot have an infinite capacity, and flux transport could not be allowed there, either. To be realistic, therefore, we shall ultimately require that

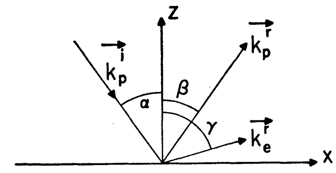


FIG. 1. Incident *P* wave, reflected *P* and *EM* waves.

the net flux transport be zero:

$$\langle(n_0+n_1)\mathbf{v}\cdot\mathbf{z}\rangle=0, \tag{2.3}$$

where the brackets signify a time average.

The above model is inconsistent, since we have ignored thermal motions but have assumed that an electron with velocity $\mathbf{v}$ due to passage of a wave can penetrate the wall according to Eq. (2.2). To some extent, thermal motions can be included by analyzing the thermal fluctuations in density into a set of plasma waves. Each of these sets up an elementary sheath, according to the discussions below. Thus, even in the absence of a source of coherent waves, there will be a sheath next to the wall.

III. REFLECTION COEFFICIENTS

A. Incident Plasma Wave

As shown in Fig. (1), a plane *P* wave is incident at the angle α. Assume that there is a reflected *P* wave at the angle β and a reflected *EM* wave at the angle γ. The three waves have the same frequency, and the three propagation directions are coplanar.

Using superscripts i and r for incident and reflected waves, we write the electric fields as follows:

$$\begin{aligned}\mathbf{E}_p{}^i&=E_p{}^i(\mathbf{k}_p{}^i/k_p)\exp(i\mathbf{k}_p{}^i\cdot\mathbf{r}),\\ \mathbf{E}_p{}^r&=E_p{}^r(\mathbf{k}_p{}^r/k_p)\exp(i\mathbf{k}_p{}^r\cdot\mathbf{r}),\\ \mathbf{E}_e{}^r&=E_e{}^r(\mathbf{y}\times\mathbf{k}_e{}^r/k_e)\exp(i\mathbf{k}_e{}^r\cdot\mathbf{r}).\end{aligned} \tag{3.1}$$

The velocity fields are

$$\begin{aligned}\mathbf{v}_p{}^i&=-[ie\omega/(m\omega_p{}^2)]\mathbf{E}_p{}^i,\\ \mathbf{v}_p{}^r&=-[ie\omega/(m\omega_p{}^2)]\mathbf{E}_p{}^r,\\ \mathbf{v}_e{}^r&=-[ie/(m\omega)]\mathbf{E}_e{}^r.\end{aligned} \tag{3.2}$$

The density fields are

$$\begin{aligned}n_1{}^i&=-i(\epsilon_0/e)\mathbf{k}_p{}^i\cdot\mathbf{E}_p{}^i,\\ n_1{}^r&=-i(\epsilon_0/e)\mathbf{k}_p{}^r\cdot\mathbf{E}_p{}^r.\end{aligned} \tag{3.3}$$

The magnetic field is

$$\mathbf{H}^r=(\omega\mu_0)^{-1}\mathbf{k}_e{}^r\times\mathbf{E}_e{}^r. \tag{3.4}$$

Application of the boundary condition (2.1) gives

$$\alpha=\beta,\quad \sin\gamma=(k_p/k_e)\sin\alpha=(c/v_0)\sin\alpha, \tag{3.5}$$

as the phase requirements. The character of the reflected electromagnetic wave will evidently be different for $(c/v_0)\sin\alpha$ less than and greater than unity. In the latter case $\cos\gamma$ is imaginary, and the wave becomes a surface wave propagating in the x direction, and

[4] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 511; P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), p. 1814.

[5] P. M. Morse and K. U. Ingard, *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1961), Vol. XI/1, pp. 34–99.

exponentially attenuated in the z direction. Since $(c/v_0)\gg 1$, the surface wave will be the usual situation, and the freely propagating wave will exist only when the incident plasma wave is nearly normal to the surface. Such surface waves are also found in the reflections set up by a planar density discontinuity in a plasma.[6]

