Bremsstrahlung Accompanying High-Energy π Meson-Proton Collisions*†

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The radiation from charged π mesons colliding with protons is calculated. This radiation, in the case of high energies and small emission angles, is produced largely externally with respect to the range of the nuclear force, and therefore the process may be calculated with a certain degree of accuracy, in spite of the fact that strong interactions are involved. The results depend on the experimentally determined parameters of elastic pion-proton scattering (i.e., the complex phase shifts). These are obtained from an analysis of certain elastic scattering data, and are used for a numerical calculation of the bremsstrahlung.

I. INTRODUCTION

A CERTAIN class of high-energy inelastic diffraction processes has been investigated some years ago,[1] which comprises phenomena ranging from purely electrodynamic effects[2] to those involving strong interactions (such as radiative nuclear scattering processes of π mesons[3]). All these effects have in common that, due to the predominant emission of the reaction products into the forward direction at high energies, and the corresponding smallness of the longitudinal momentum transfer, the region of space in which the processes take place is large. For the latter cases, this means large compared to the region of strong interaction; perturbation theoretic calculations are thus permissible, with the strongly interacting particles represented by *free* scattered waves as determined from the elastic scattering. We have investigated by this method the process first discussed by Landau and Pomeranchuk,[3] i.e., nuclear scattering of charged pions accompanied by the emission of electromagnetic radiation. High-energy π-meson beams are now available; also, sufficient data on elastic pion-proton scattering at energies of several Bev have been obtained,[4] although their interpretation is not yet complete. We confine ourselves to considering radiative pion collisions with *protons*, in order to be able to use these elastic scattering data. This introduces an inherent inaccuracy in the calculation due to the possibility of recoil of the proton, which however, we believe, may be partly made up for by the fact that the elastic πp scattering, as a fundamental process, can eventually be better understood than the π-nucleus scattering; moreover, the coherent Coulomb scattering, which is important for heavy nuclei, will conveniently be negligible here.[5] The electromagnetic pion vertex is treated in lowest order perturbation theory. Our final results are obtained in the center-of-mass system, and are given in terms of the complex phase shifts of elastic πp scattering. For a numerical evaluation, we have analyzed the π^- scattering data at 6.80 Bev and at 1.44 Bev under the assumption of no spin flip and purely absorptive scattering (imaginary phase shifts), which was possible at both energies but is by no means unique; for that reason, the curves we present for the angular distribution of the bremsstrahlung are to some extent considered as an illustration only.

In Sec. II, the conditions for the radiation to be emitted externally are discussed, i.e., for the perturbation calculation to be applicable. The formula for the cross section on the basis of perturbation theory is obtained in Sec. III. In Sec. IV, relativistic wave functions for the elastically scattered pion-proton system, consisting of plane and scattered waves, are set up in the interaction-free region, in such a manner as to give asymptotically the S matrix of Jacob and Wick.[6] These wave functions are used in Sec. V to explicitly evaluate the differential bremsstrahlung cross section. The phase space is discussed in Sec. VI, and a tentative analysis of the elastic pion-proton scattering data is presented in Sec. VII. Finally, Sec. VIII brings a numerical evaluation of our results for the radiative scattering.

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[1] For a survey of the processes involved, see E. L. Feinberg and I. Ja. Pomeranchuk, Suppl. Nuovo cimento **3**, 652 (1956), and M. L. Good and W. D. Walker, Phys. Rev. **120**, 1855, 1857 (1960).

[2] See, for example, F. J. Dyson and H. Überall, Phys. Rev. **99**, 604 (1955); H. Überall, *ibid.* **103**, 1055 (1956); **107**, 223 (1957).

[3] L. D. Landau and I. Ja. Pomeranchuk, Zhur. Eksp. i Teoret. Fiz. **24**, 505 (1953).

[4] $\pi^- p$ scattering above 1-Bev pion energy: (a) L. M. Eisberg, W. B. Fowler, R. M. Lea, W. D. Shephard, R. P. Shutt, A. M. Thorndike, and W. L. Whittemore, Phys. Rev. **97**, 797 (1955) (1.5 Bev, 110 events); (b) W. D. Walker and J. Crussard, *ibid.* **98**, 1416 (1955) (1.5 Bev, 43 events); (c) M. Chretien, J. Leitner, N. P. Samios, M. Schwartz, and J. Steinberger, *ibid.* **108**, 383 (1957) (1.44 Bev, 1000 events); (d) G. Maenchen, W. B. Fowler, W. M. Powell, and R. W. Wright, *ibid.* **108**, 850 (1957) (5 Bev, 27 events); (e) W. D. Walker, *ibid.* **108**, 872 (1957) (4.8 Bev, 30 events); (f) R. C. Whitten and M. M. Block, *ibid.* **111**, 1676 (1958) (1.85 Bev, 64 events); (g) R. G. Thomas, *ibid.* **120**, 1015 (1960) (5 Bev, 375 events); (h) Wang Kang-Ch'ang, Wang Ts'u-Chieng, Ting Ta-Ts'ao, V. G. Ivanov, Yu. V. Katyshev, E. N. Kladnitskaya, L. A. Kolyukina, Nguyen Dinh Ty, A. V. Nikitin, S. Z. Otvinovskii, M. J. Solov'ev, R. Sosnovskii, and M. D. Shafranov, Zhur. Eksp. i Teoret. Fiz. **38**, 426 (1960) [translation: Soviet Phys.—JETP **11**, 313 (1960)] (6.8 Bev, 213 events); (i) K. W. Lai, L. W. Jones, and M. L. Perl, Phys. Rev. Letters **7**, 125 (1961) (1.5, 2.0 and 2.5 Bev; 2405, 1300, and 1080 events). $\pi^+ p$ scattering: (j) L. O. Roellig and D. A. Glaser, Phys. Rev. **116**, 1001 (1959) (1.1 Bev, 661 events).

