and thereby shift the resonance. The only states which make appreciable contributions to this are ($F=1$, $m=1$) and ($F=1$, $m=1$). The pulling effect of these states is very small, however, for the following reasons: Normally these states do not couple to the resonant mode because the static magnetic field is parallel, rather than perpendicular, to the oscillating field. In addition, the two states have effects of opposite sign, so that if care is taken to populate them equally they will have a negligible net effect even if the static magnetic field is not precisely parallel to the oscillating field.

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Angular Distribution of Relativistic Atomic K-Shell Photoelectrons*

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Using the high-energy limit of the exact Coulomb wave function for the outgoing electron, the differential cross section, correct to three orders in αZ, is calculated for the K-shell photoeffect. An analytic expression, exact in αZ, is obtained for the differential cross section for the special case in which the electron emerges in the forward direction.

I. INTRODUCTION

USING the first Born approximation, Gavrila[1] calculated the differential and total cross sections for the K-shell photoeffect to two orders in αZ. It is apparent from Gavrila's work that the αZ correction is significant even for fairly small values of Z. More recently Pratt[2] made numerical calculations of the total cross section using the high-energy limit of the exact Coulomb wave function for the ejected electron. Pratt also derived an approximation formula which gives the total cross section as a function of Z. This formula compares favorably with the exact numerical results for all values of Z.

In this paper, the differential cross section for the relativistic K-shell photoeffect is calculated by using the high-energy limit of the exact Coulomb wave function for the ejected electron. While this result is correct to three orders in αZ (i.e., to terms of relative order α^2Z^2), it is not a strict expansion in this parameter. We have used Pratt's work as a guide to determine what factors should be left unexpanded. Upon integration over the solid angles of the outgoing electron we then obtain precisely Pratt's approximate formula for the total cross section. One might then conjecture that our result would give an accurate approximate formula for the angular distribution. However, in the forward direction the terms of relative order 1 and αZ vanish, and the resulting cross section is valid only to the first nonvanishing order in αZ. This nonvanishing term makes a contribution of relative order α^2Z^2 to the total cross section. Since the terms of relative order α^2Z^2 make a very small contribution to Pratt's expression for the total cross section, we cannot use a comparison with Pratt's result for the total cross section to justify the validity of our differential cross section for electron ejection angles near the forward direction. Therefore, for this special case of photoelectrons emerging in the forward direction, we calculate the differential cross section correct to all orders in αZ.

Figure 1 shows the angular distribution of the ejected photoelectrons. Figure 2 gives the differential cross section exact in αZ for the special case of forward emission as a function of Z.

II. MATRIX ELEMENT

Neglecting radiative corrections, the matrix element for the photoeffect is

$$M=-\left(\frac{2\pi\alpha}{k}\right)^{\frac{1}{2}}\int\psi_2{}^*\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}e^{i\mathbf{k}\cdot\mathbf{r}}\psi_1(\mathbf{r})d\mathbf{r}, \tag{1}$$

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[1] M. Gavrila, Phys. Rev. **113**, 514 (1959).
[2] R. H. Pratt, Phys. Rev. **117**, 1017 (1960).

where $\mathbf{k}$ is the momentum of the incident photon and $\boldsymbol{\varepsilon}$ is its polarization direction. ψ_1 is the wave function of the bound atomic electron and ψ_2 is the wave function of an electron in the continuum states of a Coulomb field.

For the bound state, the K-shell hydrogen-like wave function is used:

$$\psi_1(\mathbf{r})=Ne^{-\lambda r}(mr)^{\gamma-1}\left(1+\frac{i\lambda}{m(1+\gamma)}\boldsymbol{\alpha}\cdot\hat{\mathbf{r}}\right)u(\mathbf{p}_1), \quad (2)$$

where

$$\lambda=\alpha Zm,$$

$$W_1=m\gamma=m(1-\alpha^2Z^2)^{\frac{1}{2}},$$

and

$$|N|^2=\frac{(1+\gamma)(\alpha Zm)^3(2\alpha Z)^{2(\gamma-1)}}{\pi\Gamma(2\gamma+1)}.$$

This wave function has been expressed in terms of the plane-wave spinor $u(\mathbf{p}_1)$ in order to facilitate the evaluation of the spin summations. In all results the momentum $\mathbf{p}_1$ is set equal to zero.

