Electron Scattering from a Quantized Liquid Drop*

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The theory of an electron interacting with a quantized liquid drop is examined, the Hamiltonian being consistently written to second order in the parameters describing the distortion. When second-order terms are kept, some care is necessary in defining the observable value of the radius. Form factors, calculated in Born approximation, for all of the transitions in the drop model are given, as well as the probabilities for real photon emission. The correction to Coulomb scattering coming from the exchange of a transverse photon is also calculated. A model is made of the contribution of the nuclear magnetization to the transverse photon exchange. In general, the Coulomb scattering will dominate. Spherically symmetric compressional oscillations of a drop with surface tension are also discussed and the form factors for transitions are calculated.

I. INTRODUCTION

WITH the development and improvement of the study of inelastic electron scattering as a tool to probe the structure of the nucleus,[1,2] it has become desirable to have a broader theoretical basis on which to interpret the experiments. With this end in mind, we have examined the theory of an electron interacting with a quantized liquid drop. This model is known to have had semiquantitative success in describing nuclear excitations, particularly if one is near closed shells and away from the regions of permanently deformed nuclei. It provides a mathematically well-defined theory with which to work, and the form factors for the transition probabilities involved (as well as the probabilities for real γ emission) are easily calculated and help to throw some light on the question of how much one can learn about a nuclear transition from the experimental form factor. It also, of course, provides a possible basis on which to interpret and correlate the experiments. For example, the electromagnetic properties of the first three excited states of ${}_6C_6{}^{12}$ can be nicely correlated in terms of this model.[3]

In Sec. II, the model is discussed, and in Sec. III, the Coulomb interaction with the electron is introduced. Section IV contains a discussion of the probabilities for real γ emission, and Sec. V is concerned with pair emission in $0^+ \rightarrow 0^+$ transitions. Section VI contains a summary of the interesting transition probabilities. In Sec. VII, the contribution of transverse photon exchange to the scattering cross section is calculated in this model and a simple model and estimate of the contribution of the nuclear magnetization is made. In Sec. VIII, the theory of the spherically symmetric oscillations of a drop with uniform compressibility and surface tension is developed; Sec. IX contains a discussion of the results.

II. THE MODEL

We shall take for a model of the nucleus the oscillating incompressible liquid drop which was originally discussed by Rayleigh and has since been applied to the nucleus by Bohr and others.[4–9] The drop is assumed to have a well-defined surface and surface tension which provides the restoring force in the mass motion and the velocity is assumed to be irrotational (this is true in the classical case if the forces which act on the drop are conservative). The surface of the drop is written as

$$R=a\left(1+\sum_{lm} q_{lm}Y_{lm}-\frac{1}{4\pi}\sum_{lm}|q_{lm}|^2\right), \qquad (2.1)$$

with

$$q_{lm}^*=(-1)^m q_{l-m}, \qquad (2.2)$$

so that the radius is real. The term quadratic in q_{lm} must be added so that a can be identified with the equilibrium radius of the drop. That is, the equilibrium volume of the drop, Ω, is given by

$$\Omega=\tfrac{4}{3}\pi a^3+O(q^3). \qquad (2.3)$$

The Lagrangian for such a drop was originally written down by Rayleigh[4]:

$$L_0=\tfrac{1}{2}\mu a^5\sum_{lm}\frac{|\dot{q}_{lm}|^2}{l}-\frac{\sigma a^2}{2}\sum_{lm}(l-1)(l+2)\times\left[1-\frac{10\gamma}{(2l+1)(l+2)}\right]|q_{lm}|^2, \qquad (2.4)$$

where σ = surface tension, μ = mass density, and

$$\gamma=\frac{3}{5}\frac{(Ze)^2}{4\pi a}\Big/4\pi\sigma a^2.$$

[$\gamma=0.040Z^2/A$ for nuclei from the semiempirical mass formula.] The sum over l in the above expressions starts at $l=2$ since $l=0$ corresponds to compressions of the drop and for $l=1$, the restoring force disappears, corresponding to a translation of the drop. The term

* Supported in part by the U. S. Air Force through the Air Force Office of Scientific Research.

[1] J. H. Fregeau and R. Hofstadter, Phys. Rev. **99**, 1503 (1955).
[2] H. Crannel, R. Helm, H. Kendall, J. Oeser, and M. Yearian, Phys. Rev. **123**, 923 (1961). (See this paper for other references.)
[3] J. D. Walecka, following paper [Phys. Rev. **126**, 663 (1962)].
[4] Lord Rayleigh, Proc. Roy. Soc. (London) **A29**, 91 (1879).
[5] M. Fierz, Helv. Phys. Acta **16**, 365 (1943).
[6] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1952).
[7] W. Jekeli, Z. Physik **131**, 481 (1952).
[8] L. J. Tassie, Australian J. Phys. **9**, 407 (1956).
[9] E. Feenberg, *Shell Theory of the Nucleus* (Princeton University Press, Princeton, New Jersey, 1955), p. 148.

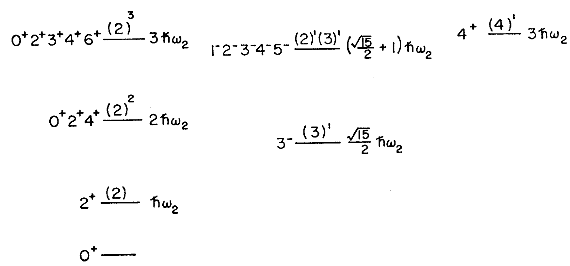

FIG. 1. Low-lying energy spectrum of a quantized liquid drop.

in γ comes from the assumption that the drop has a uniform charge density.[6]

If one now introduces the canonical momenta conjugate to the q_{lm} and then proceeds to the Hamiltonian, one finds[6]

$$H_0 = \sum_{lm} \left(\frac{1}{2B_l} |p_{lm}|^2 + \tfrac{1}{2} C_l |q_{lm}|^2 \right), \tag{2.5}$$

where the following definitions have been used:

$$\begin{aligned} B_l &\equiv \mu a^5 / l, \\ C_l &\equiv \sigma a^2 (l-1)(l+2) \left[1 - \frac{10\gamma}{(2l+1)(l+2)} \right], \\ \omega_l^2 &= C_l / B_l, \end{aligned} \tag{2.6}$$

and where

$$p_{lm} = B_l \dot{q}_{lm}^*. \tag{2.7}$$

This is the familiar Hamiltonian for an infinite system of uncoupled oscillators which represent the normal modes of oscillation of the system. If the following expansion is introduced for q_{lm}:

$$q_{lm} = \left[\frac{\hbar}{2(B_l C_l)^{\frac{1}{2}}} \right]^{\frac{1}{2}} \left[a_{lm} e^{-i\omega_l t} + (-1)^m a_{l-m}^* e^{i\omega_l t} \right] \tag{2.8}$$

[which guarantees the conjugation property of the q_{lm}'s, (2.2)], then

$$H_0 = \sum_{lm} \hbar\omega_l \tfrac{1}{2} (a_{lm} a_{lm}^* + a_{lm}^* a_{lm}). \tag{2.9}$$