[5] A. G. Sitenko, Doklady Akad. Nauk S.S.S.R. **109**, 1119 (1956) [translation: Soviet Phys.—Doklady **1**, 512 (1956)].

[6] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959).

II. OUTLINE OF METHOD AND CONDITIONS OF APPLICABILITY

The process considered is

$$\pi^{\pm}+p\rightarrow\pi^{\pm}+p+\gamma; \tag{1}$$

we shall designate the initial energies and momenta by $E_{\pi0}{}^L$, $\mathbf{p}_{\pi0}{}^L$; $E_{p0}{}^L$, $\mathbf{p}_{p0}{}^L$ in the laboratory system (superscript L; no superscript in the center-of-mass system), and the final energies and momenta by $E_\pi{}^L$, $\mathbf{p}_\pi{}^L$; $E_p{}^L$, $\mathbf{p}_p{}^L$; k^L, $\mathbf{k}^L$. We further call the angle between the emitted photon and the incoming pion, $\vartheta_k{}^L$, the angle between the scattered and the incoming pion, $\vartheta_\pi{}^L$ (all in the laboratory system), and the azimuthal angle between the $\mathbf{p}_\pi{}^L$, $\mathbf{p}_{\pi0}{}^L$ and the $\mathbf{k}^L$, $\mathbf{p}_{\pi0}{}^L$ plane, φ. We use the energy and momentum conservation theorems in the laboratory,

$$\begin{aligned}\mathbf{p}_{\pi0}{}^L&=\mathbf{p}_\pi{}^L+\mathbf{p}_p{}^L+\mathbf{k}^L,\\ E_{\pi0}{}^L+M&=E_\pi{}^L+E_p{}^L+k^L,\end{aligned} \tag{2}$$

and shall consider only high energies of incoming and scattered pions, $(\hbar=c=1)$,

$$E_{\pi0}{}^L\gg m,\quad E_\pi{}^L\gg m \tag{3}$$

(where M, m designate the proton and pion mass, respectively), and further restrict ourselves to small (forward) emission angles,

$$\vartheta_\pi{}^L\lesssim m/E_\pi{}^L\ll 1,\quad \vartheta_k{}^L\lesssim m/k^L\ll 1 \tag{4}$$

(which implies that we will also consider photons of energy $k^L\gtrsim M$ only), because at the high energies involved, the reaction products will be emitted into these small angles anyway. (This statement will be confirmed by our final results.) If the photon is emitted by the pion, then the momentum transfer to the proton (upon which the matrix element for the radiation essentially depends), decomposed into forward and sideways components,

$$p_{p\parallel}{}^L=p_{\pi0}{}^L-p_\pi{}^L\cos\vartheta_\pi{}^L-k^L\cos\vartheta_k{}^L, \tag{5a}$$

$$p_\perp{}^{L2}=p_\pi{}^{L2}\sin^2\vartheta_\pi{}^L+k^{L2}\sin^2\vartheta_k{}^L+2p_\pi{}^Lk^L\sin\vartheta_\pi{}^L\sin\vartheta_k{}^L\cos\varphi, \tag{5b}$$

can be shown by expansion, as a consequence of the assumptions (3) and (4), to have the following order of magnitude:

$$p_{p\perp}{}^L\lesssim m; \tag{6a}$$

$$p_{p\parallel}{}^L\cong T_p{}^L+\Delta, \tag{7a}$$

with

$$\Delta=\tfrac{1}{2}(E_\pi{}^L\vartheta_\pi{}^{L2}+k^L\vartheta_k{}^{L2}+m^2k^L/E_{\pi0}{}^LE_\pi{}^L)$$

and

$$T_p{}^L=E_p{}^L-M;$$

from (7a) follows

$$p_{p\parallel}{}^L\cong\frac{1}{2}\frac{p_{p\perp}{}^{L2}+2M\Delta-\Delta^2}{M-\Delta}, \tag{7b}$$

which together with (3), (4), and (6a) gives

$$p_{p\parallel}{}^L\lesssim m^2/M. \tag{6b}$$

The meaning of (6b) is[1] that the important region in space in which the emission of photons occurs has an extension of $\gtrsim M/m^2$, which may be considered large compared to the range of the strong interactions, m^{-1}. Thus, the emission process is "external," and the free-pion wave function, outside the region of interaction with the proton, will play the most important role in the calculation. This fact will be used in our evaluation of the process (1).

