Self-Consistent Determination of the Effective Radii of Heavy Nuclei for Alpha-Particle Emission*

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In attempts to use the absolute rate of natural alpha-particle decay to determine the "density of alpha particles on the nuclear surface" the penetrability of the effective barrier surrounding the nucleus plays a dominating role. A new approach to the barrier problem is proposed based on the hypothesis that for transitions of definite type—to ground states of spherical even-even nuclei beyond the double closed shells at Pb^{208}—we might expect the intrinsic emission probability measured in single particle units, whatever its absolute magnitude, about which no assumption is made, to be proportional to the surface area of the nucleus only. The dispersion in the reduced widths inferred from the emission rates depends on the cut-off radius $R=r_0A^{\frac{1}{3}}$ that is chosen and so we can define a self-consistent potential whose radius constant r_0 minimizes this dispersion. The "spherical" nuclei show a well-defined minimum to the dispersion at $r_0=1.57\pm0.06$ fermis. The deformed nuclei have a different behavior as is expected. This self-consistent potential is very close to that derived by Igo from an optical model analysis of alpha-particle scattering; the penetrabilities for the natural alpha-particle emitters calculated with the two potentials agree to within a factor of about 2. It is shown that if the hypothesis is modified to allow a smooth dependence of the intrinsic emission probability on A of the form $1+\epsilon A$ then the resulting minimum dispersions computed as a function of ϵ themselves show a minimum with ϵ very close to zero, thereby justifying the hypothesis in its simple form.

INTRODUCTION

THERE is considerable interest in the texture of the nuclear surface. Is it "smooth" in the sense that it contains no more nucleon correlations than provided by a zero-order shell-model wave function or is it "rough" in the sense that it is relatively rich in explicit nucleon clusters or density fluctuations of very short lifetime, most probably "alpha particles" from energy and symmetry considerations, that constantly melt and re-form elsewhere? There are several experimental approaches to this question that all appear to indicate that the "rough" picture is the more likely.[1] One approach is that based on an analysis of the absolute rate for alpha-particle emission from heavy nuclei. Briefly one uses a wave function describing a "smooth" surface, i.e., a zero-order shell-model wave function to predict the probability for the assembly of an alpha particle at the edge of the nucleus; then this formation probability is transformed into an absolute emission rate by calculating a barrier penetrability. If the calculated rate falls short of the experimental by a wide margin we say that an explicit if very transitory surface clustering beyond that implied by the simplest shell model is implied. It is clearly vital in this chain of argument to have confidence in our ideas about the effective barrier since the penetrability depends extremely sharply upon it. It is incidentally important, if we are to use the language of alpha-particle formation and transmission, that the effective barrier should be such that the alpha-particle kinetic energy becomes negative outside the region of alpha-particle formation which must be taken to be where the matter is. In other words, the matter distribution must sit well inside the effective barrier.

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[1] D. H. Wilkinson, in *Proceedings of the Rutherford Jubilee International Conference* (Heywood and Co., Ltd., London, 1961), p. 339.

THE ALPHA-PARTICLE BARRIER

At the moment our ideas about the effective barrier for alpha particles come from two significant sources only. The first is the phenomenon that we discuss here; natural alpha-particle emission. One commits oneself to some model that gives the absolute rate for alpha particles to impinge on the inner wall of the barrier; if one then additionally commits oneself to the form of the barrier, such as sharp cutoff or rounded edge, the "radius of the nucleus for alpha-particle emission" is determined by the absolute decay rate. This method we must completely reject for our present purpose, of course, since it is precisely with the problem of the absolute alpha-particle formation probability inside the nucleus that we are concerned, and so we must make no absolute assumptions about it. Our second method is the optical-model analysis of alpha-particle elastic and inelastic interaction. The detailed work of Igo[2] has shown that a consistent description of such alpha-particle interaction with nuclei ranging from argon to lead and at energies up to 50 Mev can be made with a potential whose real part is the Coulomb potential plus

$$V_\alpha=-1100 \exp[-(r-1.17A^{\frac{1}{3}})/0.574] \text{ Mev},$$

where the distances are in fermis. This potential is determined only in the surface reigon, which is adequate for our present purpose, and, of course, is only one of a continuum of ways in which the interaction can be parameterized, if one of the most simple and reasonable. The use of this potential for calculating the barrier against natural alpha-particle emission is made hazardous by the very different energy involved in such emission as compared with that used for establishing the potential; a velocity dependence of the potential may well be anticipated. To use this potential is the best direct estimate that can be made at the moment. The penetrabilities for barriers constructed with the