Since the Hamiltonian is in canonical form, it is a simple matter to quantize the motion. One writes the commutation relations

$$\begin{aligned} [a_{lm}, a_{l'm'}^*] &= \delta_{ll'} \delta_{mm'}, \\ [a_{lm}, a_{l'm'}] &= [a_{lm}^*, a_{l'm'}^*] = 0 \end{aligned} \tag{2.10}$$

and interprets the a_{lm} and a_{lm}^* as annihilation and creation operators. Their matrix elements follow from the commutation relations and are

$$\begin{aligned} (n_{lm}+1 | a_{lm}^* | n_{lm}) &= (n_{lm}+1)^{\frac{1}{2}} \\ (n_{lm}-1 | a_{lm} | n_{lm}) &= (n_{lm})^{\frac{1}{2}}. \end{aligned} \tag{2.11}$$

We can also make use of the fact that the transition from classical mechanics to quantum mechanics necessitates a specification of the ordering of the operators,[10,11] to write

$$H_0 = \sum_{lm} \hbar\omega_l a_{lm}^* a_{lm} = \sum_{lm} \hbar\omega_l N_{lm}. \tag{2.12}$$

In this way, the energy of the lowest state [hereafter referred to as the vacuum, and denoted by $|0)$ but always meant to mean a uniformly charged sphere of radius a] is defined to be zero. The a_{lm}^* will then create from the vacuum, states of discrete energy $\hbar\omega_l$. In the limit of light nuclei, the term in γ in C_l is small and can be omitted. Therefore, one has

$$\omega_l^2 / \omega_2^2 = l(l-1)(l+2)/8, \tag{2.13}$$

and the ratio of the energy spacings are functions only of l. The predictions for the first few excited states are [the configurations are denoted by $(l)^N$] shown in Fig. 1. The angular momenta of these states has been discussed in detail by Feenberg,[9] and the parity of each surfon is given by $(-1)^l$. We note particularly the existence of the low-lying collective 3^- state and $4^+ (l=4)$ state. There is strong evidence, particularly from the electron scattering experiments of Kendall, that such states exist generally in nuclei,[12] indicating that the oscillating drop model may have a greater degree of validity then has hitherto been supposed.

III. COULOMB INTERACTION

We next consider the interaction of the drop with an electron. We write the interaction as

$$H_1 = -\frac{e^2}{4\pi} \int \frac{\rho_N(\mathbf{x}) \rho_e(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{x} d\mathbf{y}, \tag{3.1}$$

where $\rho_e(\mathbf{y})$ is the electron charge density operator,

$$\rho_e(\mathbf{y}) = \bar{\psi}(\mathbf{y}) \gamma_4 \psi(\mathbf{y}), \tag{3.2}$$

and $\rho_N(\mathbf{x})$ is the nuclear charge density operator,

$$\rho_N(\mathbf{x}) = \frac{3Z}{4\pi a^3} \theta \left[a \left(1 + \sum_{lm} q_{lm} Y_{lm} - \frac{1}{4\pi} \sum_{lm} |q_{lm}|^2 \right) - r \right]. \tag{3.3}$$

$\theta(x)$ is the step function:

$$\begin{aligned} \theta(x) &= 1 \quad \text{for} \quad x > 0 \\ &= 0 \quad \text{for} \quad x < 0. \end{aligned}$$

Now an electron will also interact with an oscillating charge distribution through the exchange of transverse photons. For collective excitations, however, it was shown by Schiff that the Coulomb interactions dominate.[13] We shall actually calculate the contribution of

[10] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd ed., p. 57.
[11] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., p. 389.
[12] A. M. Lane and E. D. Pendlebury, Nuclear Phys. **15**, 39 (1960).
[13] L. I. Schiff, Phys. Rev. **96**, 765 (1954).

the transverse photons in Sec. VIII in this model and will show that under the appropriate conditions they are negligible. We therefore limit ourselves to the Coulomb interaction. If the following representations are now introduced,

$$\frac{1}{|\mathbf{x}-\mathbf{y}|}=\frac{4\pi}{(2\pi)^3}\int\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{q}^2}d\mathbf{q},\tag{3.4}$$

$$e^{i\mathbf{q}\cdot\mathbf{x}}=4\pi\sum_{LM}i^LY_{LM}^*(\Omega_q)Y_{LM}(\Omega_x)j_L(qx),\tag{3.5}$$

then H_1 reduces to

$$H_1=\left(\frac{e^2}{4\pi}\right)\left(\frac{2}{\pi}\right)\sum_{LM}i^L\int\frac{d\mathbf{q}}{\mathbf{q}^2}Y_{LM}^*(\Omega_q)\mathfrak{M}_{LM}(q)\times\int e^{-i\mathbf{q}\cdot\mathbf{y}}\rho_e(\mathbf{y})d\mathbf{y},\tag{3.6}$$

where $\mathfrak{M}_{LM}(q)$, the Lth multipole of the nuclear charge distribution is defined by

$$\mathfrak{M}_{LM}(q)\equiv\int d\mathbf{x}\,\rho_N(\mathbf{x})Y_{LM}(\Omega_x)j_L(qx).\tag{3.7}$$

These moments can be calculated in the drop model by writing

$$\mathfrak{M}_{LM}(q)=\frac{3Z}{4\pi a^3}\int Y_{LM}(\Omega_x)d\Omega_x\int_0^{a'}x^2j_L(qx)dx,\tag{3.8}$$

where

$$a'\equiv a\left(1+\sum_{lm}q_{lm}Y_{lm}-\frac{1}{4\pi}\sum|q_{lm}|^2\right),$$

and expanding the radial integral about the point a, keeping terms up through order q_{lm}^2. One obtains[14]

$$\begin{aligned}\mathfrak{M}_{LM}(q)=&\,\delta_{L0}\delta_{M0}\frac{3Z}{(4\pi)^{\frac{1}{2}}}\left(4\pi\frac{j_1(qa)}{qa}-j_0(qa)\sum_{lm}|q_{lm}|^2\right)\\&+\frac{3Z}{4\pi}j_L(qa)q_{LM}^*+\frac{3Z}{8\pi qa}\left[\frac{\partial}{\partial\rho}\rho^2j_L(\rho)\right]_{\rho=qa}\\&\times\sum_{lm}\sum_{l'm'}\left(\frac{(2l+1)(2l'+1)(2L+1)}{4\pi}\right)^{\frac{1}{2}}\\&\times\begin{pmatrix}l&l'&L\\0&0&0\end{pmatrix}\begin{pmatrix}l&l'&L\\m&m'&M\end{pmatrix}q_{lm}q_{l'm'}.\end{aligned}\tag{3.9}$$

In most of our applications we will deal with nuclei for which

$$Ze^2/4\pi\hbar c\equiv Z\alpha=Z/137\tag{3.10}$$

is small enough that it will suffice to use first Born approximation to compute the scattering amplitude.[15] The electron transition charge density can then be written

$$\rho_e(\mathbf{y})=e^{-i(\mathbf{K}_2-\mathbf{K}_1)\cdot\mathbf{y}}\bar{u}(\mathbf{K}_2)\gamma_4u(\mathbf{K}_1).\tag{3.11}$$