The left-hand sides of (6a,b) may become even smaller, and our method correspondingly more accurate, if $\vartheta_\pi{}^L$, $\vartheta_k{}^L$ are taken smaller, and if φ approaches π. They will, however, not become smaller automatically as $E_{\pi0}{}^L$, $E_\pi{}^L$, or k^L increase [and with the $\vartheta_{\pi,k}{}^L$ always restricted by (4)], which would be the case for scattering by a heavy nucleus that takes up no recoil energy T^L. This is what we referred to in the introduction as "inherent inaccuracy" of our method, and is caused by the proton recoil, or the appearance of $p_{p\perp}{}^L$ in (7b). As a consequence, to obtain the same accuracy of calculation as in a nuclear scattering, the angular range of $\vartheta_{\pi,k}{}^L$ would have to be more restricted.

The argument for the radiation process to take place externally is not applicable to the case that the proton radiates; the momentum transferred to the pion is not small, and the radiation will then be mostly "internal." This seems to form an obstacle to our calculations, as at pion energies of several Bev, the recoiling proton may give off an amount of radiation which is not necessarily small compared to that from the pion. (This can be estimated, e.g., from a calculation of the proton radiation in the center-of-mass system, similar to that for the pion radiation below, which may serve as an order-of-magnitude check.) However, it seems certain that the radiation from the proton, if it is really of this order, will still be rather isotropic in the laboratory system. One can see this in a number of ways. One way is to note that in the laboratory, at the scattering angles and energies considered, Eqs. (3) and (4), the proton will have nonrelativistic velocities (from kinematics), and its radiation therefore will have no particular forward peaking. If the total radiation is of the order of the pion radiation, and the latter one is concentrated into angles $\vartheta_k{}^L\lesssim m/k^L\lesssim m/M$, then the background of proton radiation into the experimentally considered forward angles is $\lesssim\frac{1}{2}(m/M)$ and negligible. Another argument would be to calculate the proton radiation by the "external" method (regarded as an order-of-magnitude estimate). We convinced ourselves in this way that the proton radiation in the laboratory is actually fairly isotropic; for the angles of the pion radiation, the results are $\vartheta_k{}^L\lesssim m/M$, which will be seen again later on. The background of proton radiation can possibly be identified and confirmed experimentally, and subtracted out by extrapolation.

III. BREMSSTRAHLUNG CROSS SECTION

The differential cross section for the emission of radiation by the pion is given by the standard expression

$$d\sigma=v^{-1}(2\pi)^{-8}\int|\langle f|H'|i\rangle|^2\delta(E_p+E_\pi+k-E_{p0}-E_{\pi0})d^3p_pd^3p_\pi d^3k \quad (8)$$

and will be evaluated by us in the center-of-mass (rather, momentum) system of the initial pion and proton (so there are no superscripts L on the momenta and energies); v is the initial pion-proton relative velocity. For the electromagnetic interaction Hamiltonian of the charged pion, treated in lowest order perturbation, we have

$$H'=\int\mathcal{H}'d^3r \quad (9)$$

with the Hamiltonian density

$$\mathcal{H}'=ie[\phi^*\nabla\phi-(\nabla\phi^*)\phi]\cdot\mathbf{A}, \quad (10)$$

and after expanding the pion field ϕ and electromagnetic field $\mathbf{A}$ in plane waves and taking the appropriate creation operators, this becomes

$$H'=-2e\sum_{\mathbf{p}_{\pi0}''\mathbf{p}_\pi''\mathbf{k}''}\left(\frac{2\pi}{k''}\right)^{\frac{1}{2}}\frac{\mathbf{p}_{\pi0}''\cdot\mathbf{e}^*}{(4E_{\pi0}''E_\pi'')^{\frac{1}{2}}}a^\dagger(\mathbf{p}_\pi'')\times a(\mathbf{p}_{\pi0}'')a_e^\dagger(\mathbf{k}'')\delta_{\mathbf{p}_{\pi0}'',\mathbf{p}_\pi''+\mathbf{k}''}; \quad (11)$$

$\mathbf{e}$ is the photon polarization unit vector, and $a^\dagger(\mathbf{p}_\pi'')$ creates a pion with momentum $\mathbf{p}_\pi''$, etc. A point interaction is assumed for the time being.

The initial and final states appearing in (8) are written in the form

$$|i\rangle=\sum_{\mathbf{p}_{p0}'\mathbf{p}_{\pi0}'\lambda_0'}c_{\lambda_0'}^\dagger(\mathbf{p}_{p0}')a^\dagger(\mathbf{p}_{\pi0}')G_{\lambda_0'}^{(+)}(\mathbf{p}_{p0}',\mathbf{p}_{\pi0}')|0\rangle, \quad (12a)$$

$$\langle f|=\sum_{\mathbf{p}_p'\mathbf{p}_\pi'\lambda'}\langle 0|a_e(\mathbf{k})c_{\lambda'}(\mathbf{p}_p')a(\mathbf{p}_\pi')G_{\lambda'}^{(-)\dagger}(\mathbf{p}_p',\mathbf{p}_\pi'), \quad (12b)$$