[2] G. Igo, Phys. Rev. **115**, 1665 (1959).

Igo potential have been tabulated by Rasmussen[3]; using these penetrabilities one concludes that the nucleus is "rough," that explicit "alpha-particle" clustering in the surface must take place.[1]

THE SELF-CONSISTENT BARRIER

An independent approach to the barrier problem is urgently needed. To be complementary to that given by the interaction data it should refer to the conditions of natural alpha-particle emission; direct information cannot be obtained at these energies, and so we are forced back to a re-examination of the emission data themselves.

The approach examined here is based on the hypothesis that if we restrict ourselves to that class of alpha-particle transitions linking the ground states of even-even nuclei, i.e., that do not call for the concentration of orbital angular momentum in the alpha-particle nucleus system then the intrinsic probability or reduced width for alpha particle emission expressed in single-particle units will be proportional to the amount of matter available for the assembly of alpha particles, i.e., to the nuclear surface area. We make no assumption whatever about the absolute probability of alpha-particle formation, i.e., invoke no dynamical model. This hypothesis is reasonable so long as we indeed restrict ourselves to transitions of this special type, so long as we do not cross closed shells and so long as the nuclei we consider are spherical or close to it. When nuclei are strongly deformed the alpha-particle formation probability is presumably a function of the coordinates of the nuclear surface and no model-free starting point can be found; we should further expect the emission probability to be dependent on the accompanying change of eccentricity.

The intrinsic emission probabilities are inferred from the absolute emission rates via a penetrability contribution. As we change the barrier by changing the effective nuclear radius we greatly affect the absolute penetrabilities, but we also change the relative penetrabilities for the different alpha particles with which we are concerned and so change the relative intrinsic emission probabilities for the various alpha-particle transitions that we infer from the absolute rates. Our hypothesis then is that the correct effective nuclear radius is the one that minimizes the dispersion in the inferred relative intrinsic probabilities in single-particle units per unit area of surface. An important test of this method is the existence or otherwise of a well-defined minimum to the dispersion and, at the minimum, of a distribution of inferred relative intrinsic probabilities that is reasonably narrow and well-defined. If such a barrier that minimizes the dispersion exists, we shall call it the self-consistent barrier and regard it as the best description of the effective alpha-particle–nucleus interaction, that we can make without guidance from more detailed information than we enjoy at the present time.

In order to reduce the free parameters of the determination to zero, we present the analysis in terms of a barrier with a sharp cutoff, viz., a Coulomb potential to a distance $r=R$ within which the potential abruptly cuts off to a large negative value. This is an old-fashioned and unsophisticated sort of barrier but we adopt it in preference to a more "realistic" one because any improvement involves us in at least three parameters instead of one and so deprives the approach of its automatic character and also because, as we shall see, the penetrabilities estimated with the sharp cut-off barrier and with a "realistic" one (that based on the Igo potential) differ only very slightly within the context of the present comparison between wholly different approaches to the nuclear radius. To avoid a further possible parameter we shall use the form $R=r_0A^{\frac{1}{3}}$ (where A is appropriate to the daughter nucleus) rather than an expression involving a constant to represent the "size of the alpha particle."

At this point we may interpolate another comment on the desirability of keeping away from deformed nuclei: the penetrability of the deformed barrier clearly depends on the coordinates of the nuclear surface as does the alpha-particle formation probability so even if we wished to make assumptions about the latter, which we do not, we should have to have a full understanding of the deformed barrier which we do not either. If we apply the method of minimizing the dispersion of relative intrinsic probabilities to deformed nuclei we should not expect it to work. Indeed another test of the present approach will be a clear change in its results as between "spherical" and deformed nuclei. The former, following the evidence of level schemes, we shall take to be $N\leqslant 138$.