The electrons are treated as relativistic Dirac particles, while we shall assume that the electron energies and nuclear mass are such that nuclear recoil can be neglected. The cross section for electron scattering, summed and averaged over electron and nuclear spins, can then be calculated from the square of the matrix element of H_1 in the usual fashion and yields[13,16]

$$\frac{d\sigma}{d\Omega}(J_f\leftarrow J_i)=\left(\frac{K_2}{K_1}\right)(4\pi\sigma_M)\sum_{L=0}^{\infty}\frac{1}{2J_i+1}\times|(J_f\|\mathfrak{M}_L(\Delta)\|J_i)|^2\tag{3.12}$$

where

$$\begin{aligned}\sigma_M&\equiv(4\alpha^2/\Delta^4)K^2\cos^2(\theta/2),\\\Delta^2&=(\mathbf{K}_1-\mathbf{K}_2)^2=4K^2\sin^2(\theta/2).\end{aligned}\tag{3.13}$$

The approximations have been made that the electrons are relativistic so that $E=\hbar Kc$ and that the energy transferred to the nucleus is small so that $K_1=K_2=K$. Also, the Wigner-Eckart theorem has been used for the matrix elements of $\mathfrak{M}_{LM}$:

$$(J_fM_f|\mathfrak{M}_{LM}|J_iM_i)=(-1)^{J_f-M_f}\begin{pmatrix}J_f&L&J_i\\-M_f&M&M_i\end{pmatrix}(J_f\|\mathfrak{M}_L\|J_i)\tag{3.14}$$

to permit summation over magnetic quantum numbers, leaving the reduced matrix elements as on the right of Eq. (3.14).

There is still an important point to be discussed with respect to $\mathfrak{M}_{LM}$. Since we have chosen to write our Hamiltonian consistently to terms of order q_{lm}^2, we face the problem that the vacuum expectation value of $\mathfrak{M}_{LM}$ is infinite. This comes from the fact that bilinear combinations of the q_{lm} contain terms

$$\sum_{lm}\frac{\hbar}{2(B_lC_l)^{\frac{1}{2}}}a_{lm}a_{lm}^*$$

and

$$\left(0\left|\sum_{lm}\frac{\hbar}{2(B_lC_l)^{\frac{1}{2}}}a_{lm}a_{lm}^*\right|0\right)=\sum_{lm}\frac{\hbar}{2(B_lC_l)^{\frac{1}{2}}}=\infty.\tag{3.15}$$

This is not surprising since we started with a continuous system and therefore have an infinite number of degrees of freedom. Physically, there is a cutoff, $l_{\max}$, determined by the requirement that we cannot have disturbances

[14] The notation used is that of A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[15] The Coulomb corrections to first Born approximation for inelastic differential cross sections have been calculated by Pratt and Walecka in "Distorted Wave Born Approximation" (to be published).

[16] K. Alder, A. Bohr, T. Huus, B. Mottleson, and A. Winther, Revs. Modern Phys. **28**, 476 (1956).

whose wavelength is smaller than the interparticle distance. We can, however, avoid specific reference to $l_{\max}$ by again making use of the fact that the transition from classical to quantum mechanics necessitates a specification of the ordering of the operators. We *define* our interaction to be normal ordered; this means all the annihilation operators are to be put to the right of all the creation operators. This Hamiltonian, together with the canonical commutation rules, can be taken to define the theory[17] which certainly has the correct classical limit. Doing this, one finds

$$(0|\mathfrak{M}_{00}|0)=\frac{3Z}{(4\pi)^{\frac{1}{2}}}\frac{j_1(\Delta a)}{\Delta a}, \tag{3.16}$$

$$d\sigma/d\Omega)_{\rm el}=Z^2\sigma_M\left(\frac{3j_1(\Delta a)}{\Delta a}\right)^2, \tag{3.17}$$

which is the elastic scattering cross section for a uniformly charged sphere of radius a. This relation allows us to determine experimentally the radius, a, of the vacuum charge distribution, and this radius can be used to compute the inelastic cross sections.

The above procedure may seem quite arbitrary to the reader. We could have approached the question from a different point of view. We write as an identity,

$$\mathfrak{M}_{LM}(\Delta)=(0|\mathfrak{M}_{LM}(\Delta)|0)+\mathfrak{M}_{LM}(\Delta)-(0|\mathfrak{M}_{LM}(\Delta)|0). \tag{3.18}$$

The last two terms form the definition of the normal ordered product of operators in $\mathfrak{M}_{LM}$ and we write this as

$$:\delta\mathfrak{M}_{LM}:\equiv\mathfrak{M}_{LM}(\Delta)-(0|\mathfrak{M}_{LM}(\Delta)|0). \tag{3.19}$$

If we now actually compute $(0|\mathfrak{M}_{LM}|0)$ in terms of the operators as they stand in expression (3.9) for $\mathfrak{M}_{LM}$, we find

$$(0|\mathfrak{M}_{LM}|0)=\frac{3Z}{(4\pi)^{\frac{1}{2}}}\frac{j_1(\Delta a_0)}{\Delta a_0}\times\left[1-\frac{(\Delta a_0)^2}{8\pi}(0|\sum_{lm}|q_{lm}|^2|0)\right]\delta_{L0}\delta_{M0}. \tag{3.20}$$

We must, however, be careful with the interpretation of this expression. Owing to the presence of the vacuum fluctuations, the effective radius of the vacuum charge distribution has been changed. It is for this reason that we have used a_0 in the above expression. If we define the *renormalized* radius of the vacuum charge distribution by demanding

$$\lim_{\Delta\to 0}(0|\mathfrak{M}_{00}|0)=\frac{Z}{(4\pi)^{\frac{1}{2}}}\left(1-\frac{(\Delta a)^2}{10}\right), \tag{3.21}$$

which comes from expanding Eq. (3.16), then, by identifying Eqs. (3.21) and (3.20) we find

$$a=a_0\left(1+\frac{5}{8\pi}(0|\sum_{lm}|q_{lm}|^2|0)\right). \tag{3.22}$$

Reinserting this in Eq. (3.20), and always keeping only up to order q_{lm}^2, we have[18]

$$(0|\mathfrak{M}_{00}|0)=\frac{3Z}{(4\pi)^{\frac{1}{2}}}\left[\frac{j_1(\Delta a)}{\Delta a}+\frac{\Delta a j_3(\Delta a)}{8\pi}(0|\sum_{lm}|q_{lm}|^2|0)\right]. \tag{3.23}$$

With any reasonable value of $l_{\max}$, the second term is small and will not contribute until high-momentum transfers are reached. We note, then, that for low-momentum transfers, the two methods of dealing with the vacuum fluctuations yield *identical* results. In both cases, the actual radius a must be determined by making a fit to the experimental elastic scattering cross section. If the method discussed first is adopted, then $l_{\max}$ never enters explicitly. It is, of course, only as good as the fit of (3.17) to the cross section. We then have

$$\mathfrak{M}_{LM}(\Delta)=\frac{3Z}{(4\pi)^{\frac{1}{2}}}\frac{j_1(\Delta a)}{\Delta a}\delta_{L0}\delta_{M0}+:\delta\mathfrak{M}_{LM}(\Delta): \tag{3.24}$$

and, together with the commutation rules, (2.10), this defines the theory.

Note added in proof. The second method, Eq. (3.23), contains the interesting possibility of describing the smearing of the nuclear surface by the quantum fluctuations of the surface coordinates.