where, e.g., $c_{\lambda'}(\mathbf{p}_p')$ annihilates a proton of momentum $\mathbf{p}_p'$ and spin component λ'. Momentum distributions of the initial and final pion-proton systems are calculated from

$$G_{\lambda_0'}^{(+)}(\mathbf{p}_{p0}',\mathbf{p}_{\pi0}')=\int u_{\lambda_0'}^\dagger(\mathbf{p}_{p0}')e^{-i(\mathbf{p}_{p0}'\cdot\mathbf{r}_p+\mathbf{p}_{\pi0}'\cdot\mathbf{r}_\pi)}\times\Psi^{(+)}(\mathbf{p}_{p0},\mathbf{p}_{\pi0},\lambda_0)d^3r_pd^3r_\pi, \quad (13a)$$

$$G_{\lambda'}^{(-)\dagger}(\mathbf{p}_p',\mathbf{p}_\pi')=\int\Psi^{(-)\dagger}(\mathbf{p}_p,\mathbf{p}_\pi,\lambda)u_{\lambda'}(\mathbf{p}_p')\times e^{i(\mathbf{p}_p'\cdot\mathbf{r}_p+\mathbf{p}_\pi'\cdot\mathbf{r}_\pi)}d^3r_pd^3r_\pi, \quad (13b)$$

where $u_{\lambda'}(\mathbf{p}_p')$ is the free-proton spinor (properly normalized), and $\Psi^{(+)}(\mathbf{p}_{p0},\mathbf{p}_{\pi0},\lambda)$ the space wave function of the pion-proton system, corresponding to the initial momenta $\mathbf{p}_{p0}$, $\mathbf{p}_{\pi0}$, and proton spin component λ_0 (these will all be contained as parameters in $G_{\lambda_0}^{(+)}$, although not explicitly written), to be evaluated in the interaction-free region, according to the method to be used. Superscripts ($\pm$) indicate out- or ingoing scattered waves appropriate for an initial or final state, respectively.[7] The Ψ's will be worked out in the following section. With (12a,b), the matrix element in (8) then comes out as follows:

$$\langle f|H'|i\rangle=-2e\left(\frac{2\pi}{k}\right)^{\frac{1}{2}}\sum_{\mathbf{p}_p'\mathbf{p}_\pi'\lambda'}\sum_{\mathbf{p}_{p0}'\mathbf{p}_{\pi0}'\lambda_0'}G_{\lambda'}^{(-)\dagger}(\mathbf{p}_p',\mathbf{p}_\pi')\times\frac{\mathbf{p}_{\pi0}'\cdot\mathbf{e}^*}{(4E_{\pi0}'E_\pi')^{\frac{1}{2}}}\delta_{\mathbf{p}_{\pi0}',\mathbf{p}_\pi'+\mathbf{k}'}\delta_{\mathbf{p}_{p0}',\mathbf{p}_p'}\delta_{\lambda_0'\lambda'}\times G_{\lambda_0'}^{(+)}(\mathbf{p}_{p0}',\mathbf{p}_{\pi0}')=\mathbf{e}^*\cdot\mathbf{B}, \quad (14)$$

containing Kronecker δ symbols. The last equality defines $\mathbf{B}$. The cross section is then, after summing over the final proton spin and photon polarization directions and averaging over the initial proton spin:

$$d\sigma=(2v)^{-1}(2\pi)^{-8}\sum_{\lambda_0\lambda}\int(|\mathbf{B}|^2-|\mathbf{B}\cdot\hat{k}|^2)\times\delta(E_p+E_\pi+k-E_{p0}-E_{\pi0})d^3p_pd^3p_\pi d^3k, \quad (15)$$

with the photon propagation unit vector $\hat{k}$.

IV. RELATIVISTIC WAVE FUNCTIONS FOR THE PION-PROTON SYSTEM IN THE INTERACTION-FREE REGION

The relativistic wave functions for the elastic scattering problem of a spin zero and a spin one-half particle can be worked out, not only in the asymptotic region, but in the whole interaction-free region, by a slight extension of the theory of Jacob and Wick,[6] which is based on a description of the spin by helicity quantum numbers (i.e., spin components along the direction of propagation).

In the center-of-momentum system, the two-particle wave function of a plane-wave Dirac particle with propagation vector $\mathbf{p}_p=\mathbf{p}$ in the $+z$ direction and spin component $\lambda=\pm\frac{1}{2}$ along the z axis, and a plane-wave scalar particle with propagation vector $\mathbf{p}_\pi=-\mathbf{p}$ in the $-z$ direction is

$$\Phi(p,0,0,\lambda)=\left(\frac{E_p+M}{2E_p}\right)^{\frac{1}{2}}\begin{pmatrix}1\\ p_p\sigma_z/(E_p+M)\end{pmatrix}\times\chi_{z,\lambda}e^{i(pz-Et)}, \quad (16)$$

using Pauli spinors

$$\chi_{z,\frac{1}{2}}=\begin{pmatrix}1\\0\end{pmatrix},\quad \chi_{z,-\frac{1}{2}}=\begin{pmatrix}0\\1\end{pmatrix},$$

