also those in which the uncertainty in the double-scattering corrections is most important.

ACKNOWLEDGMENTS

I should like to express my sincere appreciation to Professor Paul Martin, who suggested this problem to me, and whose help and encouragement were invaluable. It is also a pleasure to thank Dr. J. N. Palmieri and Edward Prenowitz for making available the Gammel-Thaler scattering amplitudes calculated by their machine program, and to acknowledge the helpful discussions which I have enjoyed with Professor Richard Wilson, Dr. Alan Cromer, Dr. Arthur Kuckes, Dr. E. H. Thorndike, and Dr. J. Lefrancois.

PHYSICAL REVIEW VOLUME 126, NUMBER 2 APRIL 15, 1962

Remarks on the Relativistic Kepler Problem*

L. C. BIEDENHARN
Duke University, Durham, North Carolina

(Received November 16, 1961; revised manuscript received December 18, 1961)

By means of a new representation, the Dirac-Coulomb spherical wave functions are treated in a manner which brings out the close formal similarity between these solutions and the spherical wave solutions for the free-electron problem. The radial functions in the new representation have the same form as the nonrelativistic radial Coulomb functions, but with an irrational orbital "angular momentum," $l(\gamma)$. This representation is utilized to deduce a general recursion relation for radial Coulomb eigenfunctions, and show the existence of the Coulomb helicity operator as a constant of the motion. The advantages and properties of this formulation are discussed briefly.

I. INTRODUCTION AND SUMMARY

THE problem of a Dirac electron in a pure Coulomb field ($\alpha Z/r$), as was first shown by Darwin in 1928, is one of the few problems involving the Dirac equation with external fields, which permits of an "exact" solution—exact, that is, within the restriction to the one-particle theory (unquantized fields) for a point nucleus of large mass.[1] The fundamental importance of this elementary problem, and the necessity for exploring the implications of the solutions need no emphasis.

It is the purpose of the present work to re-examine and rederive the Dirac-Coulomb solutions in a representation not hitherto discussed in the literature. This representation is chosen in order to diagonalize (in Dirac $\boldsymbol{\rho}$ space) two operators, Γ (Sec. III) and $\mathcal{R}$ (Sec. IV), of central importance to the Dirac-Coulomb problem. The first of these operators Γ was introduced by Martin and Glauber[2]; the second operator $\mathcal{R}$ was introduced much earlier by Johnson and Lippmann in a brief note.[3] The operator $\mathcal{R}$, as we shall show, has the significance of a generalized helicity operator, and is a constant of the motion for the relativistic Kepler problem; the operator Γ is more difficult to categorize briefly, but is connected with a generalization of the operator $\rho_3 K$ (where K is Dirac's operator) and is not a constant of the motion.

The importance of the representation S which diagonalizes the operator Γ lies in the fact that it enables us to treat the Dirac-Coulomb eigenfunctions as the precise analogs to the spherical wave solutions of the free (Dirac) electron. Moreover, in this representation the radial wave functions are surprisingly simple, being of precisely the same form as the radial functions in the nonrelativistic Coulomb problem. The transformation to the representation S makes it evident that the integer orbital angular momentum of the free-electron problem, becomes in the relativistic Coulomb problem a noninteger (irrational) "orbital angular momentum." In neither the integer nor the noninteger case is the orbital angular momentum sharp, yet it is conceptually helpful in understanding the problem.

The plan of the present paper is to discuss (Sec. II) the free-electron (plane wave) problem first, employing techniques which permit generalization to the Dirac-Coulomb problem (Sec. III). A basic result of this

* Supported in part by the Army Research Office (Durham) and the National Science Foundation.

[1] The classic treatment is that of Arnold Sommerfeld, *Atombau und Spektrallinien* (Friedrich Vieweg und Sohn, Braunschweig, 1939), Vol. II, Chap. 4, p. 209ff. See also H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Academic Press Inc., New York, 1957); and M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1961).

[2] P. C. Martin and R. J. Glauber, Phys. Rev. **109**, 1307 (1958). These authors were concerned with a specific calculation, and primarily with the discrete spectrum, and did not discuss the representation that diagonalized Γ.

[3] M. H. Johnson and B. A. Lippmann, Phys. Rev. **78**, 329(A) (1950). The operator actually introduced by Johnson and Lippmann differs trivially from $\mathcal{R}$, however. The use of this operator was originally suggested to Martin and Glauber by K. A. Johnson (cf. footnote 6 of reference 2).

treatment of the Dirac-Coulomb problem is the generalized recursion operator for the radial eigenfunctions, Eq. (25). This relation furnishes a concise treatment not only of the Dirac-Coulomb radial functions, but also applies to all special cases, including plane waves.

Section IV utilizes the general radial recursion operator to deduce and define the Coulomb helicity operator $\mathcal{R}$. The final section, V, discusses some advantages and properties of the representation S.

The structure of the solutions to motion in a Coulomb field are not without intrinsic interest, for this structure exhibits invariance properties whose origins lie quite deep. For the nonrelativistic problem, this symmetry is that of rotational invariance in a space of four dimensions.[4] It is well known that this symmetry is spoiled for the relativistic case—yet for the Dirac electron, the symmetry is not completely spoiled and it is of interest to inquire as to the implications of such invariance as still exists. Although such questions motivated the present work, they may be disregarded here; we hope to discuss them more systematically elsewhere.