Since we are restricting ourselves to high-energy incident photons, the final energy of the electron will be much greater than its rest mass energy m. Now in the limit $pr\rightarrow\infty$, the exact Coulomb wave function reduces to a distorted plane wave. For a final state this is

$$\psi_2(\mathbf{r})=e^{i\mathbf{p}\cdot\mathbf{r}}(pr+\mathbf{p}\cdot\mathbf{r})^{i\nu}u_2(\mathbf{p}), \quad (3)$$

where $\nu=\alpha Z$. We are justified in using this wave function for the final state of the electron provided that in the neighborhood of the origin there is no contribution to the matrix element when the exact Coulomb wave function is used. That this is so has recently been demonstrated.[3,4]

III. DIFFERENTIAL CROSS SECTION

For high energy, the differential cross section for two K-shell electrons is

$$d\sigma=\frac{pW}{(2\pi)^2}\sum|M|^2d\Omega, \quad (4)$$

where $\sum$ represents the sum over initial and final electron spin states and W is the energy of the outgoing electron. Using the matrix element given by Eq. (1) together with Eqs. (2) and (3), we obtain

$$\sum|M|^2=\frac{2\pi\alpha}{k}|N|^2\left\{|I_1|^2+\frac{\lambda^2}{m^2(1+\gamma)^2}\mathbf{I}_2\cdot\mathbf{I}_2^*+\frac{2\lambda}{m(1+\gamma)}\operatorname{Re}(\hat{p}\cdot\mathbf{I}_2-2\boldsymbol{\varepsilon}\cdot\hat{p}\boldsymbol{\varepsilon}\cdot\mathbf{I}_2)I_1^*\right\}, \quad (5)$$

where

$$I_1=-(\partial/\partial\lambda)I, \quad (6a)$$

$$\mathbf{I}_2=\nabla_qI, \quad (6b)$$

and

$$I=\int\frac{e^{-\lambda r}}{r}(mr)^{\gamma-1}e^{-i\mathbf{q}\cdot\mathbf{r}}(pr+\mathbf{p}\cdot\mathbf{r})^{-i\nu}d\mathbf{r}, \quad (6c)$$

with $\mathbf{q}=\mathbf{p}-\mathbf{k}$. The basic integral, I, has been calculated correct to three orders in αZ.[5] (The procedure used in the evlauation of I is outlined in the Appendix.) Using this result and performing the differentiations indicated by Eqs. (6a) and (6b), we obtain

$$I_1=C\left\{\frac{1}{q^2+\lambda^2}\left(\frac{2\lambda}{q^2+\lambda^2}+\frac{i\nu}{\lambda-i\hat{p}\cdot\mathbf{q}}-\frac{2i\nu\lambda}{q^2+\lambda^2}\right)-\frac{(1-\gamma)}{q^2}\left[-\frac{\pi}{2q}+\frac{3\lambda}{q^2}-\frac{2\lambda}{q^2}\ln\frac{q}{m}-\frac{\nu}{2q}\left(2i\pi\ln2-L_2(\hat{p}\cdot\hat{q})+L_2(-\hat{p}\cdot\hat{q})+\ln(\hat{p}\cdot\hat{q})\ln\frac{1+\hat{p}\cdot\hat{q}}{1-\hat{p}\cdot\hat{q}}-i\pi\ln\frac{1+\hat{p}\cdot\hat{q}}{\hat{p}\cdot\hat{q}}-\frac{2}{\hat{p}\cdot\hat{q}}\ln\frac{\hat{p}\cdot\mathbf{q}}{m}-2i\pi+\frac{\pi^2}{2}+\frac{i\pi}{\hat{p}\cdot\hat{q}}\right)\right]\right\}, \quad (7)$$