[7] H. A. Bethe, L. Maximon, and F. Low, Phys. Rev. 91, 417 (1953); G. Breit and H. A. Bethe, *ibid.* 93, 888 (1954).

and relative coordinates $\mathbf{r}=\mathbf{r}_p-\mathbf{r}_\pi$, as well as total energy $E=E_p+E_\pi$, where $E_p=(p_p^2+M^2)^{\frac{1}{2}}$, $E_\pi=(p_\pi^2+m^2)^{\frac{1}{2}}$. For $\mathbf{p}$ in the general direction (θ,ϕ), one applies the rotation operator[8] for a rotation through an angle θ around the axis $\hat{z}\times\hat{p}$,

$$R_{\phi,\theta,-\phi}=R_z(\phi)R_y(\theta)R_z(-\phi), \tag{17}$$

where the rotation operator around the ith axis is

$$R_i(\beta)=e^{-i\beta J_i}=e^{-i\beta(L_i+\frac{1}{2}\sigma_i)}, \tag{18}$$

and defines the corresponding wave function as

$$\begin{aligned}\Phi(p,\theta,\phi,\lambda)&=R_{\phi,\theta,-\phi}\Phi(p,0,0,\lambda)\\&=e^{i\lambda\phi}R_{\phi,\theta,0}\Phi(p,0,0,\lambda),\end{aligned} \tag{19}$$

which comes out to be

$$\Phi(p,\theta,\phi,\lambda)=e^{i\lambda\phi}\left(\frac{E_p+M}{2E_p}\right)^{\frac{1}{2}}\begin{pmatrix}1\\ \mathbf{p}_p\cdot\boldsymbol{\sigma}/(E_p+M)\end{pmatrix}\times\chi_{p_p,\lambda}e^{i(\mathbf{p}\cdot\mathbf{r}-Et)}, \tag{20}$$

with Pauli spinors

$$\chi_{p_p,\frac{1}{2}}=\begin{pmatrix}\cos\frac{1}{2}\theta e^{-i\frac{1}{2}\phi}\\ \sin\frac{1}{2}\theta e^{i\frac{1}{2}\phi}\end{pmatrix},\quad \chi_{p_p,-\frac{1}{2}}=\begin{pmatrix}-\sin\frac{1}{2}\theta e^{-i\frac{1}{2}\phi}\\ \cos\frac{1}{2}\theta e^{i\frac{1}{2}\phi}\end{pmatrix}, \tag{21}$$

corresponding to given spin components along the $\mathbf{p}$ direction (helicities). Wave functions which are eigenfunctions of the total angular momentum with eigenvalues J, M can be obtained by Wigner's[9] prescription:

$$\begin{aligned}\Phi(p;JM,\lambda)&=\left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}}\frac{1}{2\pi}\int_0^{2\pi}d\alpha\int_0^{\pi}\sin\beta d\beta\int_0^{2\pi}d\gamma\\&\qquad\times D_{M\lambda}{}^{J*}(\alpha,\beta,\gamma)R_{\alpha,\beta,\gamma}\Phi(p,0,0\,\lambda)\\&=\left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}}\int D_{M\lambda}{}^{J*}(\phi,\theta,-\phi)\Phi(p,\theta,\phi,\lambda)d\Omega,\end{aligned} \tag{22}$$

where $d\Omega=\sin\theta d\theta d\phi$, and where we have used the representations of the rotation group,[8]

$$D_{M\lambda}{}^J(\alpha,\beta,\gamma)=e^{-iM\alpha}d_{M\lambda}{}^J(\beta)e^{-i\lambda\gamma}. \tag{23}$$

The normalization is as follows:

$$\langle\Phi(p,\theta',\phi',\lambda')|\Phi(p,\theta,\phi,\lambda)\rangle=\delta(\cos\theta-\cos\theta')\delta(\phi-\phi')\delta_{\lambda\lambda'}$$
$$\langle\Phi(p;J'M',\lambda')|\Phi(p;JM,\lambda)\rangle=\delta_{JJ'}\delta_{MM'}\delta_{\lambda\lambda'}.$$

Wave functions of the type (22) have to be used, as the S matrix is characterized by the total angular momentum J (and also by p and by initial and final helicities); it can be written for the case of pion-proton scattering[10]

$$\langle\lambda'|S^J(p)|\lambda\rangle=\tfrac{1}{2}[e^{2i\delta_{l+}}+(-1)^{1-\lambda-\lambda'}e^{2i\delta_{(l+1)-}}]. \tag{24}$$

Here, the scattering phase shifts $\delta_{l\pm}$ were introduced, which correspond to partial waves with total angular momentum $J=l\pm\frac{1}{2}$ and orbital parity $(-1)^l$. We claim then that the free wave function of the scattering pion-proton system in the center-of-momentum frame, with the protons coming in along the $+z$ axis, can be represented as:

$$\begin{aligned}\Psi^{(+)}(p,0,0,\lambda)&=\Phi(p,0,0,\lambda)\\&+\sum_{J\lambda'}\Phi_{\text{out}}(p;J\lambda,\lambda')\left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}}\langle\lambda'|S^J(p)-1|\lambda\rangle.\end{aligned} \tag{25}$$