IV. RADIATIVE DECAY

We are also interested in computing the rates of radiative γ transitions between the different levels in our model. To do this, we include a term in the Hamiltonian

$$H_2=-\int \mathbf{j}_N(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})d\mathbf{x}, \tag{4.1}$$

with

$$\nabla\cdot\mathbf{A}(\mathbf{x})=0 \tag{4.2}$$

for the transverse radiation field. We can now proceed in the standard fashion to make a multipole analysis of $\mathbf{A}(\mathbf{x})$ and to get the transition probabilities from the square of the matrix elements of H_2.[19] If we restrict ourselves to long-wavelength transitions,

$$(Ka)\ll 1, \tag{4.3}$$

[17] *Note.* It is evident that this reordering of factors in no way changes the commutation relations of H with any combination of creation and destruction operators so that the equations of motion are unaltered. The Hermiticity of H is similarly unaffected.

[18] *Note.* We may always replace a_0 by a in $:\delta\mathfrak{M}_{LM}:$ to order q_{lm}^2. The two procedures therefore yield identical results for $:\delta\mathfrak{M}_{LM}:$ and differ only in their treatment of the elastic scattering from the ground state, or equivalently, in the experimental identification of the radius a.

[19] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Chap. XII and Appendix B.

then the probability per unit time for making an electric multipole transition of order L depends only on $\nabla\cdot\mathbf{j}_N(\mathbf{x})$ and can be written with the aid of the continuity equation,

$$\nabla\cdot\mathbf{j}_N(\mathbf{x})+1/c[\partial\rho_N(\mathbf{x})/\partial t]=0, \tag{4.4}$$

as

$$\omega_L{}^{\mathrm{el}}(J_f\leftarrow J_i)=\frac{8\pi\alpha(Kc)K^{2L}}{[(2L+1)!!]^2}\left(\frac{L+1}{L}\right)\frac{1}{2J_i+1}\times|(J_f\|Q_L\|J_i)|^2, \tag{4.5}$$

where

$$Q_{LM}=\int\rho_N(\mathbf{x})Y_{LM}(\Omega_x)x^L d\mathbf{x}. \tag{4.6}$$

Q_{LM} can be obtained from $\mathfrak{M}_{LM}$ by

$$Q_{LM}=\lim_{\Delta\to0}\frac{(2L+1)!!}{\Delta^L}\mathfrak{M}_{LM}(\Delta). \tag{4.7}$$

The probability per unit time of making a magnetic multipole transition of order L is given in the long-wavelength limit by

$$\omega_L{}^{\mathrm{mag}}(J_f\leftarrow J_i)=\frac{8\pi\alpha(Kc)K^{2L}}{[(2L+1)!!]^2}\left(\frac{L+1}{L}\right)\frac{1}{2J_i+1}\times|(J_f\|M_L\|J_i)|^2, \tag{4.8}$$

where

$$M_{LM}=\frac{1}{L+1}\int[\mathbf{x}\times\mathbf{j}_N(\mathbf{x})]\cdot[\nabla x^L Y_{LM}(\Omega_x)]d\mathbf{x}. \tag{4.9}$$

This expression may be evaluated by using the equation for the current[4]:

$$\mathbf{j}_N(\mathbf{x})=\frac{3Z}{4\pi ac}\sum_{lm}\frac{\dot{q}_{lm}}{l}[\nabla(r/a)^l Y_{lm}]\theta(a-r), \tag{4.10}$$

and gives

$$M_{LM}=0. \tag{4.11}$$

This means that the magnetic transitions are at least quadratic in the deformation parameter. Since the current contains a factor $(a/c)\dot{q}_{lm}$, we can also expect that the magnetic multipole rates will be down by a factor $(Ka)^2$ from the corresponding electric transitions. We also note that there are *no* $M1$ transitions between the states $(2^2)_{2^+}\to(2^1)_{2^+}$ to second order in the deformation since M_{LM} cannot change the number of surfons by one.

We cannot actually write out the current operator to terms quadratic in the deformation parameter at this stage because the current depends on the equations of motion and surface boundary condition which have only been developed to first order. We will therefore content ourselves with the observation that the electric transitions are expected to be the dominant ones. We note that we *were* able to write the electric moments consistently to second order in the long-wavelength limit (4.7) since in that case all we needed was the *divergence* of the current which we could relate back to the charge density through the continuity equation.

V. PAIR EMISSION IN $0^+\to0$ TRANSITIONS

Since a single γ-ray transition $0^+\to0$ is absolutely forbidden by angular momentum selection rules, the most probable process for the electromagnetic decay of a 0^+ state will be the emission of an electron-positron pair (provided that there is more than 1.02 Mev available, of course).[20] The probability for such a process is easily calculated from H_1. The electron charge density has a term in it which creates an electron-positron pair, that is,

$$(\mathbf{K}_+\mathbf{K}_-|\rho_e(\mathbf{y})|0)=e^{-i(\mathbf{K}_++\mathbf{K}_-)\cdot\mathbf{y}}\bar{u}(\mathbf{K}_+)\gamma_4 v(-\mathbf{K}_-). \tag{5.1}$$

Taking the matrix element of H_1 between the states $|0^+)$ and $|0,\mathbf{K}_+,\mathbf{K}_-)$, squaring it, and summing over the final lepton spins, one is led in the familiar fashion to the transition rate

$$\omega_{e^\pm}(0\leftarrow0^+)=\frac{2\alpha^2\hbar c^2}{\pi^2}\int\int\frac{d\mathbf{K}_+d\mathbf{K}_-}{\Delta^4}|(0|\mathfrak{M}_{00}(\Delta)|0^+)|^2\times\left(1+\hbar^2c^2\frac{\mathbf{K}_+\cdot\mathbf{K}_-}{E_+E_-}-\frac{m_e{}^2c^4}{E_+E_-}\right)\delta(E_++E_--\epsilon) \tag{5.2}$$

with ϵ the energy of the transition and

$$\Delta^2\equiv(\mathbf{K}_++\mathbf{K}_-)^2. \tag{5.3}$$

If we again work in the long-wavelength limit, $\Delta a\ll1$, we can write

$$\mathfrak{M}_{00}(\Delta)=Z-\frac{\Delta^2}{6(4\pi)^{\frac{1}{2}}}\int x^2\rho_N(\mathbf{x})d\mathbf{x}. \tag{5.4}$$

Z cannot cause transitions between different states. Defining

$$\mathrm{M.E.}\equiv\left(0\left|\int x^2\rho_N(\mathbf{x})d\mathbf{x}\right|0^+\right) \tag{5.5}$$

and assuming $m_e{}^2c^4/\epsilon^2\ll1$, we find[20]

$$\omega_{e^\pm}(0\leftarrow0^+)=\frac{\alpha^2(Kc)}{135\pi}(K^4|\mathrm{M.E.}|^2). \tag{5.6}$$

VI. SUMMARY OF MOMENTS AND TRANSITION PROBABILITIES

The transitions between the states 0 (vacuum), 2^+, 3^-, and the degenerate triplet $(2^2)_{0^+}$, $(2^2)_{2^+}$, $(2^2)_{4^+}$ have been summarized in Table I. The two-surfon states of

[20] J. R. Oppenheimer and J. S. Schwinger, Phys. Rev. **56**, 1066 (1939).