II. STRUCTURE OF THE DIRAC PLANE WAVE

The plan of the present work is to compare in detail the Dirac-Coulomb problem and the Dirac plane wave in order to show explicitly the similar structure of both. Since it would defeat our purpose simply to refer to the many places where one may find the required answers, we shall develop the desired results from the beginning.

The Dirac plane wave Ψ is a solution to the equation

$$(\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}+\rho_3E/\hbar c+mc/\hbar)\Psi=0. \tag{1}$$

To solve this equation, consider the second-order (iterated) equation, that is,

$$\begin{aligned}\mathcal{O}_-\mathcal{O}_+\Phi&=(\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}+\rho_3E/\hbar c-mc/\hbar)\\&\quad\times(\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}+\rho_3E/\hbar c+mc/\hbar)\Phi\\&=\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{\mathbf{L}^2}{r^2}+k^2\right)\Phi=0.\end{aligned} \tag{2}$$

[In Eq. (2) the usual definition, $k^2=(E^2-m^2c^4)^{\frac{1}{2}}/\hbar c$, has been introduced.] Every solution of (1) is at the same time a solution of (2), but not conversely; we distinguish the solution of (2), therefore, by another symbol Φ.

The Legendre operator $\mathbf{L}^2$ may be put into spinor form using the identity

$$\mathbf{L}^2=(\boldsymbol{\sigma}\cdot\mathbf{L}+1)^2-(\boldsymbol{\sigma}\cdot\mathbf{L}+1). \tag{3}$$

With this identity, it is then clear that the second-order equation, Eq. (2), possesses as constants the commuting operators ρ_i and $\boldsymbol{\sigma}\cdot\mathbf{L}+1$. Alternatively, and conventionally, it is most convenient to classify the solutions to Eq. (2) by the eigenvalues of Dirac's operator $K=\rho_3(\boldsymbol{\sigma}\cdot\mathbf{L}+1)$ and ρ_3.

The spinor solutions to the eigenvalue problem

$$(\boldsymbol{\sigma}\cdot\mathbf{L}+1)\chi_\kappa{}^\mu=-\kappa\chi_\kappa{}^\mu, \tag{4}$$

are the two-component Pauli spinors

$$\chi_\kappa{}^\mu=\sum_\tau\,(l\,\tfrac{1}{2}\,\mu-\tau\,\tau\,|\,j\mu)Y_l{}^{\mu-\tau}(\vartheta,\varphi)\chi_{\frac{1}{2}}{}^\tau, \tag{5}$$

which, in addition to satisfying relation (4), are orthonormal, have sharp j $(=|\kappa|-\frac{1}{2})$ and j_3 $(=\mu)$, and obey the relation

$$\boldsymbol{\sigma}\cdot\hat{r}\chi_\kappa{}^\mu=-\chi_\kappa{}^\mu. \tag{6}$$

According to (3), the orbital angular momentum is given by

$$l(\kappa)=|\kappa|+\tfrac{1}{2}[\operatorname{sgn}(\kappa)-1]. \tag{7}$$

Hence we may write the solutions $\Phi_\pm$ to Eq. (2) which have sharp K and ρ_3 in the compact form:

$$\Phi_+\equiv\Phi_{\rho_3=+}=\begin{pmatrix}j_{l(-\kappa)}(kr)\chi_{-\kappa}{}^\mu\\0\end{pmatrix}, \tag{8a}$$

and

$$\Phi_-\equiv\Phi_{\rho_3=-}=\begin{pmatrix}0\\j_{l(\kappa)}(kr)\chi_\kappa{}^\mu\end{pmatrix}. \tag{8b}$$

The essential point to be utilized next is that solutions to the first-order equation (1) may be obtained from the solutions (8) by exploiting the fact that $\mathcal{O}_-$ and $\mathcal{O}_+$ commute. In fact, the solutions to Eq. (1) (with a given sign of the energy) may be associated in a one-to-one fashion with solutions of Eq. (2), having the same sign of the energy and the appropriate sign of ρ_3.[5] It is conventional to choose for electrons of positive energy (negatons) the solutions of Eq. (2) having $\rho_3=-1$ in order that the "large component" be associated with the lower component in $\boldsymbol{\rho}$ space.

With these conventions one finds that the wave function associated with Φ_- is given by

$$\Psi=(\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}+\rho_3E/\hbar c-mc/\hbar)\Phi_-\equiv\mathcal{O}_-\Phi_-. \tag{9}$$

The operator $\mathcal{O}_-$ in Eq. (9) commutes with K but not ρ_3, and hence causes "mixing" in $\boldsymbol{\rho}$ space. Explicitly this operator has the form:

$$\mathcal{O}_-=\begin{bmatrix}\left(\dfrac{E-mc^2}{\hbar c}\right) & -i\boldsymbol{\sigma}\cdot\hat{r}\left(\dfrac{\partial}{\partial r}+\dfrac{(1+\kappa)}{r}\right)\\ i\boldsymbol{\sigma}\cdot\hat{r}\left(\dfrac{\partial}{\partial r}+\dfrac{(1-\kappa)}{r}\right) & -\left(\dfrac{E+mc^2}{\hbar c}\right)\end{bmatrix}. \tag{10}$$

[4] V. Fock, Z. Physik **98**, 145 (1935); V. Bargmann, Z. Physik **99**, 576 (1936). See also, L. C. Biedenharn, J. Math. Phys. **2**, 433 (1961).