The proof consists in noting that (a) this expression is a superposition of plane waves, and thus a solution of the corresponding free-particle equation; (b) the outgoing scattered wave Φ_{out} [by which we mean: Write (22) in terms of plane waves (20), use the Rayleigh expansion

$$e^{i\mathbf{p}\cdot\mathbf{r}}=4\pi\sum_{LM}i^Lj_L(pr)Y_{LM}{}^*(\hat{p})Y_{LM}(\hat{r}),$$

where $\hat{p}\equiv\mathbf{p}/p$, etc., and replace $j_L(pr)$ by its outgoing-wave part $\frac{1}{2}h_L{}^{(1)}(pr)$] is correctly decomposed into JM eigenstates, with S-matrix elements as amplitudes; and (c) in the asymptotic region, (25) leads to the correct cross section of Jacob and Wick[11]; upon taking the asymptotic form

$$h_L{}^{(1)}(pr)\longrightarrow\frac{(-i)^L}{ipr}e^{ipr},$$

the scattered-wave part of (25) goes over into

$$\frac{1}{ip}\sum_J(J+\tfrac{1}{2})\langle\lambda'|S^J(p)-1|\lambda\rangle e^{i(\lambda-\lambda')\phi_r}d_{\lambda\lambda'}{}^J(\theta_r)\times\left[e^{i\lambda'\phi_r}\left(\frac{E_p+M}{2E_p}\right)^{\frac{1}{2}}\begin{pmatrix}1\\ p\hat{r}\cdot\boldsymbol{\sigma}/(E_p+M)\end{pmatrix}\chi_{r,\lambda'}\frac{e^{ipr}}{r}\right],$$

(with θ_r, ϕ_r the direction of $\mathbf{r}$); of this, the factor in square brackets is $1/r$ times a plane wave going out in the direction $\hat{r}$, and its coefficient is the scattering amplitude of Wick.[11]

Again, the free wave function for protons moving in along a general direction θ, ϕ can be obtained by applying (17) to (25):

$$\Psi^{(+)}(p,\theta,\phi,\lambda)=R_{\phi,\theta,-\phi}\Psi^{(+)}(p,0,0,\lambda). \tag{26}$$

This form will be used in our evaluation of the G's in (12). We do not really need an explicit form of $\Psi^{(+)}$, but we shall write it down for the sake of general

[8] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

[9] E. P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Edwards Brothers, Inc., Ann Arbor, Michigan, 1954), Chap. XII, Eq. (2).

[10] See reference 6, Eq. (57).

[11] See reference 6, Eq. (31).

interest:

$$\Psi^{(+)}(p,\theta,\phi,\lambda)$$
$$=\left(\frac{E_p+M}{2E_p}\right)^{\frac{1}{2}}\left\{e^{i\lambda\phi}\begin{pmatrix}1\\ \boldsymbol{\sigma}\cdot\mathbf{p}/(E_p+M)\end{pmatrix}\chi_{p,\lambda}e^{i\mathbf{p}\cdot\mathbf{r}}\right.$$
$$+\pi^{\frac{1}{2}}\sum_{l\lambda'}i^l e^{i(\lambda-\lambda')\phi}\Bigg[(l+1)^{\frac{1}{2}}\Delta_l^{(+)}d_{\lambda'\lambda}{}^{j=l+\frac{1}{2}}(\theta)$$
$$\times\begin{pmatrix}h_l^{(1)}(pr)\mathcal{Y}_{j\lambda'}{}^l(\hat{r})\\ -ip/(E_p+M)h_{l+1}{}^{(1)}\mathcal{Y}_{j\lambda'}{}^{l+1}\end{pmatrix}$$
$$+(-1)^{\lambda+\frac{1}{2}}l^{\frac{1}{2}}\Delta_l^{(-)}d_{\lambda'\lambda}{}^{j=l-\frac{1}{2}}(\theta)$$
$$\left.\times\begin{pmatrix}h_l^{(1)}(pr)\mathcal{Y}_{j\lambda'}{}^l(\hat{r})\\ ip/(E_p+M)h_{l-1}{}^{(1)}\mathcal{Y}_{j\lambda'}{}^{l-1}\end{pmatrix}\Bigg]\right\}e^{-iEt};\quad(27)$$

here

$$\Delta_l^{(\pm)}=e^{2i\delta_{l\pm}}-1=2ie^{i\delta_{l\pm}}\sin\delta_{l\pm},\qquad(28)$$

and $\mathbf{p}=\mathbf{p}_p=-\mathbf{p}_\pi$, $E_p=(p^2+M^2)^{\frac{1}{2}}$. Note that the $\mathbf{p}_p$, $\mathbf{p}_\pi$ used in this section are not identical with our final momenta, e.g., in (8), but are momenta in the center-of-mass system of πp scattering. We have used the relation

$$R_{\alpha,\beta,\gamma}\mathcal{Y}_{jm}=\sum_{m'}D_{m'm}{}^j(\alpha,\beta,\gamma)\mathcal{Y}_{jm'},$$

and our angular momentum eigenfunctions are

$$\mathcal{Y}_{jm}{}^l(\hat{r})=\sum_{m'm''}(l\tfrac{1}{2}m'm''|l\tfrac{1}{2}jm)Y_{lm'}(\theta_r,\phi_r)\chi_{z,m''}.$$