$$\mathbf{I}_2=C\left\{\frac{1}{q^2+\lambda^2}\left(\frac{2(i\nu-1)}{q^2+\lambda^2}\mathbf{q}-\frac{\nu}{\lambda-i\hat{p}\cdot\mathbf{q}}\hat{p}\right)+\frac{(1-\gamma)}{q^4}\left(1-2\ln\frac{q}{m}\right)\mathbf{q}\right\} \quad (8)$$

where

$$C=4\pi\left(\frac{m}{2p}\right)^{i\nu}\Gamma(\gamma)\Gamma(1-i\nu)\left(\frac{\lambda^2+q^2}{m(\lambda-i\hat{p}\cdot\mathbf{q})}\right)^{i\nu},$$

and $L_2(x)$ is Euler's dilogarithm.[6] Using these results we can write the differential cross section in the form

$$d\sigma=\sigma_0Fd\Omega, \quad (9)$$

where

$$\sigma_0=4\alpha(\alpha Z)^5(W/m^3),$$

and where F is correct to three orders in αZ. However, considerable simplifications can be made in F if we confine our attention to the region where $\hat{p}\cong\hat{k}$. It is easily seen that the outgoing electrons are confined to a cone in the forward direction of half-width of order (m/W) because of the high power of q in the denominator. For θ in this region, $\hat{p}\cdot\mathbf{q}=\gamma m$ to highest order in W, and $(\boldsymbol{\varepsilon}\cdot\hat{p})^2$ will be of order (m^2/W^2) so that the term dependent on the polarization direction can be neg-

[3] R. T. Deck, C. J. Mullin, and C. L. Hammer (to be published).

[4] R. H. Pratt (reference 2) has used a somewhat different argument to justify the use of the distorted plane wave.

[5] Equations (A2), (A10), and (A21) of reference 3. The integral I given by (A2) of this work is the same as that given above if we replace $\mathbf{q}$ by $-\mathbf{q}$ and $\mathbf{p}$ by $-\mathbf{p}$. However, the results of the integration are unchanged since the vectors appear only in the combination $\mathbf{p}\cdot\mathbf{q}$.

[6] K. Mitchell, Phil. Mag. 40, 351 (1949), gives nine place tables for $L_2(x)$, $-1\leq x\leq1$.

lected. Thus we obtain

$$F=\nu^{2(\gamma-1)}\exp[-\pi\nu+2\nu\sin^{-1}\nu]\left\{\frac{2}{\mu^4}-\frac{2}{\mu^6}+\nu\pi\left(\frac{1}{\mu^7}-\frac{1}{\mu^5}\right)\right.$$
$$+\frac{\nu^2}{2}\left[\frac{6}{\mu^4}-\frac{2}{3}\frac{\pi^2}{\mu^4}-4\frac{\ln 2}{\mu^4}+\frac{3}{\mu^6}+\frac{11}{12}\frac{\pi^2}{\mu^6}\right.$$
$$+4\frac{\ln 2}{\mu^6}-\frac{8}{\mu^8}+2\left(\frac{1}{\mu^5}-\frac{1}{\mu^7}\right)\left(L_2(1/\mu)\right.$$
$$\left.\left.\left.-L_2(-1/\mu)+\ln\mu\ln\frac{\mu+1}{\mu-1}-\frac{\pi^2}{2}\right)\right]\right\}, \quad (10)$$

where

$$\mu^2=\frac{q^2+\lambda^2}{m^2}\simeq 1+\frac{W^2}{m^2}\theta^2.$$

The factor which is left unexpanded in αZ is precisely the same factor which appears in Pratt's approximate formula for the total cross section. In the forward direction $\mu=1$ so that the first two orders of αZ ($\nu=\alpha Z$) drop out, leaving our result valid only to lowest non-vanishing order in this region. However, the basic integral I given in Eq. (6c) can be evaulated exactly for the case $\hat{p}=\hat{k}$ and is given in the Appendix, Eq. (A8). Using the results given by Eq. (A8) and performing the differentiations indicated in Eqs. (6a) and (6b), we obtain