[5] This statement assumes that $(E-mc^2)$ does not vanish, in order that the mapping be one-to-one.

Since $\frac{1}{2}(1+\rho_3)\Psi$ must satisfy Eq. (2) this shows that there exists, according to (10), an operator identity for the spherical Bessel functions. Indeed, one knows that

$$\left(\frac{\partial}{\partial r}+\frac{(1+\kappa)}{r}\right)j_{l(\kappa)}(kr)=k\,\mathrm{sgn}(\kappa)j_{l(-\kappa)}(kr). \quad (11)$$

Hence the solution Ψ to Eq. (1) associated with the solution Φ_- of the second-order equation is given by

$$\Psi=\begin{pmatrix} ik\,\mathrm{sgn}(\kappa)j_{l(-\kappa)}(kr)\chi_{-\kappa}^{\mu} \\ -(\hbar c)^{-1}(E+mc^2)j_{l(\kappa)}(kr)\chi_{\kappa}^{\mu}\end{pmatrix}. \quad (12)$$

These solutions are (aside from normalization) just the usual spherical wave solutions for the free electron; they have sharp K, j_3, and parity. The present derivation is not completely new; in fact, a related derivation and an accompanying notation (based upon Sommerfeld's) were presented by the author several years ago.[6]

III. STRUCTURE OF THE DIRAC-COULOMB WAVE

The Dirac equation for motion in the Coulomb field $(\alpha Z/r)$ differs ostensibly very little from Eq. (1); this equation is

$$[\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}+\rho_3(E/\hbar c-\alpha Z/r)+mc/\hbar]\Psi=0. \quad (13)$$

(αZ positive corresponds to a repulsive field.)

Just as before, the first step is to obtain the corresponding iterated equation. This equation is easily found to be

$$\mathcal{O}_-\mathcal{O}_+\Phi=\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{\mathbf{L}^2}{r^2}-\frac{2\alpha ZE}{\hbar cr}+\frac{(\alpha Z)^2}{r^2}+\frac{i\alpha Z\rho_1\boldsymbol{\sigma}\cdot\hat{r}}{r^2}+k^2\right)\Phi=0. \quad (14)$$

Introducing Dirac's operator K, by means of (3), one finds that (14) takes the form

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{K^2-(\alpha Z)^2}{r^2}-\frac{2\alpha ZE}{\hbar cr}+\frac{1}{r^2}(\rho_3K+i\alpha Z\rho_1\boldsymbol{\sigma}\cdot\hat{r})+k^2\right]\Phi=0. \quad (15)$$

Let us now introduce the operator, Γ, defined by the relation

$$\Gamma=\rho_3K+i\alpha Z\rho_1\boldsymbol{\sigma}\cdot\hat{r}. \quad (16)$$

Since ρ_3K and $\rho_1\ \boldsymbol{\sigma}\cdot\hat{r}$ anticommute, it follows that $\Gamma^2=K^2-\alpha^2Z^2$. This operator already appears in (15). We may therefore write Eq. (15) in the very suggestive form

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{\Gamma(\Gamma-1)}{r^2}-\frac{2k\eta}{r}+k^2\right)\Phi=0. \quad (17)$$

[In Eq. (17) the relativistic analog of Sommerfeld's (nonrelativistic) parameter $\eta_{\mathrm{N.R.}}=Ze^2/\hbar v$ has been introduced by the usual definition: $\eta=\alpha ZE/\hbar ck$.]

Equation (17) suggests that one ought to introduce a noninteger "orbital angular momentum" $l(\gamma)$ in formal analogy to the integer orbital angular momentum $l(\kappa)$ of Eqs. (2) and (7).[7] To accomplish this one first takes Γ to have a sharp value. Let $\Gamma\rightarrow\gamma=\pm|(\kappa^2-\alpha^2Z^2)^{\frac{1}{2}}|$, and then define the "orbital angular momentum" $l(\gamma)$ by the relation $\Gamma^2-\Gamma=l(l+1)$, i.e., the analog to $\mathbf{L}^2=(\boldsymbol{\sigma}\cdot\mathbf{L}+1)^2-(\boldsymbol{\sigma}\cdot\mathbf{L}+1)$ of Eq. (3). The explicit solution is the analog to Eq. (7):

$$l(\gamma)=|\gamma|+\tfrac{1}{2}[\mathrm{sgn}(\gamma)-1]. \quad (18)$$

(To do this without ambiguity, it is necessary to note that Γ commutes with K.)

Thus, exactly as for the plane-wave case, we find that the second-order Dirac-Coulomb equation possesses an additional commuting operator Γ which also commutes with K. In this sense, Γ is to be considered a generalization of the operator ρ_3K of the plane-wave case.