TABLE I. Squares of matrix elements for electromagnetic transitions in the oscillating drop model.

Transition	$\frac{1}{2J_i+1}\lvert(J_f\Vert\mathfrak{M}_J(\Delta)\Vert J_i)\rvert^2\left(\frac{16\pi^2}{9Z^2}\right)$ $J=\lvert J_f-J_i\rvert$	$\frac{1}{2J_i+1}\lvert(J_f\Vert Q_J\Vert J_i)\rvert^2\left(\frac{16\pi^2}{9Z^2}\right)$ $J=\lvert J_f-J_i\rvert$	$\lvert\text{M.E.}\rvert^2\left(\frac{16\pi^2}{9Z^2}\right)$
$0\leftrightarrow 2^+$	$5\left(\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}\right)[j_2(\Delta a)]^2$	$\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}a^4$	
$0\leftrightarrow (2^2)0^+$	$\frac{5}{8\pi}\left(\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}\right)^2[\Delta a j_1(\Delta a)]^2$		$10\left(\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}\right)^2 a^4$
$0\leftrightarrow (2^2)2^+$	$\frac{100}{7\pi}\left(\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}\right)^2\left[j_2(\Delta a)-\frac{\Delta a}{4}j_3(\Delta a)\right]^2$	$\frac{20}{7\pi}\left(\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}\right)^2 a^4$	
$0\leftrightarrow (2^2)4^+$	$\frac{405}{7\pi}\left(\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}\right)^2\left[j_4(\Delta a)-\frac{\Delta a}{6}j_5(\Delta a)\right]^2$	$\frac{45}{7\pi}\left(\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}\right)^2 a^8$	
$0\leftrightarrow 3^-$	$7\left(\frac{\hbar}{2(B_3C_3)^{\frac{1}{2}}}\right)[j_3(\Delta a)]^2$	$\frac{\hbar}{2(B_3C_3)^{\frac{1}{2}}}a^6$	
$2^+\leftarrow (2^2)0^+$		$2\left(\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}\right)a^4$	
$2^+\leftarrow (2^2)2^+$		$2\left(\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}\right)a^4$	
$2^+\leftarrow (2^2)4^+$		$2\left(\frac{\hbar}{2(B_2C_2)^{\frac{1}{2}}}\right)a^4$	

definite angular momentum have been constructed by using[9,21]

$$|l_1l_2JM)=(2J+1)^{\frac{1}{2}}\sum_{m_1m_2}\begin{pmatrix} l_1 & l_2 & J \\ -m_1 & -m_2 & M\end{pmatrix}\times a_{l_1m_1}{}^*a_{l_2m_2}{}^*|0). \tag{6.1}$$

We note that if $l_1=l_2$, then the states with J odd vanish through the symmetry properties of the $3-j$ symbol, which is just the statement that two bosons (we have used the commutation rules for Bose particles) must exist in a symmetric state. In this case, we also need an extra $1/\sqrt{2}$ in the normalization.

VII. CONTRIBUTION OF THE TRANSVERSE PHOTON EXCHANGE TO ELECTRON SCATTERING

We now return to the problem of calculating the contribution of the exchange of a transverse photon to the cross section for electron scattering. We include in the interaction both H_2 and a similar term for the electron:

$$H_2'=-ie\int\bar{\psi}(\mathbf{y})\gamma\psi(\mathbf{y})\cdot\mathbf{A}(\mathbf{y})d\mathbf{y}. \tag{7.1}$$

Then by combining the first-order contribution from the Coulomb potential (of order e^2) with the second-order contribution from the interaction with $\mathbf{A}$, it is not difficult to show that the cross section for scattering into an angle θ is given by[13,16]

$$\frac{d\sigma}{d\Omega}(J_f\leftarrow J_i)=\left(\frac{K_2}{K_1}\right)\frac{8\pi\alpha^2}{\Delta^4}\left[2\mathbf{K}^2\cos^2(\theta/2)\sum_{L=0}^{\infty}\frac{1}{2J_i+1}\times|(J_f\Vert\mathfrak{M}_L(\Delta)\Vert J_i)|^2+\mathbf{K}^2[1+\sin^2(\theta/2)]\times\sum_{L=1}^{\infty}\frac{1}{2J_i+1}[|(J_f\Vert T_L{}^{\text{el}}\Vert J_i)|^2+|(J_f\Vert T_L{}^{\text{mag}}\Vert J_i)|^2]\right]. \tag{7.2}$$

The electron has again been assumed to be relativistic and to lose only a negligible portion of its energy to the nucleus. The transverse multipole operators are the familiar expressions:

$$T_{LM}{}^{\text{el}}=\frac{1}{\Delta}\int d\mathbf{x}[\mathbf{j}_N(\mathbf{x})\cdot(\boldsymbol{\nabla}\times j_L(\Delta x)\mathfrak{Y}_{LL1}{}^M)+\Delta^2 j_L(\Delta x)\mathfrak{Y}_{LL1}{}^M\cdot\boldsymbol{\mu}_N(\mathbf{x})] \tag{7.3}$$

$$T_{LM}{}^{\text{mag}}=\int d\mathbf{x}[\boldsymbol{\mu}_N(\mathbf{x})\cdot(\boldsymbol{\nabla}\times j_L(\Delta x)\mathfrak{Y}_{LL1}{}^M)+j_L(\Delta x)\mathfrak{Y}_{LL1}{}^M\cdot\mathbf{j}_N(\mathbf{x})]. \tag{7.4}$$

The possibility of a contribution from an extra magnetization density $\boldsymbol{\mu}_N(\mathbf{x})$ has been included for completeness. We shall return to it later in this section.

[21] D. W. Robinson, Nuclear Phys. 25, 459 (1961).

It should be noted that since the transverse photon terms start with $L=1$ they cannot contribute to the elastic scattering from the ground state in our model, since it is a $0 \to 0$ transition. The second thing to be noted is the quite general property that the Coulomb scattering and transverse photon scattering are multiplied by different angular functions coming from the electron spinors, the first by $2\cos^2(\theta/2)$ and the second by $1+\sin^2(\theta/2)$. This means it is possible, if one wants to, to make the transverse photon contribution the dominant one by appropriate choice of angles (i.e., the backward direction). We shall show, however, that in our model, the Coulomb contribution will dominate almost everywhere in the inelastic processes.