Simultaneous application of conditions (2.1) and (2.2) gives the reflection coefficients:

$$\frac{E_p{}^r}{E_p{}^i}=-\frac{(bY_b+\cos\alpha-aY_a\cos\alpha)[\csc^2\alpha-(c/v_0)^2]^{\frac{1}{2}}+(c/v_0)(aY_a-X)\sin\alpha}{(bY_b-\cos\alpha+aY_a\cos\alpha)[\csc^2\alpha-(c/v_0)^2]^{\frac{1}{2}}+(c/v_0)(aY_a-X)\sin\alpha}, \tag{3.6}$$

$$\frac{E_e{}^r}{E_p{}^i}=\frac{2(1-aY_a)\cos\alpha}{(bY_b-\cos\alpha+aY_a\cos\alpha)[\csc^2\alpha-(c/v_0)^2]^{\frac{1}{2}}+(c/v_0)(aY_a-X)\sin\alpha}, \tag{3.7}$$

where

$$a=im\omega_p{}^2/(e\omega),\quad b=n_0k_p/\omega.$$

The average power flow into the boundary is the real part of the expression $-\frac{1}{2}(\mathbf{E}\times\mathbf{H}^*+mv_0{}^2n_1\mathbf{v}^*)\cdot\mathbf{z}$ [II]. Since $\mathbf{E}\times\mathbf{z}=0$, we have

$$P=-\tfrac{1}{2}\operatorname{Re}\{mv_0{}^2n_1\mathbf{v}^*\cdot\mathbf{z}\}\ \text{watts/m}^2. \tag{3.8}$$

In terms of the reflection coefficients, this can be written (for $\sin\alpha<v_0/c$)

$$\begin{aligned}P=&\tfrac{1}{2}\epsilon_0v_0(1-x)^{\frac{1}{2}}|E_p{}^i|^2\\&\times\{X^{-1}\cos\alpha[1-(E_p{}^r/E_p{}^i)(E_p{}^r/E_p{}^i)^*]\\&+(c/v_0)\sin\alpha(E_e{}^r/E_p{}^i)^*(1+E_p{}^r/E_p{}^i)\}.\end{aligned} \tag{3.9}$$

When $\sin\alpha>v_0/c$, the last term in Eq. (3.9) is imaginary, and must be dropped. Eq. (3.9) can in turn be written in terms of the real and imaginary parts of Y_a and Y_b, but this expression becomes rather complicated.

B. Incident Electromagnetic Wave

An incident plane *EM* wave can be decomposed into two components, one polarized with the electric vector parallel to the boundary, and the other polarized in the plane of incidence. The former has $\mathbf{z}\cdot\mathbf{v}=\mathbf{z}\cdot\mathbf{E}=0$ everywhere and is not interesting. The problem, in this case, reduces to the usual one of a dielectric-metal boundary, and P waves are not generated.

Let the incident wave be polarized in the plane of incidence, as in Fig. 2. There will be reflected *EM* and P waves, as shown. The electric fields are

$$\begin{aligned}\mathbf{E}_e{}^i&=E_e{}^i[\mathbf{y}\times\mathbf{k}_e{}^i/k_e]\exp(i\mathbf{k}_e{}^i\cdot\mathbf{r}),\\ \mathbf{E}_e{}^r&=E_e{}^r[\mathbf{y}\times\mathbf{k}_e{}^r/k_e]\exp(i\mathbf{k}_e{}^r\cdot\mathbf{r}),\\ \mathbf{E}_p{}^r&=E_p{}^r[\mathbf{k}_p{}^r/k_p]\exp(i\mathbf{k}_p{}^r\cdot\mathbf{r}).\end{aligned} \tag{3.10}$$

The velocity fields are

$$\begin{aligned}\mathbf{v}_e{}^i&=-[ie/(m\omega)]\mathbf{E}_e{}^i,\\ \mathbf{v}_e{}^r&=-[ie/(m\omega)]\mathbf{E}_e{}^r,\\ \mathbf{v}_p{}^r&=-[ie\omega/(m\omega_p{}^2)]\mathbf{E}_p{}^r.\end{aligned} \tag{3.11}$$

The density field is

$$n_1{}^r=-i(\epsilon_0/e)\mathbf{k}_p{}^r\cdot\mathbf{E}_p{}^r. \tag{3.12}$$

The magnetic fields are

$$\begin{aligned}\mathbf{H}^i&=(\omega\mu_0)^{-1}\mathbf{k}_e{}^i\times\mathbf{E}_e{}^i,\\ \mathbf{H}^r&=(\omega\mu_0)^{-1}\mathbf{k}_e{}^r\times\mathbf{E}_e{}^r.\end{aligned} \tag{3.13}$$