That this is actually so can be seen by diagonalizing the operator Γ. The transformation which accomplishes the diagonalization is readily found to be

$$S=\exp[-\tfrac{1}{2}\rho_2\boldsymbol{\sigma}\cdot\hat{r}\tanh^{-1}(\alpha Z/K)], \quad (19)$$

where

$$S\Gamma S^{-1}=\rho_3K|[1-(\alpha Z/K)^2]^{\frac{1}{2}}|.$$

(In defining the operator S it should be noted that $\rho_2\ \boldsymbol{\sigma}\cdot\hat{r}$ commutes with K.)

In the representation defined by the transformation S, the operator $\Gamma/|\gamma|\mathrm{sgn}K$ is now just the ρ_3 of the plane-wave problem.

The solutions of Eq. (17) are easily obtained, for it is clear that this equation [for $\Gamma^2-\Gamma\rightarrow l(l+1)$] is exactly in the form of the nonrelativistic radial Coulomb problem.[8] The normalized continuum wave functions[9] are thus given by

$$F_{l(\gamma),\eta}(kr)=C_l(\eta)(kr)^{l(\gamma)}e^{-ikr}\times{}_1F_1(l+1-i\eta,\,2l+2,\,2ikr), \quad (21)$$
$$C_l(\eta)=[2^le^{-\pi\eta/2}|\Gamma(l+1+i\eta)|/\Gamma(2l+2)],$$

where $l(\gamma)$ is defined by (18), and $\eta=\alpha ZE/\hbar ck$,

[6] M. E. Rose, L. C. Biedenharn, and G. B. Arfken, Phys. Rev. **85**, 5 (1952); L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. **25**, 729 (1953).

[7] This important observation was also made by Martin and Glauber (reference 2).

[8] *Tables of Coulomb Functions*, National Bureau of Standards, Applied Mathematical Series No. 17 (U. S. Government Printing Office, Washington, D. C., 1952).

[9] It is useful to note that for $\eta=0$, we have the relation $F_{l,\eta=0}(kr)=j_l(kr)$. Moreover, in the limit $c\rightarrow\infty$, the $F_{l,\eta}$ become just $(kr)^{-1}$ times the usual nonrelativistic Coulomb functions as discussed by Breit and collaborators.

$k=(E^2-m^2c^2)^{\frac{1}{2}}/\hbar c$. Asymptotically one has

$$F_{l(\gamma),\eta}(kr)\sim(kr)^{-1}\sin[kr-2\eta\ln kr+\sigma_l(\eta)], \quad (22)$$

with $\sigma_l(\eta)=\arg\Gamma[l(\gamma)+1+i\eta]$.

In the representation in which Γ is diagonal, we find the explicit solutions to the iterated equation to be

$$\Phi_+=\begin{pmatrix}F_{l(-\gamma),\eta}(kr)\chi_{-\kappa}{}^{\mu}\\ 0\end{pmatrix},$$
$$\Phi_-=\begin{pmatrix}0\\ F_{l(\gamma),\eta}(kr)\chi_{\kappa}{}^{\mu}\end{pmatrix}. \quad (23)$$

These solutions have sharp K, with $\operatorname{sgn}\gamma=\operatorname{sgn}\kappa$.

The final step is to generate from solutions of the second-order equation, the corresponding solutions of the first-order equation. Just as earlier, this is accomplished by the operator $\mathcal{O}_-=\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}+\rho_3(E/\hbar c-\alpha Z/r)-mc/\hbar$, but now we must remember to transform the operator $\mathcal{O}_-$ to the new representation. That is

$$\Psi=S\mathcal{O}_-S^{-1}\Phi_-,$$

with

$$S\mathcal{O}_-S^{-1}=\rho_2\boldsymbol{\sigma}\cdot\hat{r}\left[\frac{\partial}{\partial r}+\frac{(1+\gamma\rho_3)}{r}+\frac{(k\eta\rho_3)}{\gamma}\right]+\rho_3\frac{\kappa E}{\hbar c\gamma}-\frac{m}{\hbar c}. \quad (24)$$

Explicitly, in $\boldsymbol{\rho}$ space this operator becomes

$$S\mathcal{O}_-S^{-1}=\begin{bmatrix}(\hbar c)^{-1}(E\kappa/\gamma-mc^2) & -i\boldsymbol{\sigma}\cdot\hat{r}\left(\frac{\partial}{\partial r}+\frac{(1+\gamma)}{r}+\frac{k\eta}{\gamma}\right)\\ i\boldsymbol{\sigma}\cdot\hat{r}\left(\frac{\partial}{\partial r}+\frac{(1-\gamma)}{r}-\frac{k\eta}{\gamma}\right) & -(hc)^{-1}(E\kappa/\gamma+mc^2)\end{bmatrix}. \quad (24a)$$

Since the projected components $\frac{1}{2}(1+\rho_3)\Psi$ must satisfy Eq. (17), we may deduce from Eq. (24) an operator identity for the radial wave functions, $F_{l(\gamma),\eta}(kr)$. This relation is found to be:

$$\left[\frac{k\eta}{\gamma}+\frac{d}{dr}+\frac{(1+\gamma)}{r}\right]F_{l(\gamma),\eta}(kr)=k\operatorname{sgn}(\gamma)\left|\left(1+\frac{\eta^2}{\gamma^2}\right)^{\frac{1}{2}}\right|F_{l(-\gamma),\eta}(kr), \quad (25)$$

where $\gamma=\pm|[\kappa^2-(\alpha Z)^2]^{\frac{1}{2}}|$.