We proceed to evaluate the multipole operators $T_{LM}{}^{\mathrm{el}}$ and $T_{LM}{}^{\mathrm{mag}}$ in the drop model. We shall restrict the discussion to the terms linear in q_{lm} which means we are discussing the "allowed" transitions $0 \to (2^1)_{2^+}$, $0 \to (3^1)_{3^-}$, $0 \to (4^1)_{4^+}$, etc. The first expression can, by an appropriate use of vector identities, be transformed into the following:

$$\int d\mathbf{x}\, \mathbf{j}_N(\mathbf{x}) \cdot (\boldsymbol{\nabla} \times j_L(\Delta x) \mathfrak{Y}_{LL1}{}^M)$$

$$= \frac{1}{i[L(L+1)]^{\frac{1}{2}}} \int d\mathbf{x} [(\boldsymbol{\nabla} \cdot \mathbf{j}_N(\mathbf{x})) \times (1+\mathbf{x}\cdot\boldsymbol{\nabla}) j_L(\Delta x) Y_{LM}(\Omega_x) - \Delta^2 j_L(\Delta x) Y_{LM}(\Omega_x)(\mathbf{x}\cdot \mathbf{j}_N(\mathbf{x}))]. \quad (7.5)$$

Upon insertion of the current (4.10) and the continuity equation (4.4), one finds

$$T_{LM}{}^{\mathrm{el}} = \frac{-3Z}{4\pi i} \frac{1}{\Delta c} \left(\frac{L+1}{L}\right)^{\frac{1}{2}} j_L(\Delta a) \dot{q}_{LM}{}^*. \quad (7.6)$$

Similarly, it is easy to show that the leading term in $T_{LM}{}^{\mathrm{mag}}$ vanishes so that we have, to this order,

$$T_{LM}{}^{\mathrm{mag}} = 0.$$

This means that the cross section for the allowed transitions is given by

$$\frac{d\sigma}{d\Omega}(J \leftarrow 0) = \left(\frac{K_2}{K_1}\right)(4\pi\sigma_M) \left[\frac{9Z^2}{16\pi^2}(2J+1) \frac{\hbar}{2(B_J C_J)^{\frac{1}{2}}} [j_J(\Delta a)]^2\right] \times \left\{1 + \left(\frac{J+1}{J}\right)\left[\frac{1+\sin^2(\theta/2)}{2\cos^2(\theta/2)}\right]\left[\frac{\omega_J}{\Delta c}\right]^2\right\}. \quad (7.7)$$

Thus, the correction to the scattering goes as the square of the ratio of the energy transferred to the nucleus to the momentum transfer and except for the very forward or backward angles, this correction is a small number in the Stanford scattering experiments.[1,2]

It is also of interest from the experimental point of view to ask when one should begin to see contributions to the transverse photon exchange coming from the nuclear magnetization.[22] The model discussed here has no magnetization. We can make an attempt to answer this question by making a crude model of what the magnetization should look like. We assume that the magnetization is distributed uniformly over the nucleus just as the charge density. Since it must be a vector, we assume it to be proportional to the angular momentum operator. Also, since the charge density and angular momentum do not commute (they are both taken to be operators in the surfon space), we symmetrize in these operators. This gives for the magnetization density

$$\boldsymbol{\mu}_N(\mathbf{x}) = \frac{\lambda}{eZ}\left(\frac{e\hbar}{2Mc}\right)[\rho_N(\mathbf{x})\mathbf{J}]_{\mathrm{sym}}, \quad (7.8)$$

where λ is the total moment in nuclear magnetons, and

$$\boldsymbol{\mu}_N = e\int d\mathbf{x}\, \boldsymbol{\mu}_N(\mathbf{x}) = \lambda\left(\frac{e\hbar}{2Mc}\right)\mathbf{J}. \quad (7.9)$$

We proceed next to calculate the moment operators $T_{LM}{}^{\mathrm{el}}$ and $T_{LM}{}^{\mathrm{mag}}$. Again we restrict the discussion to allowed transitions, and with a little work one obtains

$$T_{LM}{}^{\mathrm{el}} = \left(\frac{\lambda}{Z}\right)\left(\frac{\hbar\Delta}{2Mc}\right)\left(\frac{3Z}{4\pi}\right) j_L(\Delta a) \times [q_L{}^* \odot J]_{\mathrm{sym}}{}^{LM} \quad (7.10)$$

$$T_{LM}{}^{\mathrm{mag}} = i\left(\frac{\lambda}{Z}\right)\left(\frac{\hbar\Delta}{2Mc}\right)\left(\frac{3Z}{4\pi}\right)\left[-\left(\frac{L}{2L+1}\right)^{\frac{1}{2}} j_{L+1}(\Delta a) \times [q_{L+1}{}^* \odot J]_{\mathrm{sym}}{}^{LM} + \left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}} j_{L-1}(\Delta a) \times [q_{L-1}{}^* \odot J]_{\mathrm{sym}}{}^{LM}\right], \quad (7.11)$$

where the symbol $\odot$ means tensor product and is defined by

$$[q_l{}^* \odot J]^{LM} \equiv \sum_{m_1 m_2} (l m_1 1 m_2 | l 1 L M) q_{l m_1}{}^* J_{m_2} \quad (7.12)$$

in the usual spherical component notation. $T_{LM}{}^{\mathrm{mag}}$ cannot create the right surfon to contribute to the allowed transition (i.e., they are all electric transitions). Using the relation

$$(JM | [q_J \odot J]_{\mathrm{sym}}{}^{JM} | 0) = \frac{\hbar}{2(B_J C_J)^{\frac{1}{2}}} \left(\frac{J(J+1)}{4}\right)^{\frac{1}{2}} \quad (7.13)$$

[22] *Note.* We mean here by magnetization everything that is not included in the current operator; for example, the contribution from the intrinsic magnetic moments of the nucleons.

we find that Eq. (7.7) for the cross section still holds if we replace

$$\left[\frac{\omega_J}{\Delta c}\right]^2 \rightarrow \left[\frac{-\omega_J}{\Delta c}+\left(\frac{\lambda}{Z}\right)\left(\frac{\hbar\Delta}{2Mc}\right)\left(\frac{J}{2}\right)\right]^2. \tag{7.14}$$

This expression is still small for moderate momentum transfers $\hbar\omega_J \ll \hbar\Delta c \ll 2Mc^2$; however, the contribution from the magnetization grows with Δ. The magnetization term is also reduced since magnetic moments (spins) tend to pair off in nuclei, and the factor λ/Z is usually less than one. There may be other mechanisms for the magnetic contribution, other than surface oscillations of the magnetization, which could make this term larger but, in general, if

$$\hbar\omega_J \ll \hbar\Delta c \ll 2Mc^2 \tag{7.15}$$

we will expect the collective Coulomb transitions to dominate the scattering.

VIII. COMPRESSION OF OSCILLATIONS OF A LIQUID DROP WITH SURFACE TENSION

So far the discussion has been limited to the surface oscillations of an incompressible liquid drop. Although this type of excitation is expected to have the lower energy, it is also of some interest to investigate the properties of compressional oscillations, particularly in the case of $0\rightarrow 0^+$ transitions where they may be expected to be most evident. In particular, one would like to know what the form factor for such a transition would look like with the hope of perhaps identifying such transitions experimentally. We shall limit our present discussion to the spherically symmetric oscillations of a liquid drop which has a uniform compressibility and surface tension. We shall neglect the effect of the charge on the motion, which should again be a good approximation for light nuclei.[23] The assumption of uniform compressibility can be expressed by saying

$$\mu(dp/d\mu)=b, \tag{8.1}$$

where p is the pressure, μ the density, and b, the bulk modulus, is constant. Integrating this relation gives

$$p=b\ln(\mu/\mu_\infty), \tag{8.2}$$

where μ_∞ is the equilibrium density of a very large drop. The equations of motion for small oscillations about the equilibrium density of the drop, μ_0, are[24]