METHOD AND RESULTS

If we eliminate transitions across and below the closed shells at Pb^{208} the material to be analyzed comprises 33 cases (as tabulated by Rasmussen[3]). Of these, 16 involve "spherical" end products, i.e., have $N\leqslant 138$ and the rest are deformed in both initial and final states. These two sets have been analyzed separately by the same method which will now be described.

We have chosen to express the intrinsic emission probabilities in terms of the formulation of alpha-particle decay due to Thomas.[4] Lumping together constant factors of no concern for the present study we obtain the inferred reduced width: Reduced width$=$const $\times(1/kRt)(d\log G_0/d\log r)_{r=R}{}^2G_0{}^2(R)$, where k is the wave number of the final alpha-particle–nucleus system, t is the half-life, and $G_0(r)$ is the usual irregular Coulomb function. As an excellent approximation to $G_0(r)$ we have used the Langer [5] modification to the WKB solu-

[3] J. O. Rasmussen, Phys. Rev. **113**, 1593 (1959); **115**, 1675 (1959).

[4] R. G. Thomas, Progr. Theoret. Phys. (Kyoto) **12**, 253 (1954).

[5] R. E. Langer, Phys. Rev. **51**, 669 (1937).

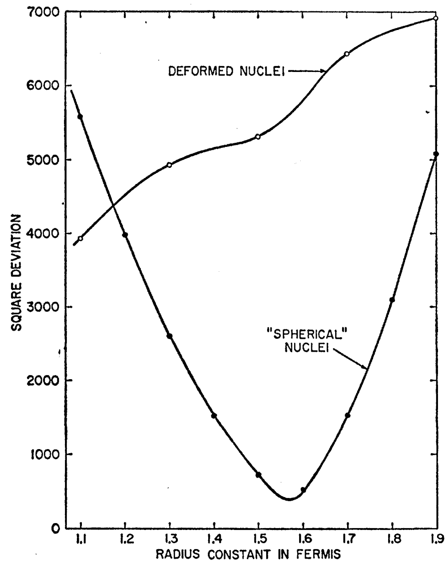

FIG. 1. The behavior of the square deviation

$$\Sigma_i(\log_{10}\xi_i - \langle\log_{10}\xi\rangle_{\mathrm{av}})^2$$

as a function of the radius constant r_0 for the ground-state alpha-particle emission from even-even nuclei beyond the closed shells. ξ is the reduced width divided by the product of a single-particle reduced width and the surface area.

tion. These reduced widths we must now compare with the appropriate single-particle unit which we take from the formulation of Thomas:

$$\text{Single-particle reduced width} = (\text{const}/MR^2) \times\{[1-(G_0/G_0')/kR]+[z_0(G_0/G_0')/kR]^2\}^{-1}.$$

M is the reduced mass of the system and z_0 is a zero of the spherical Bessel function that describes the alpha particle's radial wave function on the single-particle model used for defining these units. Notice that the energy dependence of this unit involves the choice of which zero of the Bessel function to use. Since, however, $kR(G_0'/G_0)\approx 20$ and $z_0 = n\pi$ the sensitivity to n of the relative single-particle widths for different k is very slight for the low values of n that are reasonable choices (we have used $n=1$ as is standard practice). Again we must notice that the present method of approach to the radius problem always involves only relative widths and probabilities and never absolute values, so uncertainty in the absolute value of the single-particle unit is irrelevant. (Indeed in the constant term in the above expression, occurs the quantity z_0^2 which makes major changes to the single-particle width but does not affect the present method.) Each inferred reduced width must be divided by the product of this single-particle unit and the nuclear surface area. It is to this quotient, ξ, that our hypothesis now refers and whose dispersion must be minimized as a function of the radius constant $r_0 = RA^{-\frac{1}{3}}$. The dispersion is defined as the square deviation, $\sum_i(\log_{10}\xi_i - \langle\log_{10}\xi\rangle_{\mathrm{av}})^2$. If the dispersion were of a Gaussian character this sum should be taken over all numbers of the set. As it is, we have chosen to allow for a small possible element of aberration by excluding from the entry under each value of r_0 that transition out of the 16 or 17 having the largest value of $(\log_{10}\xi_i - \langle\log_{10}\xi\rangle_{\mathrm{av}})^2$. This does not affect the results materially and merely sharpens a little the definition of the minimum in the dispersion.