The identity given in Eq. (25) constitutes a complete definition of the Dirac-Coulomb radial functions. For γ negative, we have a "raising operator"; for γ positive, a "lowering operator"; together, of course, they imply Eq. (17). The plane-wave case as well as the nonrelativistic Coulomb case, are all contained in Eq. (25) as special cases.

It remains to detail the actual solutions in this representation. One finds that:

$$\Psi=\begin{pmatrix}ik\operatorname{sgn}(\kappa)F_{l(-\gamma),\eta}(kr)\chi_{-\kappa}{}^{\mu}\\ -(\hbar c)^{-1}(\kappa E/\gamma+mc^2)F_{l(\gamma),\eta}(kr)\chi_{\kappa}{}^{\mu}\end{pmatrix}. \quad (26)$$

The solution given by Eq. (26) has positive energy E and sharp values of K $(=\kappa)$, $j(=|\kappa|-\frac{1}{2})$, and j_3 $(=\mu)$, as well as parity $[=(-)^{l(\kappa)}]$. {Note that the convention $\operatorname{sgn}(\gamma)=\operatorname{sgn}(\kappa)$ has been adopted, i.e., $\gamma=\kappa|[1-(\alpha Z/\kappa)^2|^{\frac{1}{2}}$.} Aside from normalization, this solution is precisely that corresponding to the usual result, transformed, however, to the new representation.

The complete analogy between the Dirac-Coulomb solution given by Eq. (26), and the Dirac plane wave result of Eq. (12) is manifest.

IV. COULOMB HELICITY OPERATOR

The general radial recursion operator as given by Eq. (25) is not only a most concise formulation for the radial Coulomb functions, but has the interesting consequence that it implies an additional constant of the motion for the Dirac-Coulomb problem, and for all its limiting cases as well.

That this is possible may be seen in the following way. The general recursion operator has the property that it changes a radial function appropriate to γ into a radial function appropriate to $-\gamma$. Since the operator $\boldsymbol{\sigma}\cdot\hat{r}$ similarly changes the Pauli central field spinor $\chi_\kappa{}^\mu$ into $\chi_{-\kappa}{}^\mu$, it is then clear that a product of these operators will produce from an eigenfunction belonging to κ an eigenfunction belonging to $-\kappa$. Since, however, the energy is the same for both $\pm\kappa$ (the familiar Dirac-Coulomb degeneracy), we see that this procedure, by construction, furnishes an additional operator which necessarily commutes with the Dirac-Coulomb Hamiltonian.

It is useful to carry out the indicated procedure for the various limiting cases of the general recursion operator, since in this way a clearer view of the method is achieved. Consider first the plane-wave (free particle) limit. We see from Eqs. (25) and (4), that the operator $\mathcal{R}_f$, defined by

$$\mathcal{R}_f\equiv(\hbar k)^{-1}(\boldsymbol{\sigma}\cdot\hat{r})\left(\frac{\partial}{\partial r}+\frac{1+\rho_3K}{r}\right),$$

has by construction the desired properties that

$$\begin{aligned} &\text{(a)} \qquad \mathcal{R}\psi_\kappa{}^\mu=\psi_{-\kappa}{}^\mu,\\ &\text{(b)} \qquad [H,\mathcal{R}]=0,\\ &\text{(c)} \qquad [\mathcal{R},K]_+=0, \end{aligned} \tag{27}$$

and

$$\text{(d)} \qquad [\mathcal{R},\Pi]_+=0,$$

(where Π is the parity operator). The operator $\mathcal{R}_f$, however, is quite obvious, for it is none other than the helicity operator, $\boldsymbol{\sigma}\cdot\hat{p}$.

Consider next the less trivial example of a nonrelativistic spin-$\frac{1}{2}$ particle in a Coulomb field. The eigenfunctions for this case are the two-component spinors given in Eq. (23). We may construct the operator $\mathcal{R}$ now to be

$$\begin{aligned}\mathcal{R}_{\mathrm{N.R.}}&=(\hbar k)^{-1}(1+\eta^2/\kappa^2)^{-\frac{1}{2}}\boldsymbol{\sigma}\cdot\hat{r}[\partial/\partial r+\boldsymbol{\sigma}\cdot\mathbf{L}/r\\&\qquad+\alpha Zmc/\hbar(\boldsymbol{\sigma}\cdot\mathbf{L}+1)]\\&=(\alpha Zmc/\hbar k)[(\boldsymbol{\sigma}\cdot\mathbf{L}+1)^2+\eta^2]^{-\frac{1}{2}}\\&\qquad\times\left(\boldsymbol{\sigma}\cdot\hat{r}+\frac{(\boldsymbol{\sigma}\cdot\mathbf{L}+1)(\boldsymbol{\sigma}\cdot\mathbf{p})}{i\alpha Zmc}\right).\end{aligned} \tag{28}$$

This operator has the same abstract properties, [Eq. (27), (a) through (d)], as the operator $\mathcal{R}_f$ previously.