$$I_1=c\{F_1+(2-i\nu)^{-1}(t_1-t_1/t_2)F_2\}, \quad (11)$$

$$\mathbf{I}_2=c\hat{q}\{-iF_1+i(2-i\nu)^{-1}(t_1+t_1/t_2)F_2\}, \quad (12)$$

where

$$c=\frac{4\pi\Gamma(2+\gamma-i\nu)}{m^3(1-i\nu)}\left(\frac{m}{2p}\right)^{i\nu}t_2^{-\gamma+i\nu-2},$$

$$F_1={}_2F_1(1,1+\gamma-i\nu;2-i\nu;t_1-t_1/t_2),$$

$$F_2={}_2F_1(2,2+\gamma-i\nu;3-i\nu;t_1-t_1/t_2).$$

The quantities t_1, t_2, and t_3 are defined by the equations following Eq. (A3) of the Appendix. The differential cross section for $\theta=0$ can now be written to all orders in αZ.

$$d\sigma=\frac{32\pi\alpha W}{m^6\gamma^2}|\Gamma(1+\gamma-i\nu)|^2|N|^2\exp[-\pi\nu+2\nu\sin^{-1}\nu]$$
$$\times\left|F_1+\frac{(1+\gamma+i\nu)}{(2-i\nu)(\gamma-i\nu)}F_2\right|^2 d\Omega. \quad (13)$$

Using standard relations between hypergeometric functions,[7] we can write the cross section in terms of one hypergeometric function.

$$d\sigma=\frac{8\pi\alpha W}{m^6\gamma^4}|\Gamma(1+\gamma-i\nu)|^2|N|^2\exp[-\pi\nu+2\nu\sin^{-1}\nu]$$
$$\times\left|(1-i\nu)\left(\frac{\nu-i\gamma}{\nu+i\gamma}\right)^{\gamma}-(1-i\nu-2\gamma)\right.$$
$$\left.\times{}_2F_1\left(1-i\nu,1-\gamma;2-i\nu;\frac{2\gamma}{\gamma-i\nu}\right)\right|^2 d\Omega. \quad (14)$$

Since the modulus of $2\gamma/(\gamma-i\nu)$ is greater than 1, the hypergeometric function must be analytically continued in order to represent the function in its usual series expansion. This is accomplished by a standard relation.[8] Thus writing

$$d\sigma=\sigma_0F_0d\Omega,$$

we find

$$F_0=\frac{(1+\gamma)}{\nu^2\gamma^3}\frac{|\Gamma(\gamma-i\nu)|^2}{\Gamma(2\gamma)}(2\nu)^{2(\gamma-1)}$$
$$\times\exp[-\pi\nu+2\nu\sin^{-1}\nu]\left|1-\frac{(1-2\gamma-i\nu)}{2\gamma^2(\gamma+i\nu)}\right.$$
$$\times\left[\frac{\Gamma(1-i\nu)\Gamma(1+\gamma)}{\Gamma(1+\gamma-i\nu)}\left(\frac{2\gamma}{\gamma-i\nu}\right)^{i\nu}\left(\frac{i\nu+\gamma}{i\nu-\gamma}\right)^{-\gamma}\right.$$
$$\left.\left.-{}_2F_1\left(1,i\nu;1+\gamma;\frac{\gamma+i\nu}{2\gamma}\right)\right]\right|^2. \quad (15)$$

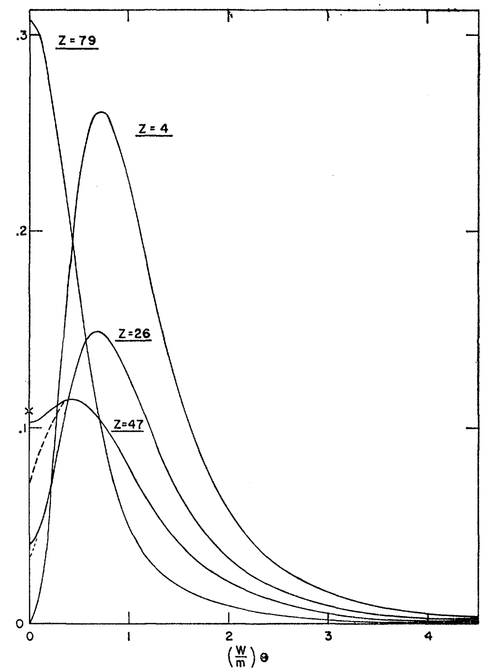

FIG. 1. F is plotted vs $(W/m)\theta$ for various values of Z. θ is in radians. The dashed curve is the extrapolation to the exact value for $\theta=0$. The ✕ marks the exact value for $Z=79$.