V. EVALUATION OF THE CROSS SECTION

To obtain the momentum wave functions G of (13), to be used in (14) for calculating the cross section (15), we shall take wave functions of the type (26). First we find $G^{(+)}$, Eq. (13a). We note that the free-proton spinor $u_{\lambda_0'}(\mathbf{p}_{p0}')$ should be defined in the same way as our rotated plane-wave function (20) is, i.e., should include a phase factor $\exp(i\lambda_0'\phi_0')$, where $\mathbf{p}_{p0}'=(p_{p0}',\theta_0',\phi_0')$. Also, $\mathbf{p}_{p0}=-\mathbf{p}_{\pi0}=\mathbf{p}_0=(p_0,\theta_0,\phi_0)$, as our calculation proceeds in the center-of-momentum system. After that, the evaluation is straightforward. One uses $\Psi^{(+)}(\mathbf{p}_{p0},\mathbf{p}_{\pi0},\lambda_0)$ as obtained from (26) and (25). Then, the plane-wave term leads to Kronecker deltas, and in the scattered-wave term, one transforms the integration into one over the relative coordinates $\mathbf{r}=\mathbf{r}_p-\mathbf{r}_\pi$ and the generalized "center-of-mass coordinates"

$$\mathbf{R}=(\alpha_p\mathbf{r}_p+\alpha_\pi\mathbf{r}_\pi)/(\alpha_p+\alpha_\pi),\qquad(29)$$

where the choice of α_p, α_π is a matter of convenience, except that $\alpha_p+\alpha_\pi\neq0$. One also defines $\mathbf{p}_0'$, $\mathbf{P}_0'$ by $\mathbf{p}_{p0}'\cdot\mathbf{r}_p+\mathbf{p}_{\pi0}'\cdot\mathbf{r}_\pi=\mathbf{p}_0'\cdot\mathbf{r}+\mathbf{P}_0'\cdot\mathbf{R}$, and gets, of course, a factor $\delta_{\mathbf{P}_0',0}\equiv\delta_{\mathbf{p}_{p0}',-\mathbf{p}_{\pi0}'}$. As outgoing waves Φ_{out} occur, and therefore factors $h_L^{(1)}(p_{p0}r)$, we have the following radial integrals

$$\int_0^\infty r^2j_l(p_{p0}'r)h_l^{(1)}(p_{p0}r)dr=\frac{i}{p_{p0}^2-p_{p0}'^2}\frac{p_{p0}'^l}{p_{p0}^{l+1}},\qquad(30)$$

using a formula[12]

$$(\alpha^2-\beta^2)\int xZ_p(\alpha x)\bar{Z}_p(\beta x)dx$$
$$=\beta xZ_p(\alpha x)\bar{Z}_{p-1}(\beta x)-\alpha xZ_{p-1}(\alpha x)\bar{Z}_p(\beta x)$$

with Z, $\bar{Z}$ any two cylinder functions. We also made the integral, which oscillates for $r\to\infty$, convergent by replacing $\beta\to\beta+i\epsilon$, $\epsilon\to+0$. The following two relations have also been used:

$$e^{i\lambda_0\phi_0}d_{\lambda_0\lambda_0'}{}^J(\theta_0)=\left(\frac{8\pi}{2J+1}\right)^{\frac{1}{2}}\chi_{p_{p0},\lambda_0'}{}^\dagger\mathcal{Y}_{J,\lambda_0}{}^{l=J-\frac{1}{2}}(\hat{p}_{p0}),\qquad(31)$$

which can be derived with the help of Eq. (A7) of reference 6, and

$$2\chi_{p_{p0},\lambda_0}\chi_{p_{p0},\lambda_0}{}^\dagger-1=(-1)^{\lambda_0-\frac{1}{2}}\hat{p}_{p0}\cdot\boldsymbol{\sigma}.\qquad(32)$$

As the result, we find

$$G_{\lambda_0'}{}^{(+)}(\mathbf{p}_{p0}',\mathbf{p}_{\pi0}')$$
$$=\delta_{\mathbf{p}_{p0}'+\mathbf{p}_{\pi0}',0}\{\delta_{\mathbf{p}_{p0}',\mathbf{p}_{p0}}\delta_{\lambda_0'\lambda_0}+f_{\lambda_0'\lambda_0}(\mathbf{p}_{p0}',\mathbf{p}_{p0})\},\qquad(33)$$