To put this operator into a more recognizable form, one notes that $[\boldsymbol{\sigma}\cdot\mathbf{L}+1,\ \boldsymbol{\sigma}\cdot\mathbf{p}]_+=0$ and hence

$$(\boldsymbol{\sigma}\cdot\mathbf{L}+1)(\boldsymbol{\sigma}\cdot\mathbf{p})=\tfrac{1}{2}i\boldsymbol{\sigma}\cdot(\mathbf{L}\times\mathbf{p}-\mathbf{p}\times\mathbf{L}).$$

Thus the operator $\mathcal{R}_{\mathrm{N.R.}}$ takes the form

$$\mathcal{R}_{\mathrm{N.R.}}=(\alpha Zmc/\hbar k)[(\boldsymbol{\sigma}\cdot\mathbf{L}+1)^2+\eta^2]^{-\frac{1}{2}}\boldsymbol{\sigma}\cdot\mathbf{A}, \tag{29}$$

where

$$\mathbf{A}=\hat{r}+(2\alpha Zmc)^{-1}(\mathbf{L}\times\mathbf{p}-\mathbf{p}\times\mathbf{L}).$$

The vector $\mathbf{A}$ is the familiar Runge-Lenz vector which is a vector invariant of the nonrelativistic Coulomb field. Thus the pseudoscalar invariant $\mathcal{R}$ for a Pauli particle in a Coulomb field is the spin measured with respect to the vector $\mathbf{A}$.[10]

It is clear now that the operator $\mathcal{R}$ is a generalized helicity operator, for which the name "Coulomb helicity operator" seems reasonable.

It remains to derive the relativistic form of the Coulomb helicity operator, $\mathcal{R}$. From Eq. (25) it is clear that the desired operator is

$$\begin{aligned}\mathcal{R}=(\hbar k)^{-1}(\gamma^2+\eta^2)^{-\frac{1}{2}}\boldsymbol{\sigma}\cdot\hat{r}&\left(\frac{\alpha Z}{\hbar c}\rho_3\tilde{H}+|\gamma|\rho_3\,\mathrm{sgn}(K)\right.\\&\left.\times\left[\frac{\partial}{\partial r}+\frac{1-\rho_3|\gamma|\,\mathrm{sgn}(K)}{r}\right]\right).\end{aligned} \tag{30}$$

(Note that in this equation all quantities, e.g., k, η, that relate to the energy operator refer to the transformed Hamiltonian, $\tilde{H}=SHS^{-1}$.)

By construction, the (relativistic) Coulomb helicity operator $\mathcal{R}$ satisfies Eq. (27) (a) through (d) [where H in Eq. (b) refers to $\tilde{H}$]. The operator $\mathcal{R}$ is thus an additional constant of the motion for the Dirac-Coulomb problem, whose existence is directly related to the twofold degeneracy of states having opposite signs for Dirac's operator K.

The operator $\mathcal{R}$ is rather complicated in appearance, even though its properties are easily seen from its derivation. Although the representation S is particularly adapted to deriving and interpreting the Coulomb helicity operator, this operator assumes a simpler form in the usual representation. Transforming to this representation, one finds

$$\begin{aligned}\mathcal{R}'\equiv S^{-1}\mathcal{R}S&=(\hbar k)^{-1}[K^2-(\alpha Z)^2+\eta^2]^{-\frac{1}{2}}\\&\times\left\{\rho_3\boldsymbol{\sigma}\cdot\hat{r}\left[\alpha ZH+\Gamma\left(\frac{\partial}{\partial r}+\frac{1-\Gamma}{r}\right)\right]\right\}.\end{aligned} \tag{31}$$

The operator in brackets can be simplified by further manipulation and one finds the result

$$(\alpha Z)^{-1}\{\cdots\}=\left[\boldsymbol{\sigma}\cdot\hat{r}+\frac{K\rho_1(H-\rho_3mc^2)}{\alpha Zmc}\right]. \tag{32}$$

The operator given in Eq. (32) is also a constant of the motion and differs only trivially from the operator $\mathcal{R}$. That such an operator exists was first shown by Johnson and Lippmann,[3] who gave the result contained in Eq. (32). By means of the representation S we have succeeded in further interpreting this important result of Johnson and Lippmann.

It might be noted that our derivation indicates that it would be profitable to consider a problem intermediate in complexity between the two-component Pauli particle in a Coulomb field and the Dirac-Coulomb problem. Such a situation results if we apply Eq. (25) to a Dirac particle which in the representation S has integer orbital angular momentum. This case is none other than the Sommerfeld-Maue-Furry approximation, and our construction leads to an alternative way to approach their results. A more detailed discussion belongs elsewhere, however.