[7] A. Erdelyi, editor, *Higher Transcendental Functions, Bateman Manuscript Project* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, Sec. 2.9, p. 105, Eq. (2) and Sec. 2.8, p. 103, Eq. (34).

[8] Reference 7, Sec. 2.10, p. 108, Eq. (1).

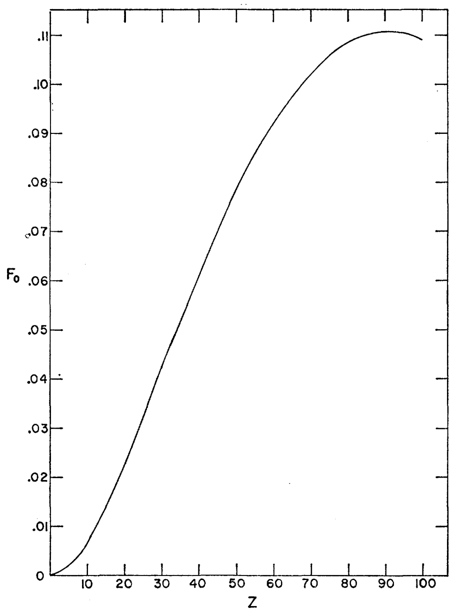

FIG. 2. F_0, the exact value of F for $\theta=0$, is plotted vs Z.

This expression is now in a form suitable for a numerical calculation.

IV. TOTAL CROSS SECTION

For the total cross section we must perform the $d\Omega$ integration over terms of the form $(\mathbf{p}\cdot\mathbf{q})^l/(q^2+\lambda^2)^n$ or over functions whose series representation has terms of this type. It is easily shown that to highest order in W, $\hat{p}\cdot\mathbf{q}$ can be set equal to γm in the integral

$$\int(\hat{p}\cdot\mathbf{q})^l/(q^2+\lambda^2)^n d\Omega,$$

provided $n>l+1$. This condition is met in our differential cross section so that we may take F given by Eq. (10) to be integrated over $d\Omega$. The integrals over the dilogarithm functions are easily carried out by integration by parts. The total cross section for the two K-shell electrons is

$$\sigma=\pi\sigma_0(m^2/W^2)\nu^{-2(1-\gamma)}\exp[-\pi\nu+2\nu\sin^{-1}\nu]$$
$$\times\left\{1-\frac{4\pi}{15}\nu+\nu^2[2+\tfrac{1}{2}-\ln 2+\pi^2(\tfrac{1}{16}-\tfrac{1}{6}-4/45)]\right\}. \quad (16)$$

This is precisely the result that Pratt obtained.

V. CONCLUSION

We have written the differential cross section for the relativistic photoeffect in the form

$$d\sigma=\sigma_0 F d\Omega,$$

where $\sigma_0=4\alpha(\alpha Z)^5 W/m^3$ and F is given correct to three orders in αZ by Eq. (10) or exactly in αZ for $\theta=0$ by Eq. (15). In Fig. 1 we plot F vs (W/m), and the dashed curves are extrapolations to the exact value for $\theta=0$. Since F is good only to lowest-nonvanishing order in αZ in the region where $\theta\simeq 0$, it is not surprising that for $Z=79$ this lowest-order result and the exact result (marked with an $\times$) are greatly different. The cross section for $\theta=0$, correct to all orders in αZ, is given as a function of Z in Fig. 2. By comparing Figs. 1 and 2 one can estimate the behavior of the differential cross sections for angles near $\theta=0$.