Application of the conditions (2.1) and (2.2) now gives

$$\alpha=\beta,\quad \sin\gamma=(v_0/c)\sin\alpha, \tag{3.14}$$

and the reflection coefficients

$$\frac{E_e{}^r}{E_e{}^i}=-\frac{(v_0/c)(X-aY_a)\tan\alpha-(1-aY_a)[\csc^2\alpha-(v_0/c)^2]^{\frac{1}{2}}+bY_b\csc\alpha}{(v_0/c)(X-aY_a)\tan\alpha+(1-aY_a)[\csc^2\alpha-(v_0/c)^2]^{\frac{1}{2}}-bY_b\csc\alpha}, \tag{3.15}$$

$$\frac{E_p{}^r}{E_e{}^i}=\frac{2(X-aY_a)}{(v_0/c)(X-aY_a)\tan\alpha+(1-aY_a)[\csc^2\alpha-(v_0/c)^2]^{\frac{1}{2}}-bY_b\csc\alpha}, \tag{3.16}$$

The power flow across the boundary is

$$P=\tfrac{1}{2}\epsilon_0v_0(1-X)^{\frac{1}{2}}|E_e{}^i|^2\left\{\operatorname{Re}\left[\sin\alpha\left(\frac{E_p{}^r}{E_e{}^i}\right)\left(1+\frac{E_e{}^r}{E_e{}^i}\right)^*\right]-\frac{1}{X}\left(\frac{E_p{}^r}{E_e{}^i}\right)\left(\frac{E_p{}^r}{E_e{}^i}\right)^*[1-(v_0/c)^2\sin^2\alpha]^{\frac{1}{2}}\right\}. \tag{3.17}$$

Again, the expression for the power in terms of the real and imaginary parts of Y_a and Y_b is rather complicated.

IV. ELECTRON FLUX TRANSPORT AND SHEATH FORMATION

We now consider Eq. (2.3), the condition that there be no charge separation. The mean flux of electrons across the boundary is $\langle(n_0+n_1)\mathbf{v}\cdot\mathbf{z}\rangle$. We have taken n_0 constant and n_1 and $\mathbf{v}$ to vary harmonically in time. Thus the mean electron flux is proportional to the mean power flow, Eq. (3.8). If we set the mean flux equal to

[6] A. H. Kritz and D. Mintzer, Phys. Rev. **117**, 382 (1960).

zero we also set Eq. (3.9) or (3.17) equal to zero, according to the circumstances. This would then lead to a relation between α, Y_a, and Y_b which presumably does not exist. It would imply that the electron flux is always zero, whereas this is not necessarily true for a semi-infinite plasma with a boundary with an infinite sink capacity.

We resolve this problem by assuming that the fields we have used above are not the complete set which exists in the plasma. Since the system is linear, the extra fields can only be the remnants of transients which existed when the field was first excited.

Transients at frequencies $\omega>0$ and $\omega\neq\omega_p$ either propagate away from the boundary or else die out exponentially in time. In principle, a standing wave at $\omega=\omega_p$ can be a remnant of the transient, but we may assume that the medium has a small loss, which serves to reduce the standing wave to arbitrarily small values after a sufficiently long time. The static remnant ($\omega=0$) is all that is left of the transient. The small medium loss will not dissipate the static field in the way that a lossy dielectric discharges a capacitor, because we still have the requirement (2.3), and the incident field must maintain the static charge on the boundary.

Let ρ_0 be the static surface charge density on the boundary. From Eqs. (3.1), (3.14), and (4.2) of [II], this charge produces the following fields:

$$n_1{}^0=[\rho_0/(eD)]e^{-z/D}, \tag{4.1}$$

$$\mathbf{E}_p{}^0=\mathbf{z}(\rho_0/\epsilon_0)e^{-z/D}, \tag{4.2}$$

where the superscript 0 stands for the static components of the fields.

The boundary condition (2.2) must be applied to these static fields:

$$\mathbf{z}\cdot\mathbf{v}^0=\rho_0[Y_a{}^0/\epsilon_0+Y_b{}^0/(eD)]. \tag{4.3}$$

In this manner the static field produces a steady flux, $(n_0+n_1{}^0)\mathbf{z}\cdot\mathbf{v}^0$, which cancels the average flux produced by the oscillating components of the field. This flux of electrons proceeds at constant velocity away from the boundary. The effect of the static pressure is cancelled by the electric field [Eqs. (4.1) and (4.2)] so the net static force on the particles is zero.