V. FURTHER DISCUSSION

A. Some Advantages of the New Representation

The representation defined by the transformation S is designed to make obvious the close analogy between the Dirac-Coulomb spherical solutions and the Dirac plane wave spherical components. Since the transformation S is unity in the absence of the Coulomb field, it is clear that the plane-wave solutions are indeed simply special cases of the more general Dirac-Coulomb

[10] $\mathbf{A}$ behaves as an angular momentum operator and has a measurable direction only in the classical limit. In this limit, for bound states, $\mathbf{A}$ becomes the direction of the semimajor axis and of length equal to the eccentricity of the elliptic orbit.

functions, and the relationship is completely obvious without calculation (in marked contrast to the situation with the customary representation). It is equally clear that the transition to the nonrelativistic limit has been made just as obvious and natural as the usual plane-wave discussion, provided that one works within the new representation. One avoids thereby the necessity for appealing to analysis for various contiguous relations of the confluent hypergeometric functions necessary to perform either of these limits when using the customary representation.

While these advantages of the new representation are principally of utilitarian value, it is nonetheless satisfying that a fundamentally simple problem has an equally simple and readily understood solution. Because of the close relation of the new representation to the nonrelativistic and plane-wave forms, it is tempting to speculate that further understanding of the Pryce-Foldy-Wouthuysen transformation in the presence of the Coulomb field may be obtained. No investigation of this point has been attempted, however.

One can, of course, go from the solutions in the new representation back into the customary form, simply by inverting the transformation S. The usual results then appear as linear combinations of the two radial functions $F_{l(\gamma),\eta}$ and $F_{l(-\gamma),\eta}$. These can be further transformed by means of the contiguous relations into the usual (Sommerfeld) form.

Rather than transform the S-representation solutions into the Sommerfeld form, it is more advantageous to transform the desired operators themselves into the S representation. This is straightforward and need not be discussed here.

Goertzel[11] has shown how to translate the Wigner-Eisenbud dispersion theory into a form appropriate to discussing Dirac electrons (without quantization). It is useful to note that the new representation greatly facilitates the introduction of penetration factors and the like for this problem—in a manner formally analogous to the functions introduced for the nonrelativistic problem. It is to be expected that such a treatment will present methodological advantages.

Feynman and Gell-Mann[12] have proposed that the invariance to transformation by ρ_1 of the iterated equation, Eq. (14), for arbitrary electromagnetic fields might be exploited for beta decay. These authors postulate that projected solutions of the iterated equation having ρ_1 sharp may be taken as fundamental, and this then implies the currently accepted chirally invariant formulation of the Fermi interaction. Since the use of the iterated equation is closely related to the present work, one might expect the S representation to be useful in the Feynman–Gell-Mann formulation of beta decay.

It will be seen at once that the operator Γ also commutes with ρ_1. Rather than taking K sharp, as in Eq. (15), one may instead project out two-component solutions of the iterated Dirac equation which have ρ_1 sharp. These solutions are automatically in the form given by the new representation. Thus the Feynman–Gell-Mann proposal leads directly to a very much simpler formulation for treating beta decay in the presence of an external Coulomb field.[13] A more systematic treatment of this formulation is in preparation.

B. Nature of the Transformation S

The transformation $S=\exp[-\frac{1}{2}\rho_2\boldsymbol{\sigma}\cdot\hat{r}\tanh^{-1}(\alpha Z/K)]$ is defined by the requirement that this transformation is to diagonalize the operator $\Gamma=\rho_3K+i\alpha Z\rho_1\boldsymbol{\sigma}\cdot\hat{r}$. It is not immediately clear just what this diagonalization implies, and the purpose of the present section is to discuss those properties of the transformation which have been found to date.

It is useful to note that Γ is not a Hermitian operator. It is, however, symmetric in the representation where K is diagonal, and therefore a transformation to diagonal form exists. Besides commuting with ρ_1 and K, the operator Γ also commutes with Wigner time reversal $(i\sigma_2K_0)$ and with the parity operator Π $(\rho_3\times\text{reflection})$. As might be expected, Γ does not commute with charge conjugation $(i\gamma_2K_0)$.

It can be easily verified that the operator S does indeed bring Γ to diagonal form. Although this constitutes an adequate demonstration, it is useful to note further that the transformation S is well defined in that: (1) the inverse of Dirac's operator K—since the eigenvalues of K are nonzero—involves no difficulties, and (2) the inverse hyperbolic tangent may be restricted to the single branch where the hyperbolic cosine is positive.

The operator K has the property that $K^2=\mathbf{J}^2+\frac{1}{4}$. We may thus write the series for $\tanh^{-1}(\alpha Z/K)$ in the form

$$\tanh^{-1}(\alpha Z/K)=\left(\frac{\alpha ZK}{\mathbf{J}^2+\frac{1}{4}}\right)1+\frac{1}{3}\left(\frac{(\alpha Z)^2}{\mathbf{J}^2+\frac{1}{4}}\right)+\cdots$$
$$=\alpha ZKf(\mathbf{J}^2). \tag{33}$$

Introducing this into the expression for S, one finds that

$$S=\exp[-\tfrac{1}{2}\alpha Zf(\mathbf{J}^2)\rho_2\rho_3\boldsymbol{\sigma}\cdot\hat{r}(\boldsymbol{\sigma}\cdot\mathbf{L}+1)]$$
$$=\exp[\tfrac{1}{2}\alpha Zf(\mathbf{J}^2)\rho_1(\boldsymbol{\sigma}\cdot\hat{r}\times\mathbf{L}-i\boldsymbol{\sigma}\cdot\hat{r})]. \tag{34}$$