$$\partial\eta/\partial t=-(b/\mu_0)\nabla\eta, \tag{8.3}$$

$$\partial\eta/\partial t=-\nabla\cdot\mathbf{v}, \tag{8.4}$$

where $\mathbf{v}$ is the velocity field, and $\mu\equiv\mu_0(1+\eta)$. Taking the divergence of the first equation and the time derivative of the second leads to

$$\nabla^2\eta=(1/s^2)\partial^2\eta/\partial t^2 \tag{8.5}$$

$$s^2\equiv b/\mu_0. \tag{8.6}$$

The boundary condition to be imposed on this equation is that the compressional pressure at the surface of the drop balance the inward pressure coming from the surface tension, or

$$b\ln[\mu(R)/\mu_\infty]=2\sigma/R. \tag{8.7}$$

In the case of equilibrium, this equation gives μ_0 in terms of σ:

$$\ln(\mu_0/\mu_\infty)=2\sigma/ab\equiv\lambda, \tag{8.8}$$

where the equilibrium radius a is related to μ_0 and the mass of the drop m by

$$3m/4\pi a^3=\mu_0. \tag{8.9}$$

There is one more constraint on the motion which relates R and μ and that is the conservation of the total mass of the drop

$$\int_0^R \mu d\mathbf{x}=m. \tag{8.10}$$

Now, by writing

$$R=a(1+q) \tag{8.11}$$

and expanding first the boundary condition (8.7) about the point a

$$\eta(a)=-\lambda q, \tag{8.12}$$

and then the conservation equation (8.10) also about a,

$$\int_0^a \eta d\mathbf{x}+4\pi a^3 q=0, \tag{8.13}$$

we find that the boundary condition which the solutions to Eq. (8.5) has to satisfy is

$$\eta(a)=\frac{\lambda}{4\pi a^3}\int_0^a \eta d\mathbf{x}. \tag{8.14}$$

If we look for spherically symmetric solutions to Eq. (8.5) which are harmonic in time and finite at the origin, we find

$$\eta=Nj_0(kr)e^{\pm i\omega t}, \tag{8.15}$$

where the eigenvalue k,

$$\omega=ks, \tag{8.16}$$

is determined from the boundary condition (8.14)

$$j_0(k_n a)=\lambda j_1(k_n a)/k_n a. \tag{8.17}$$

We next construct the total energy of the system, H.[25] By computing the potential energy in the drop due

[23] Compressional oscillations of a charged drop have been considered by K. Woeste, Z. Physik **133**, 370 (1952).

[24] G. Joos, *Theoretical Physics* (Hafner Publishing Company, New York, 1950), p. 211.

[25] The approach in this section is based on that used by F. Bloch in computing the energy lost to atomic excitation by a charged particle passing through matter [F. Bloch, Z. Physik **81**, 363 (1932)]. The author is very grateful to Professor Bloch for discussions of this problem.

to pressure-volume work, we find

$$H=\int d\mathbf{x}\left[\tfrac{1}{2}\mu\mathbf{v}^2+b\left(\frac{\mu}{\mu_\infty}-\ln\frac{\mu}{\mu_\infty}-1\right)\right]+4\pi\sigma R^2. \quad (8.18)$$

The last term is the energy of the surface. For the equilibrium situation we set $\mathbf{v}=0$ which by Eq. (8.3) implies that μ is constant, and we minimize H with respect to μ. This leads again to Eq. (8.8). Now expanding H about its equilibrium value and keeping terms through second order in η,[26] yields

$$H_1\equiv H-H_0=\frac{b}{2}\left[\int_0^a d\mathbf{x}\left(\frac{\mathbf{v}^2}{s^2}+\eta^2\right)\right.$$
$$\left.-\frac{\lambda}{4\pi a^3}\int_0^a d\mathbf{x}\int_0^a d\mathbf{x}'\,\eta(x)\eta(x')\right]. \quad (8.19)$$

Upon introducing the expansion

$$\begin{aligned}\eta&=\sum_n\left(\frac{2\pi\hbar\omega_n}{3ms^2}\right)^{\frac{1}{2}}F_n(x)\left(a_ne^{-i\omega_nt}+a_n{}^*e^{i\omega_nt}\right),\\ F_n(x)&=\left(\frac{1}{2\pi j_1{}^2(k_na)}\right)^{\frac{1}{2}}\left(\frac{k_na}{(k_na)^2-3\lambda+\lambda^2}\right)^{\frac{1}{2}}j_0(k_nx),\end{aligned} \quad (8.20)$$

and the corresponding equation for v_r ($\mathbf{v}=v_r\hat{r}$ from the spherical symmetry) obtained from Eq. (8.3)

$$v_r=s^2\sum_n\left(\frac{2\pi\hbar\omega_n}{3ms^2}\right)^{\frac{1}{2}}\left(\frac{dF_n}{dx}\right)\frac{1}{i\omega_n}\left(a_ne^{-i\omega_nt}-a_n{}^*e^{i\omega_nt}\right), \quad (8.21)$$

one finds after some manipulation and the use of Eq. (8.17)

$$H_1=\sum_n\hbar\omega_n\tfrac{1}{2}(a_n{}^*a_n+a_na_n{}^*). \quad (8.22)$$

Since this expression is now in canonical form, we can identify H with the Hamiltonian and proceed to quantize the motion. We write

$$\begin{aligned}[a_n,a_{n'}{}^*]&=\delta_{nn'},\\ [a_n,a_{n'}]&=[a_n{}^*,a_{n'}{}^*]=0,\end{aligned} \quad (8.23)$$

and can again interpret these operators as creation and destruction operators. The low-lying spectrum of H_1 is shown in Fig. 2. The values of the frequencies are the solutions to Eq. (8.17).

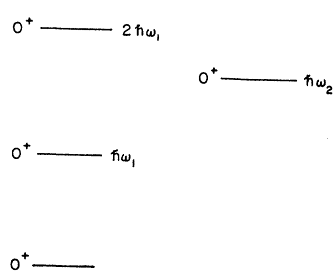

FIG. 2. Low-lying energy spectrum for spherically symmetric compressional oscillations of a quantized liquid drop.

[26] To do this one must write out Eq. (8.10) to second order.