From Eq. (2.3) we now have

$$n_0\mathbf{v}^0\cdot\mathbf{z}=-\langle n_1{}^\omega\mathbf{v}^\omega\cdot\mathbf{z}\rangle, \tag{4.4}$$

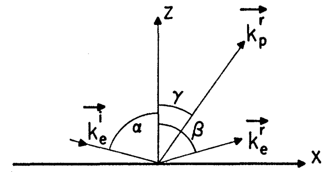
where the superscript ω stands for the components at frequency ω, and where we have neglected $n_1{}^0$ in comparison with n_0. By Eq. (3.8),

$$P_{\mathrm{ac}}=-\tfrac{1}{2}\,\mathrm{Re}\{mv_0{}^2n_1{}^\omega\mathbf{v}_1{}^{\omega*}\cdot\mathbf{z}\}$$

is the average power associated with the first order ac components of the wave. The static surface charge density thus is given by

$$\rho_0=\frac{DP_{\mathrm{ac}}}{mn_0v_0{}^2[Y_a{}^0/\epsilon_0+Y_b{}^0/(eD)]}. \tag{4.5}$$

FIG. 2. Incident EM wave, reflected EM and P waves.



It is reasonable to assume that $P_{\mathrm{ac}}>0$, i.e., that the wall is passive. By their definitions, $Y_a{}^0$ and $Y_b{}^0$ must be negative. The surface charge is thus negative, and, from Eq. (4.1), $n_1{}^0$, the first-order static excess of electrons in the plasma, is also negative. The quantity $n_1{}^0$ thus comprises a positive sheath next to the boundary; the deficiency of electrons decreases exponentially with a scale on the order of a Debye length.

The total average power absorbed at the wall (per unit area), P_t, is now given by the following expression, [II], evaluated at $z=0$.

$$\begin{aligned}P_t&=-mv_0{}^2\langle(n_1{}^0+n_1{}^\omega)(\mathbf{v}^0+\mathbf{v}^\omega)\rangle\cdot\mathbf{z}\\&=-mv_0{}^2(n_1{}^0\mathbf{v}^0+\langle n_1{}^\omega\mathbf{v}^\omega\rangle)\cdot\mathbf{z}.\end{aligned}$$

By Eq. (4.4), $\mathbf{v}^0$ is a second order quantity, so the term $n_1{}^0\mathbf{v}^0$ is negligible compared with $\langle n_1{}^\omega\mathbf{v}^\omega\rangle$. Thus $P_t\approx P_{\mathrm{ac}}$, and the static fields, which neutralize the flux transport, have only a third order effect on the total energy absorbed at the boundary. (The ac energy is a second order term.) The equilibrium reflection process can thus be described in terms of electrons making inelastic collisions with an impenetrable wall. The net flux transport is zero. The incident electrons have low density and high velocity. The emergent electrons have a high density but such a low velocity that they carry negligible energy.

A. Discussion

We have shown that whenever there is a net power flow into the wall, there will be a sheath, connected with the requirement that the net electron flux be zero. This result, as we have obtained it, depends on the condition $\mathbf{n}\times\mathbf{E}=0$, for it is only with this condition that the ac power flow is proportional to the electron flux. It does not, however, directly depend on the admittance assumption. We could have begun the discussion by assuming static and ac velocity components of undetermined amplitudes. The requirement of no charge separation would then have led to a nonzero static velocity, proportional to the power absorbed at the wall. This static velocity would have had to be connected with a steady field of force which pulled the electrons out of the boundary. Finally, from the Klein-Gordon equation, any one-dimensional static field must have the exponential screening form. The *strength* of the sheath, on the other hand, does depend directly on the admittance parameters, as in Eq. (4.5).

The sheath formation in the transient period of a reflection process can be described in the same terms as used ordinarily when describing a sheath in terms of

randomly moving particles.[7] The electrons move much more rapidly than the ions (we have assumed the ions to be stationary) and, at the beginning of the process, there is a net flux of electrons to the wall. These electrons accumulate on the boundary and produce a static field which opposes the electron flux until, in equilibrium, there is no net current flow into the wall.

The thermal fluctuations in electron density that exist in the plasma can be regarded as a set of plasma waves. Each of them, when striking a metal boundary, will be reflected according to the processes discussed above. Thus a sheath, with thickness on the order of a Debye length, will be set up next to the wall, and power will flow from the waves into the wall. This is equivalent to the normal description of the cooling of a plasma by its being in contact with the container. We have obtained the results, however, from the linear continuum theory, rather than from a statistical consideration of random electron and ion motions.

V. A SCATTERING THEOREM

Let V be a source-free volume bounded by the surface S with outward normal $\mathbf{n}$. The medium inside V is different from that outside, where there is a homogeneous plasma. In particular, S may be a metal surface. Let there be an external harmonic source.