This result indicates that for states of a definite angular momentum magnitude, the transformation S is in the form of a Lorentz transformation, i.e., $\exp(\frac{1}{2}\chi\rho_1\boldsymbol{\sigma}\cdot\hat{n})$. To the extent to which such a conclusion is valid one finds that the angle χ of the transformation is given by $\alpha Zf(\mathbf{J}^2)(\mathbf{J}^2+\frac{1}{4})^{\frac{1}{2}}$, and the direction is given by $\hat{n}=(\hat{r}\times\mathbf{L}-i\hat{r})/(\mathbf{J}^2+\frac{1}{4})^{\frac{1}{2}}$. The fact that the direction is

[11] G. Goertzel, Phys. Rev. **73**, 1463 (1948).

[12] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[13] L. C. Biedenharn, Nuovo cimento **22**, 1097 (1961).

complex, and moreover an operator, indicates that the conclusion is not completely justified. We may, however, avoid these difficulties by considering instead the classical limit of this expression, in order to obtain some insight into the nature of the transformation S.

In this limit, $\mathbf{J}$ ($\approx \mathbf{L}$) is large and becomes an observable vector. The transformation then takes the form

$$S \rightarrow \exp[\tfrac{1}{2}(\alpha Z/L)\rho_1 \boldsymbol{\sigma}\cdot\hat{r}\times\hat{L}]. \tag{35}$$

Thus one sees that the classical limit of the transformation S is a Lorentz transformation along a direction tangent to the electron orbit with the velocity $v=(\alpha Z/L)c$.

In the relativistic Kepler problem the classical orbits are confined to a plane but do not have a fixed orientation, the perihelion advancing at a regular rate. This precession arises from the variation of mass with velocity, combined with angular momentum conservation. [If the problem also involves spin (through a magnetic moment) this will also contribute to an orbit precession.] The orbits, as well as the precession rate, are functions of the angular momentum. In his discussion of the relativistic Kepler problem, Sommerfeld remarks that in a properly chosen rotating coordinate system, the orbits, for a given angular momentum, are ellipses—exactly as in the nonrelativistic Kepler problem. Sommerfeld showed that the polar angle ψ measured in the rotating system must be $[1-(\alpha Z/L)^2]^{\frac{1}{2}}$ times the angle in the fixed system. Expressed in terms of the time required to increase the angles by 2π, the rotating system shows the longer period in the ratio $[1-(\alpha Z/L)^2]^{-\frac{1}{2}}$.

Returning to the Lorentz transformation given in (35), one notes that this transformation causes a time dilation between the fixed and transformed systems, in the Dirac problem, by exactly the same factor: $[1-(v/c)^2]^{-\frac{1}{2}}=[1-(\alpha Z/L)^2]^{-\frac{1}{2}}$. Thus it seems reasonable to conclude that the transformation S—at least in a classical approximation—is effectively a Lorentz transformation for the Dirac problem in some sense equivalent to Sommerfeld's rotating coordinate system.

The existence of this close relationship between the transformation S and Sommerfeld's treatment of the relativistic Kepler problem is helpful in understanding the long-standing puzzle as to reasons why Sommerfeld's quantization of the relativistic Kepler problem (by the Wilson-Sommerfeld quantization rule) gives exactly the correct (Dirac) energy levels.

The transformation S is not unitary, but this is in accord with the fact that Lorentz transformations for the Dirac equation are also not unitary. The proper requirement is that $(\bar{\Psi}\Psi)$ remain invariant, which is indeed the case, since $\rho_3 S^\dagger \rho_3 = S^{-1}$.

It is interesting to note that the operator $\Gamma=(\boldsymbol{\sigma}\cdot\mathbf{L}+1)+i\alpha Z\rho_1\boldsymbol{\sigma}\cdot\hat{r}$ is quite similar in appearance to an angular momentum operator appropriate to rotations in four-dimensional space. In R_4, the generators of the rotations are the vector operators $\mathbf{L}$ and $\mathbf{A}$, with the commutation rules $\mathbf{L}\times\mathbf{L}=i\mathbf{L}$ and $\mathbf{A}\times\mathbf{A}=i\mathbf{L}$. The principal quantum number operator N may be defined as $N=\boldsymbol{\sigma}\cdot\mathbf{L}+1+\boldsymbol{\sigma}\cdot\mathbf{A}$, and from the commutation rules it follows that $\boldsymbol{\sigma}\cdot\mathbf{L}+1$ and $\boldsymbol{\sigma}\cdot\mathbf{A}$ anticommute. The analogy to Γ is thus rather close, but not complete, for a complete analogy would require $\hat{r}$ to possess noncommuting components.

ACKNOWLEDGMENTS

We are indebted to many colleagues for helpful discussions on this work. In particular, we would like to thank Professor Eugen Merzbacher, Professor Frank Tangherlini, Dr. T. A. Griffy, and Dr. Pieter Brussaard for their interest and helpfulness.