APPENDIX

The basic integral necessary in the work of this paper is

$$I=\int\frac{e^{-\lambda r}}{r}(mr)^{\gamma-1}e^{-i\mathbf{q}\cdot\mathbf{r}}(pr+\mathbf{p}\cdot\mathbf{r})^{-i\nu}d\mathbf{r}. \quad (\mathrm{A1})$$

Using Hankel's representation

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_{-\infty}^{0+}e^t t^{-z}dt, \quad |\arg t|\leq\pi,$$

and setting $t=\xi s$ where ξ is real positive, we obtain

$$\xi^{z-1}=\frac{\Gamma(z)}{2\pi i}\int_{-\infty}^{0+}e^{\xi s}s^{-z}ds. \quad (\mathrm{A2})$$

Using this representation for $(mr)^{\gamma-1}$ and $(pr+\mathbf{p}\cdot\mathbf{r})^{-i\nu}$ in the integral I, the integration over $\mathbf{r}$ becomes trivial and we obtain

$$I=\frac{\Gamma(1-i\nu)\Gamma(\gamma)}{2\pi mp}\int_{-\infty}^{0+}\frac{t^{-\gamma}}{t_3-t}dt\lim_{R\to\infty}\int_{-R}^{0+}\frac{s^{i\nu-1}}{s-s_0(t)}ds, \quad (\mathrm{A3})$$

where

$$s_0(t)=(m/2p)[(t_1-t)(t_2-t)/(t_3-t)],$$
$$t_1=t_2^*=(\lambda-iq)/m,$$
$$t_3=(\lambda-i\hat{p}\cdot\mathbf{q})/m.$$

We can close the path of integration for s with a circle of radius R. As $R\to\infty$ there will be no contribution from the integration along the circle. Reversing the path of integration and applying Cauchy's theorem, we obtain

$$I=-2i\frac{\Gamma(1-i\nu)\Gamma(\gamma)}{m^2}\left(\frac{m}{2p}\right)^{i\nu}$$
$$\times\int_{-\infty}^{0+}\frac{t^{-\gamma}}{(t_2-t)(t_1-t)}\left[\frac{(t_1-t)(t_2-t)}{t_3-t}\right]^{i\nu}dt. \quad (\mathrm{A4})$$

By expanding the integrand in terms of αZ, this integral has been evaluated to three orders in αZ as mentioned earlier. The integral can be evaluated exactly in terms of generalized hypergeometric functions. However, for the special case in which $\hat{p}=\hat{q}$ the integral can be written in terms of hypergeometric functions. In this case $t_1=t_3$

and

$$I=-2i\frac{\Gamma(1-i\nu)\Gamma(\gamma)}{m^2}\left(\frac{m}{2p}\right)^{i\nu}\times\int_{-\infty}^{0+}t^{-\gamma}(t_1-t)^{-1}(t_2-t)^{i\nu-1}. \quad \text{(A5)}$$

In this contour of integration, the origin is approached below the cut where the phase of t is $-\pi$, the origin is encircled, and then $-\infty$ is approached along the top of the cut where the phase of t is π. Splitting the integral into three parts, two along the cut and one around the origin, we find that that the contribution from the origin is zero and the two parts along the cut combine to give

$$I=-4\sin(\pi\gamma)t_1^{2-\gamma}t_2^{i\nu-1}\frac{\Gamma(1-i\nu)\Gamma(\gamma)}{m^2}\left(\frac{m}{2p}\right)^{i\nu}\times\int_0^{\infty}s^{-\gamma}(s+1)^{-1}\left(\frac{t_1}{t_2}s+1\right)^{i\nu-1}ds. \quad \text{(A6)}$$

Now

$${}_2F_1(a,b;c;1-z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\times\int_0^{\infty}s^{b-1}(1+s)^{a-c}(1+sz)^{-a}ds,$$

with

$$\mathrm{Re}\,c>\mathrm{Re}\,b>0, \quad |\arg z|<\pi. \quad \text{(A7)}$$

Thus we obtain

$$I=\frac{4\pi\Gamma(1+\gamma-i\nu)}{m^2(1-i\nu)}\left(\frac{m}{2p}\right)^{i\nu}t_2^{-\gamma-1+i\nu}\times{}_2F_1\left(1,1+\gamma-i\nu;2-i\nu;\frac{t_2-t_1}{t_2}\right). \quad \text{(A8)}$$