The results are shown in Fig. 1. It is seen that for the "spherical" nuclei a clear and well-defined minimum to the dispersion is given for a value of the radius constant of 1.57 fermis. For the deformed nuclei the behavior of the dispersion is totally different, and although there is a feeble attempt at a minimum in the same general region of r_0 as for the spherical nuclei no sense is to be made of it. Examination of the behavior of the deviations $(\log_{10}\xi_i - \langle\log_{10}\xi\rangle_{\mathrm{av}})^2$ for the individual cases is very interesting: among the "spherical" nuclei there is a strong tendency for the individual deviations to show a form generally similar to that of the assembly of "spherical" nuclei in Fig. 1. As soon as $N=138$ is passed, however, the pattern changes and the straggling behavior characteristic of the assembly of deformed nuclei of Fig. 1 takes its place.

The difference in behavior between the two sets of nuclei is so great, and it sets in so abruptly around $N=138$, that it is not possible to deny its association with the onset of marked deformation. Conversely this expected difference between the behavior for "spherical" and deformed nuclei lends good credibility to the basic hypothesis. This must now be further tested by examining the distribution of the deviations of the individual transitions found at the minimum for the "spherical" nuclei. This is shown as an ideogram in Fig. 2 not quite at the minimum but at $r_0=1.6$ fermis, the nearest computed point. (Note that what is displayed in Fig. 2 is actually the deviations of the reduced widths or intrinsic emission probabilities rather than of the quotient ξ. These two quantities are rather closely proportional to each other since the single-particle reduced width in Thomas' formulation is almost inversely proportional to R^2, which factor is removed in the definition of ξ on

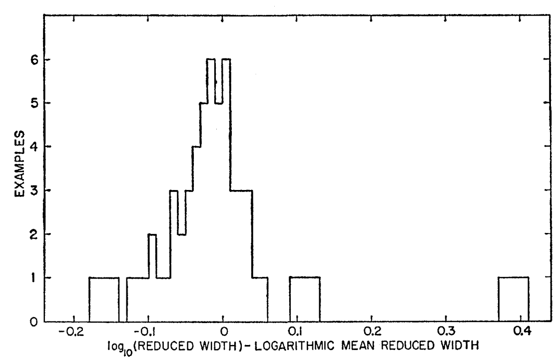

FIG. 2. Deviations of the individual reduced widths for ground-state alpha-particle emission to even-even "spherical" nuclei beyond the closed shells for a radius constant $r_0=1.6$ fermis.

multiplying by the surface area.) The distribution is indeed extraordinarily narrow with a width at half maximum of about 20%. The case with a value of $\log_{10}\xi - \langle \log_{10}\xi \rangle_{av} \approx 0.4$ is Rn^{218} (parent body) and is regarded as aberrant within the license defined above even though it deviates by less than a factor of 2.5 from the mean. It would be rather surprising if so tight a grouping as this were not significant, particularly when it is remembered that the lifetimes from which these intrinsic values are deduced cover a range of 10^{20} to 1 or so. We might with confidence simultaneously conclude that there can be nothing basically wrong with the methods by which the penetrability is calculated and that we may, for example, dismiss models of alpha-particle transmission which call for extreme polarization of the alpha particle as it traverses the barrier.