The equation for the charge density is now

$$\rho=\frac{3Z}{4\pi a^3}(1+\eta(x))\theta(R-r). \quad (8.24)$$

and evaluating the monopole transition moment from the ground to the first excited state yields

$$\mathfrak{M}_{00}(0^+\leftarrow 0)=\frac{3Z}{4\pi}\sqrt{2}\left(\frac{2\pi\hbar\omega_1}{3ms^2}\right)^{\frac{1}{2}}\left(\frac{1}{(k_1a)^2-3\lambda+\lambda^2}\right)^{\frac{1}{2}}$$
$$\times\left(\frac{\Delta^2}{k_1{}^2-\Delta^2}\right)\left(j_0(\Delta a)-\lambda\frac{j_1(\Delta a)}{\Delta a}\right). \quad (8.25)$$

We can determine M.E. from

$$\text{M.E.}=\lim_{\Delta\to 0}\left[\frac{-6(4\pi)^{\frac{1}{2}}}{\Delta^2}\mathfrak{M}_{00}(\Delta)\right] \quad (8.26)$$

and find

$$\text{M.E.}=\frac{-18Z}{(2\pi)^{\frac{1}{2}}}\left(\frac{2\pi\hbar\omega_1}{3ms^2}\right)^{\frac{1}{2}}\left(\frac{1}{(k_1a)^2-3\lambda+\lambda^2}\right)^{\frac{1}{2}}$$
$$\times\left(\frac{1-\lambda/3}{k_1{}^2}\right). \quad (8.27)$$

In the case $\lambda\to 0$, the equations simplify, for then the solutions to Eq. (8.17) are

$$k_n=n\pi/a,\quad (\lambda\to 0). \quad (8.28)$$

For example,

$$\text{M.E.}=\frac{-6\sqrt{3}}{\pi^2}(\hbar Za)\left[\frac{1}{(AM)\hbar\omega_1}\right]^{\frac{1}{2}} \quad (8.29)$$

which is the value given by Schiff.[27] We therefore see that the form factor characteristic of this type of transition is

$$\frac{\Delta^2}{k_1{}^2-\Delta^2}\left[j_0(\Delta a)-\lambda\frac{j_1(\Delta a)}{\Delta a}\right] \quad (8.30)$$

or

$$\frac{(\Delta a)^2}{\pi^2-(\Delta a)^2}j_0(\Delta a),\quad (\lambda\to 0). \quad (8.31)$$

It is interesting to ask to what extent such a theory could be applicable to low-lying 0^+ states in nuclei. The nuclear compressibility is usually defined by[28]

$$Ab_N\equiv(R^2d^2E/dR^2)_{R=a} \quad (8.32)$$

where E is the total nuclear energy. b can be related to b_N by the use of $dE=PdV$, and one finds

$$b=(A/12\pi a^3)b_N. \quad (8.33)$$

Using a value of[28]

$$b_N\cong 100\text{ Mev}, \quad (8.34)$$

[27] L. I. Schiff, Phys. Rev. **98**, 1281 (1955).
[28] L. Wilets, Phys. Rev. **101**, 201 (1956).

which appears to be a reasonable average of many estimates of this number,[28] and $K_1 \cong \pi/a$, one finds

$$\hbar\omega_1 \cong \frac{56}{A^{\frac{1}{3}}} \text{ Mev.} \qquad (8.35)$$

which indicates that these levels must lie very high in light nuclei.[29] Schiff has also shown that if ω_1 is made small enough to fit experiment, such a model gives far too large a value for M.E. in the low-lying $0^+ \to 0$ transitions in ${}_6C_6{}^{12}$ and ${}_8O_8{}^{16}$.[27] However, it is known from the observed surface oscillations in nuclei that it is difficult to calculate the energy of the excitation from first principles so that it would be interesting to know whether there are any 0^+ levels with the above properties among the reasonably low-lying states of nuclei.

IX. DISCUSSION

It has been shown that the Hamiltonian for an incompressible oscillating drop interacting with an electron through the Coulomb force can be consistently written to second order in the deformation coordinates and then quantized according to the usual procedures. Some care must be taken in experimentally defining the radius of the ground state but, when this is done, one has a mathematically well-defined theory to work with. This theory has the advantage that both the form factors and γ-ray transition probabilities turn out to be very simple functions of the radius of the drop. The form factors are just simple combinations of spherical Bessel functions, and this is the basis on which the recent electron scattering experiments have been analyzed.[2] The form factors for the allowed transitions (first order in the deformation parameter) go as $j_J(\Delta a)$, while the first forbidden form factors go as

$$\frac{1}{2\Delta a}\left[\frac{d}{d\rho}\rho^2 j_J(\rho)\right]_{\rho=\Delta a} = \frac{J+2}{2}\left[j_J(\Delta a) - \frac{\Delta a}{J+2} j_{J+1}(\Delta a)\right].$$

They start off as $j_J(\Delta a)$ but have their first zero *before* that of $j_J(\Delta a)$. The ratio of cross sections for the allowed transitions is given by

$$\frac{(d\sigma/d\Omega)(J\leftarrow 0)}{(d\sigma/d\Omega)(J'\leftarrow 0)} = \frac{(2J+1)\dfrac{\hbar}{2(B_J C_J)^{\frac{1}{2}}} j_J{}^2(\Delta a)}{(2J'+1)\dfrac{\hbar}{2(B_J C_J)^{\frac{1}{2}}} j_{J'}{}^2(\Delta a)}$$

$$= \frac{2J+1}{2J'+1}\left[\frac{J(J'-1)(J'+2)}{J'(J-1)(J+2)}\right]^{\frac{1}{2}} \frac{j_J{}^2(\Delta a)}{j_{J'}{}^2(\Delta a)}$$

in this model and, if $K_1 \cong K_2 \cong K$, then the cross sections grow as K^2 for fixed momentum transfer. It should be emphasized that all the calculations have been carried out assuming that $Z\alpha \ll 1$ or that Born approximation is applicable. The Coulomb corrections to the initial and final electron wave functions can, however, be readily included for higher Z.[15] Nuclear recoil has also been ignored.

We have shown that the form factors for $0 \to 0^+$ transitions look like $\Delta a j_1(\Delta a)$ if the excitation is to a two-surfon state, or like

$$\frac{\Delta^2}{k_1{}^2 - \Delta^2}\left[j_0(\Delta a) - \lambda \frac{j_1(\Delta a)}{\Delta a}\right]$$

if the excitation corresponds to compressional oscillation. Both of these start off as Δ^2 and therefore give a finite cross section at $\Delta = 0$, since $\sigma_M \sim 1/\Delta^4$, just as the quadrupole transitions do. The first form factor has its first zero at that of $j_1(\Delta a)$ while the first zero of the second lies between that of $j_2(\Delta a)$ and $j_3(\Delta a)$ (for small λ). It is thus somewhat difficult to disentangle experimentally a monopole transition from a quadrapole transition by just looking at the form factor, although they do, of course, have different shapes. In particular, even though the compressional oscillations of a nucleus may be expected to have a rather high excitation energy, it would be interesting to see if any could be experimentally identified by their characteristic form factor (8.30).

We also conclude by a direct calculation that if $\hbar\omega_J \ll \hbar\Delta c \ll 2Mc^2$, then the cross section will be dominated by the Coulomb scattering provided one is not at the very backward angles. This confirms the general estimates of Schiff.[13] The current contribution to transverse photon exchange is small since it is proportional to the time rate of change of the charge density, giving a factor of $\omega_J/\Delta c$ in the transverse electric amplitude. The contribution of the nuclear magnetization, in the simple model made for it here, since it is proportional to the curl of the magnetization, contributes as $\hbar\Delta/2Mc$ to the same amplitude. It is also down by a factor λ/Z, which is the ratio of the number of nuclear magnetons contributing to the transition to the number of charges, Z. This number is usually less than one since magnetic moments (spins) tend to pair off in nuclei while the charges will all add. Since the transverse amplitude does not interfer with the Coulomb amplitude when one sums over nuclear spins, the corrections to the cross section will be the square of the above amplitude.

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[29] The other parameter of interest, λ is given by

$$\lambda = 6(4\pi a^2 \sigma / b_N A) \cong 1.1/A^{\frac{1}{3}}.$$