Define the set of incident fields as $\mathbf{\Gamma}^i \equiv (\mathbf{E}^i, \mathbf{H}^i, \mathbf{v}^i, n_1{}^i)$, i.e., $\mathbf{\Gamma}^i$ is the field that the source would generate in a homogeneous unbounded plasma. Define the total field $\mathbf{\Gamma}^t$ as the field that actually exists in the presence of the scatterer, and define the scattered field as $\mathbf{\Gamma}^s = \mathbf{\Gamma}^t - \mathbf{\Gamma}^i$. The sources of the scattered field are inside or on S, and $\mathbf{\Gamma}^s$ satisfies the source-free field equations for the homogeneous plasma everywhere outside S [II]. We shall not be concerned with $\mathbf{\Gamma}^s$ or $\mathbf{\Gamma}^t$ inside V.

Now define a modified scattered field $\mathbf{\Gamma}^m$, equal to $\mathbf{\Gamma}^s$ outside the scatterer, and equal to $-\mathbf{\Gamma}^i$ inside the scatterer. $\mathbf{\Gamma}^m$ evidently satisfies the source-free field equations (for the homogeneous exterior medium) everywhere except on S, where there are discontinuities in $\mathbf{\Gamma}^m$ equal to $\mathbf{\Gamma}^t$ (evaluated on the outside of S). We see from Eqs. (4.1) to (4.6) of [II] that the following set of sources on S

$$\begin{aligned} &\mathbf{J} = \mathbf{n} \times \mathbf{H}^t \quad && \mathbf{K} = \mathbf{E}^t \times \mathbf{n} \\ &\rho = \epsilon_0 \mathbf{n} \cdot \mathbf{E}^t \quad && \rho^m = \mu_0 \mathbf{n} \cdot \mathbf{H}^t \\ &Q = n_0 \mathbf{n} \cdot \mathbf{v}^t \quad && F = m v_0{}^2 n_1{}^t, \end{aligned} \tag{5.1}$$

regarded as radiating in an unbounded homogeneous plasma, generates identically the scattered field outside the scatterer, and the negative of the incident field inside the scatterer.

This theorem is a generalization of the two usual theorems which are separately applicable in electromagnetic theory and in acoustics. If we set the electronic charge $(-e)$ equal to zero, the fields and sources separate, and the two simpler theorems appear.

When S is a metal surface, $\mathbf{n} \times \mathbf{E} = 0$, and $\mathbf{K}$ and ρ^m are zero. In this case the electric current $\mathbf{J}$ is the actual current flowing on the surface. The difference from the usual free-space electromagnetic case is that Q and F are not necessarily zero.

It is of interest to apply this theorem to the reflection problem. For an incident plasma wave, Eqs. (3.1) to (3.5) and (4.1) to (4.3) give the sources on S:

$$\begin{aligned} \mathbf{J} &= -\mathbf{x}[k_e/(\omega\mu_0)]E_e{}^r \cos(k_p x \sin\alpha - \omega t), \\ \rho &= \epsilon_0[(E_p{}^r - E_p{}^i)\cos\alpha - (c/v_0)E_e{}^r \sin\alpha] \\ &\qquad \times \cos(k_p x \sin\alpha - \omega t) + \rho_0, \\ Q &= (\epsilon_0\omega/e)[(E_p{}^r - E_p{}^i)\cos\alpha - X(c/v_0)E_e{}^r \sin\alpha] \\ &\qquad \times \sin(k_p x \sin\alpha - \omega t) \\ &\qquad\quad + n_0\rho_0[Y_a{}^0/\epsilon_0 + Y_b{}^0/(eD)], \\ F &= n_0 e k_p D^2 (E_p{}^i + E_p{}^r)\sin(k_p x \sin\alpha - \omega t) \\ &\qquad + m v_0{}^2 \rho_0/(eD). \end{aligned} \tag{5.2}$$

The oscillating components of $(\mathbf{J}, \rho, Q)$ satisfy the continuity equation (1.1), as they should. The static component of Q is not connected to a singular static distribution of $\mathbf{J}$, as can be seen from the following argument. The surface distribution of Q must be regarded as the limit of a continuous distribution spread through a slab. We may take the slab of thickness s, and the source volume density as Q_v, so that $Q = sQ_v$. By using the relation $\nabla \cdot \mathbf{J} = eQ_v$ we see that $\mathbf{J}$ has only a z-component, $J_z = zeQ/s$. This current has maximum value eQ, and its integral, the two-dimensional surface current, is zero in the limit $s \to 0$.