We must ask how much weight can be placed on this value of $r_0 = 1.57$ fermis. To answer this the 16 "spherical" nuclei were split up into two groups of 8 in all ways and for each way of dividing the 16 the two groups were treated separately by the same method. The maximum and minimum values of r_0 so obtained were 1.49 and 1.65 fermis so it seems reasonable to quote an "error" of something less than this maximum spread, say ± 0.06 fermi:

$$r_0 = 1.57 \pm 0.06 \text{ f.}$$

CRITIQUE OF THE HYPOTHESIS

The treatment so far has been based on the primitive hypothesis that the alpha-particle reduced widths are proportional only to the product of the appropriate single-particle unit and the nuclear surface area, i.e., almost a constant. Although this has worked out well it awaits justification. This may be based on the observation that on leaving Pb^{208} and moving towards heavier elements we are beginning on a very large and complicated shell cluster analogous to that which runs through the rare earths. Indeed the onset of strong deformation at $N = 138$ analogous to that at $N = 90$ in the rare earths confirms the comparison of the two regions of the periodic table. It is these circumstances that make reasonable the hypothesis which has found encouragement in the tight grouping of Fig. 2. We must now ask, however, whether some relaxation of the primitive hypothesis in order to permit some slow change of intrinsic emission probability with A might not work even better, i.e., lead to an even tighter grouping, an even lower minimum dispersion. To investigate this we now assume that the reduced widths should be further multiplied by a factor of the form $1 + \epsilon A$ and repeat this analysis, computing the dispersion or square deviation as shown before in Fig. 1 for $\epsilon = 0$ but now for each value of ϵ and so also determining the associated r_0 that minimizes the dispersion as a function of ϵ.

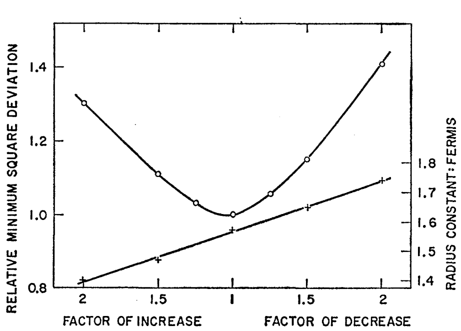

FIG. 3. The relative minimum square deviation as a function of the factor by which the intrinsic emission probability is multiplied or divided as between the lightest and heaviest of the 16 "spherical" nuclei, the changes in the intervening nuclei being linearly interpolated (upper curve with circles); also the value of r_0 for which the minimum square deviation is found for each value of the additional factor (lower curve with crosses).

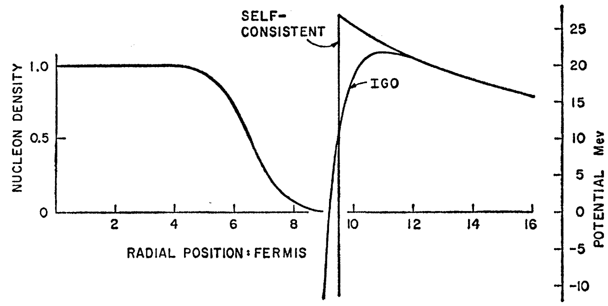

FIG. 4. Comparison of the self-consistent and Igo potentials between Ra^{222} and an alpha particle; also the matter distribution as inferred from electron scattering.

The results of this supplementary investigation are displayed in Fig. 3 which shows the minimum value of the dispersion, relative to that found for $\epsilon = 0$, as a function of the factor by which the reduced widths are additionally changed as between the lightest to the heaviest of the 16 "spherical" nuclei, the intermediate ones being linearly interpolated according to $1 + \epsilon A$. It is seen that the smallest minimum dispersion is indeed found very close to $\epsilon = 0$ and from the accompanying graph of the values of r_0 that give the minimum dispersions it is seen that the additional uncertainty introduced into the above-quoted value of r_0 by the lack of definition of the minimum in Fig. 3 is small.

In view of the clear indication of Fig. 3 and the limited range of material, we cannot consider it profitable at this stage to hypothesize more complicated forms for the dependence of the intrinsic emission probability on A.

COMPARISON WITH THE IGO POTENTIAL

We may now compare the self-consistent barrier $r_0 = 1.57$ fermis with that deriving from an Igo potential already presented. Figure 4 shows the situation for Ra^{222}. The two potentials are compared and the matter distribution as inferred from the electron scattering data[6] of Hofstadter and his colleagues (assuming that the protons and neutrons have the same distribution) is also

[6] D. G. Ravenhall, Revs. Modern Phys. 30, 430 (1958).

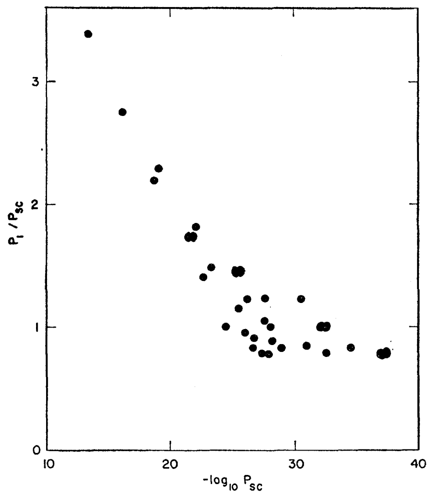

FIG. 5. Ratio of the penetrability calculated with the Igo to that calculated with the self-consistent potential on a radius constant $r_0=1.57$ fermis for the latter for ground-state transitions from even-even nuclei beyond the closed shells.

shown. It is seen both that the self-consistent and Igo potentials are remarkably close together, and that the matter distribution indeed lies comfortably inside the effective potential.

Another comparison is between the penetrabilities computed for the Igo and self-consistent potentials. This is shown in Fig. 5 as a function of the penetrability. This figure contains all the cases treated, "spherical" and deformed alike, since it is a straightforward comparison between computations and the use of actual transitions is here merely illustrative. (Note that the comparison in Fig. 5 is not quite on equal terms since the penetrabilities have been calculated for the Igo potential with the pure WKB method while those for the self-consistent potential incorporate the Langer modification. However, this latter is worth only a factor of 2 or less in those practical cases and so changes the picture only slightly. The correction increases the penetrability and so removing it would increase the discrepancy shown in Fig. 5. The correction is largest when the penetrability is smallest, so removing it will tend to flatten the trend of the points in Fig. 5 and we can say with good accuracy that the two potentials differ in penetrability by a factor of about 2 in all practical cases. There is certainly no strong dependence of the relative penetrabilities for transitions of different energy on the form of potential chosen such as is often claimed. The agreement is quite astonishing and of course must be largely fortuitous. Moving r_0 to either end of the range quoted above shifts the "self-consistent" penetrability by a factor of about 5 either way.

It seems difficult, in view of the complete lack of freedom in the determination of the self-consistent barrier, to disregard this coincidence between the answer it gives and that given by the independently-determined Igo potential. Perhaps we may now have confidence in these alpha-particle penetrabilities to about an order of magnitude. In saying this we leave out of account the possible importance of the imaginary part of the optical-model alpha-particle–nucleus potential which will tend to break up the alpha particle at the way out and so lower the effective penetrability.[7] If present ideas about the imaginary part are used this effect is worth a factor of 4 or 5. It makes no effective contribution to the relative transmission for alpha particles of different energy, however, and so is not relevant for our present approach which depends solely on the relative and not the absolute transmissions.

It would now be interesting to apply this method to a potential of the rounded type but that of course would be a computing problem of a wholly different order. It is clear that the Igo potential is not so successful in satisfying the hypothesis as the present sharp-cutoff potential since for it the square deviation takes the value 2420 (again rejecting the largest contributor) as compared with 400 at the minimum of Fig. 1. (If the Langer modification appropriate to the sharp-cutoff potential is crudely applied to the pure WKB results for the Igo potential, this dispersion falls to 1856, so it is unlikely that the failure of the Igo potential to realize the low minimum value of the dispersion is due to the shortcomings of the WKB solution.)

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[7] J. Osada, D. R. de Oliveira, N. Martins, and T. Miyazima, Nuovo cimento **20**, 845 (1961).