The fields that these surface sources would radiate if they existed by themselves in an unbounded plasma can be found from Eqs. (5.1) to (5.7) of [II]. It is easy to show that these fields are identically the reflected fields, in the region $z > 0$; and the negative of the incident field, in the region $z < 0$.

VI. DIPOLE ANTENNA

A. Introduction

One makes a very good approximation for the radiation from a dipole antenna in free space, by assuming that there is a sinusoidal current in the wire, and that it radiates as if it were in free space, i.e., as if the metal boundary were not there. A motivation for this procedure is obtained by regarding the dipole as a scattering problem. The arms of the dipole form a scattering body, and the source is in the gap between the arms. The incident field (in the absence of the wires) is nearly negligible, so that the total field is approximately the same as the scattered field. The latter is generated by the terminating currents ($\mathbf{J}$ and $\mathbf{K}$) on the wire, radiating as if they were in free space, according to our discussion in the preceding section. But $\mathbf{K} = 0$ because

[7] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), p. 17.

$\mathbf{n}\times\mathbf{E}=0$, and $\mathbf{J}$ is the actual surface current flowing on the wire.

When we try to extend this procedure to a dipole in a plasma, we encounter serious difficulties. The assumption that the scattered field is nearly equal to the total field seems reasonable. The sources of the scattered field, however, include $\mathbf{J}$, ρ, Q, and F (the magnetic current is again zero because $\mathbf{n}\times\mathbf{E}=0$). There are three independent quantities among the four sources, and we must assume that all exist on the wire surface, as they do on the metal plane in the preceding section. The sources will include oscillating and static components. Finally, there are two modes of propagation in the plasma, with different wavelengths. If we assume, *a priori*, that the sources are sinusoidally distributed, we do not know which wavelength to use.

The static components of the sources, in the linear approximation, have no effect on the waves generated by the antenna. They are only connected to the sheath which will exist around the antenna. The existence of the sheath presumably implies an average power flow into the wire, as it did in the reflection problem, and this will have an effect on antenna efficiency.

Even if we make the approximation that $\mathbf{n}\cdot\mathbf{v}=0$ on the wire surface, we cannot readily obtain an approximate solution. This boundary condition makes $Q=0$, but leaves F on the surface, to contribute to both the P and EM modes. We have no simple way of estimating the relative *strengths* of $\mathbf{J}$ and F; and indeed, we have no experience for making a guess for the *form* of F.

In this connection we may note that the source F in Eq. (5.2) produces fields of the same order of magnitude as the set $(\mathbf{J},\rho,Q)$. It clearly is not permissable to ignore the acoustic sources.

B. Integral Equations

It is necessary to have boundary conditions for the surface of the wire. The simplest ones are $\mathbf{n}\times\mathbf{E}=0$, and $\mathbf{n}\cdot\mathbf{v}=0$. The condition $\mathbf{n}\times\mathbf{E}=0$ is valid if the wire has a high conductivity. We have assumed that the condition $\mathbf{n}\cdot\mathbf{v}=0$ is not correct, but it may be an adequate approximation in some circumstances.

The dipole arms form a scattering body. Assume that the incident field is negligible, so that the total field is approximately equal to the scattered field. This is valid when the gap between the dipole arms is small. According to the scattering theorem developed above, the field is the same as that produced by the set of sources (5.1) on the surface of the wire, radiating into an unbounded homogeneous plasma. Assume that $\mathbf{n}\times\mathbf{E}$ and $\mathbf{n}\cdot\mathbf{v}$ are zero on the surface of the wire. This makes Q, $\mathbf{K}$, and ρ^m zero.

The radiated plasma field is given by Eq. (8.9) of [II]. If we substitute for ρ from the continuity equation, this gives

$$n_1=\frac{1}{4\pi}\iiint_{V^+}\left(\frac{\nabla'\cdot\mathbf{J}}{i\omega eD^2}-\frac{\nabla'\cdot\mathbf{F}}{mv_0^2}\right)\frac{e^{ik_pr}}{r}dv'. \tag{6.1}$$

In Eq. (6.1), V^+ includes the volume of the dipole. The primed coordinates refer to a source point, and unprimed coordinates refer to a field point. The sources $\mathbf{J}$ and $\mathbf{F}$ are three-dimensional volume distributions, and one of the integrations will convert them into surface distributions.

We first transform Eq. (6.1) by using the divergence theorem. This gives a surface integral which is zero because the sources are zero on the surface of V^+, and the following volume integral,

$$n_1=\frac{1}{4\pi}\iiint_{V^+}\left(\frac{\mathbf{F}}{mv_0^2}-\frac{\mathbf{J}}{i\omega eD^2}\right)\cdot\nabla'\left(\frac{e^{ik_pr}}{r}\right)dv'$$

$$=\frac{1}{4\pi}\nabla\cdot\iiint_{V^+}\left(\frac{\mathbf{J}}{i\omega eD^2}-\frac{\mathbf{F}}{mv_0^2}\right)\frac{e^{ik_pr}}{r}dv'.$$

Now use the definitions for the surface distributions of $\mathbf{J}$ and $\mathbf{F}$. This gives

$$n_1=\frac{1}{4\pi}\nabla\cdot\iint_S\left(\frac{\mathbf{n}\times\mathbf{H}}{i\omega eD^2}-n_1\mathbf{n}\right)\frac{e^{ik_pr}}{r}ds', \tag{6.2}$$

where S is the surface of the antenna. Equation (6.2) is one of a set of two integral equations for the fields.

The EM field is obtained from Eq. (8.11) of [II]. The solution to this equation is

$$\mathbf{A}=\frac{\mu_0}{4\pi}\iiint_{V^+}\left(\mathbf{J}+\frac{e}{i\omega m}\mathbf{F}_e\right)\frac{e^{ik_er}}{r}dv', \tag{6.3}$$

where $\mu_0\mathbf{H}=\nabla\times\mathbf{A}$, and $\mathbf{F}_e$ is the solenoidal part of $\mathbf{F}$. Since $\mathbf{H}$ is the curl of $\mathbf{A}$, we may substitute $\mathbf{F}$ for $\mathbf{F}_e$ in (6.3) without changing $\mathbf{H}$. Thus,

$$\mathbf{H}=\frac{1}{4\pi}\nabla\times\iiint_{V^+}\left(\mathbf{J}+\frac{e}{i\omega m}\mathbf{F}\right)\frac{e^{ik_er}}{r}dv'.$$

Again use the definitions of the surface distributions. This gives

$$\mathbf{H}=\frac{1}{4\pi}\nabla\times\iint_S[\mathbf{n}\times\mathbf{H}-i(ev_0^2/\omega)n_1\mathbf{n}]\frac{e^{ik_er}}{r}ds'. \tag{6.4}$$

Equations (6.2) and (6.4) form a pair of simultaneous linear integral equations for n_1 and $\mathbf{H}$. In this paper we make no attempt to solve them.

C. Current Filament

Consider an oscillating current filament along the z-axis of a Cartesian coordinate system:

$$\left.\begin{aligned} I_z &= I_0 \cos kz \\ \rho &= (ikI_0/\omega)\sin kz \end{aligned}\right\}, \quad \text{for} \quad |z| < L/2 = \pi/(2k),$$

$$I_z = \rho = 0, \qquad \text{for} \quad |z| > L/2.$$

We expect that this current will be close to the true distribution of **J** on the surface of a thin wire dipole in a plasma, provided $2L$ is smaller than the EM mode wavelength.

The radiated fields are computed according to the methods developed in [II]. In the far field the EM mode has an electric field with only a θ component:

$$E_{e\theta} = -\frac{i\omega\mu_0 I_0 e^{ik_e r}}{2\pi kr}\,\frac{\sin\theta\cos[(\pi k_e\cos\theta)/(2k)]}{1-(k_e/k)^2\cos^2\theta}.$$

Associated with $E_{e\theta}$ is a magnetic field H_φ, and a velocity field $v_{e\theta}$.

The P mode, in the far field, has an electric field with a radial component:

$$E_{pr} = \frac{i\omega I_0 X e^{ik_p r}}{2\pi\epsilon_0 v_0^2 kr}\,\frac{\cos\theta\cos[(\pi k_p\cos\theta)/(2k)]}{1-(k_p/k)^2\cos^2\theta}.$$

Associated with E_{pr} is a velocity field v_{pr}, and a density field n_1.

The EM mode is a transverse electromagnetic field, with a pattern with a null along the axis. The P mode is a longitudinal field, with a pattern with a maximum along the axis.

The components of radiation resistance are found by integrating the radiated power densities and dividing by $\frac{1}{2}|I_0|^2$. This gives

$$R_e = 120(L/\lambda_0)^2(1-X)^{\frac{1}{2}}\Lambda_e,$$
$$R_p = 120(L/\lambda_0)^2(c/v_0)^3 X(1-X)^{\frac{1}{2}}\Lambda_p,$$

where λ_0 is the free space wavelength, and the functions Λ_e and Λ_p are

$$\begin{aligned}\Lambda_e(q) = {} & (2q)^{-1}(1+q^{-2})[\mathrm{Cin}(\pi+\pi q)-\mathrm{Cin}(\pi-\pi q)] \\ & +\pi(2q)^{-1}(1-q^{-2})[\mathrm{Si}(\pi+\pi q)-\mathrm{Si}(\pi-\pi q)] \\ & -2q^{-2}\cos^2(\pi q/2),\end{aligned}$$

$$\begin{aligned}\Lambda_p(s) = {} & (2s^3)^{-1}\{\pi[\mathrm{Si}(\pi+\pi s)-\mathrm{Si}(\pi-\pi s)] \\ & -[\mathrm{Cin}(\pi+\pi s)-\mathrm{Cin}(\pi-\pi s)] \\ & +4s(1-s^2)^{-1}\cos^2(\pi s/2)\},\end{aligned}$$

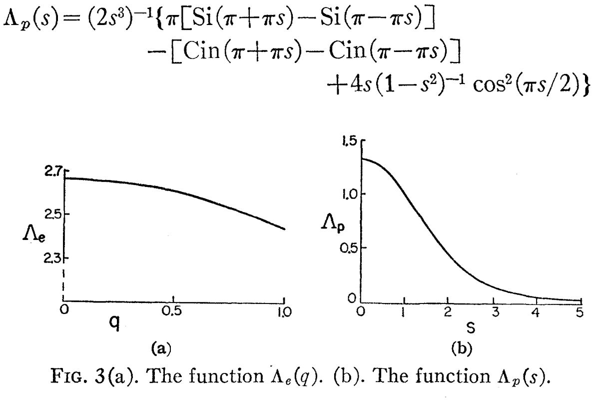


FIG. 3(a). The function $\Lambda_e(q)$. (b). The function $\Lambda_p(s)$.

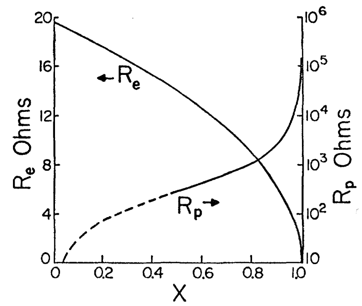


FIG. 4. Radiation resistance components for a current filament, $L/\lambda_0 = 0.25$, $c/v_0 = 100$.

where Si is the sine integral,

$$\mathrm{Cin}(x) = \int_0^x [(1-\cos x)/x]dx,$$

and

$$q = (2L/\lambda_0)(1-X)^{\frac{1}{2}},$$
$$s = (2L/\lambda_0)(c/v_0)(1-X)^{\frac{1}{2}}.$$

The functions Λ_e and Λ_p are shown in Fig. 3. When $s \gg 1$, $\Lambda_p(s) \approx \pi^2/(2s^3)$.

As an illustration of these results, we show in Fig. 4 the two radiation resistance components as a function of X, for the case $(L/\lambda_0) = 0.25$, $c/v_0 = 100$. The curve for R_p is shown dashed for $X < 0.5$, because we have assumed that the P mode will not propagate in this region [I].

The curve for R_e starts at 19.6 ohms at $X = 0$. This is the usual free-space value of radiation resistance for a $\lambda_0/4$ dipole. R_e then drops to zero as X goes to 1, as expected, since $\epsilon_r = 1 - X$. The component of radiation resistance from the P mode is zero at $X = 1$, and has the enormous value of 1.6×10^5 ohms at $X \approx 0.999$. The total radiation resistance is the sum of the two components and so it has the violent behavior of the P mode near $X = 1$. The curves, however, were calculated assuming a loss-free plasma, and the peak would be reduced if collisions were taken into account.

We wish to emphasize that the radiation resistance shown in Fig. 4 is for a *current filament*. It is not clear how close this will be to the radiation resistance of a *metal wire dipole*. We were able to separate the radiation resistance for the current filament into the two components, because the P and EM modes are independent. When all the sources for the wire dipole are taken into account, there will still be two components of the radiation resistance (P and EM), but each will be due to the combined effect of all sources.

It would be of considerable interest to measure antenna impedances in plasma. With a complete impedance calculation one could study the plasma itself. In particular, one might be able to investigate the P mode experimentally. This would be valuable because, in spite of the extensive theoretical literature on longitudinal waves in a homogeneous plasma, there is very little experimental work involving them directly.