with

$$f_{\lambda_0'\lambda_0}(\mathbf{p}_{p0}',\mathbf{p}_{p0})=e^{-i\lambda_0'\phi_0'}\left(\frac{E_{p0}+M}{2E_{p0}}\right)^{\frac{1}{2}}\left(\frac{E_{p0}'+M}{2E_{p0}'}\right)^{\frac{1}{2}}4\pi^{\frac{3}{2}}i$$
$$\times\frac{E_{p0}+E_{p0}'}{p_{p0}^2-p_{p0}'^2}\sum_{l_0\lambda_0''}e^{i(\lambda_0-\lambda_0'')\phi_0}\frac{p_{p0}'^{l_0}}{p_{p0}^{l_0+1}}\chi_{p_{p0}',\lambda_0'}{}^\dagger$$
$$\times\Bigg[\frac{(l_0+1)^{\frac{1}{2}}\Delta_{l_0}{}^{(+)}}{E_{p0}+M}d_{\lambda_0''\lambda_0}{}^{j_0=l_0+\frac{1}{2}}(\theta_0)\mathcal{Y}_{j_0\lambda_0''}{}^{l_0}(\hat{p}_{p0}')$$
$$+(-1)^{\lambda_0+\frac{1}{2}}\frac{l_0^{\frac{1}{2}}\Delta_{l_0}{}^{(-)}}{E_{p0}'+M}d_{\lambda_0''\lambda_0}{}^{j_0=l_0-\frac{1}{2}}(\theta_0)\mathcal{Y}_{j_0\lambda_0''}{}^{l_0}(\hat{p}_{p0}')\Bigg],\qquad(34)$$

with $E_{p0}'=(p_{p0}'^2+M^2)^{\frac{1}{2}}$. The first term in brackets in (33) stems, of course, from the plane wave in (25).

In the calculation of $G^{(-)}$, Eq. (13b), two points have to be noticed. The first one is that ingoing scattered waves must be used. The corresponding wave function can be obtained from $\Psi^{(+)}$, in the case of the two-particle system propagating along the z direction, by applying the time reversal operator[13] T,

$$T\psi(t)=i\begin{pmatrix}\sigma_y & 0\\ 0 & \sigma_y\end{pmatrix}\psi^*(-t),\qquad(35)$$

[12] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945), 4th ed., p. 146.

[13] See, e.g., G. C. Wick, Ann. Rev. Nuclear Sci. 8, 1 (1958).

which transforms outgoing into ingoing waves but reverses the initial momentum, and $R_y(\pi)$, which restores the initial momentum direction:

$$\Psi^{(-)}(p,0,0,\lambda)=TR_y(\pi)\Psi^{(+)}(p,0,0,\lambda). \tag{36}$$

The only effect of this is to change the sign of the phases δ_l, and to replace $h_l^{(1)}$ by $h_l^{(2)}\equiv h_l^{(1)*}$, so we have

$$\Psi^{(-)}(p,0,0,\lambda)=\Phi(p,0,0,\lambda)+\sum_{J\lambda'}\Phi_{\text{in}}(p;J\lambda,\lambda')\left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}}\langle\lambda'|S^J(p)-1|\lambda\rangle^*. \tag{37}$$

The second point to be noted is that although for the initial state of our process (1) in the center-of-momentum system, (12a), the two-body wave functions to be used in $G^{(+)}$ are also in the center-of-momentum system of the elastic scattering, this is not the case for the final state, (12b); here, momentum conservation in the bremsstrahlung process gives

$$\mathbf{p}_p+\mathbf{p}_\pi+\mathbf{k}=0, \tag{38}$$

whereas the wave functions (25) or (26) would correspond to $\mathbf{p}_p+\mathbf{p}_\pi=0$. The correct wave function to be used instead of (25) or (26) is thus the one obtained by a Lorentz transformation such that (38) is satisfied. We designate the energies and momenta in the center-of-momentum system of the *scattering* by E_{p1}, $\mathbf{p}_{p1}=-\mathbf{p}_{\pi1}$, and, in order to avoid complications from rotation matrices, we shall put $\mathbf{p}_{p1}\|z$, i.e., fix the direction of the z axis, which had been left arbitrary in the calculation of $G^{(+)}$. This is a perfectly legal thing to do, as, in the final results, everything can again be expressed by scalar vector products, which eliminates any preferential axis. We can furthermore let the $\mathbf{p}_p$, $\mathbf{p}_\pi$ plane coincide with the xz plane, which will then contain $\mathbf{k}$ also, with $-\mathbf{k}=\mathbf{P}\equiv\mathbf{p}_p+\mathbf{p}_\pi$. The wave function to be used then in the calculation of $G^{(-)}$ is that obtained from (37) by a Lorentz transformation.

The effect of this transformation on a space wave function is

$$\psi(\mathbf{r}_p,\mathbf{r}_\pi)\longrightarrow\mathfrak{L}\psi(\mathbf{r}_p,\mathbf{r}_\pi)=\psi(\mathfrak{L}\mathbf{r}_p,\mathfrak{L}\mathbf{r}_\pi)\equiv\psi'(\mathbf{r}_p,\mathbf{r}_\pi), \tag{39}$$

where

$$\left.\begin{aligned}\mathfrak{L}\mathbf{r}_p&=\mathbf{r}_p+(\boldsymbol{\beta}/\beta)[\mathbf{r}_p\cdot(\boldsymbol{\beta}/\beta)(\gamma-1)-\beta\gamma t],\\ \mathfrak{L}t&=\gamma(t-\boldsymbol{\beta}\cdot\mathbf{r}_p),\end{aligned}\right. \tag{40a}$$

$$\left.\begin{aligned}\mathfrak{L}\mathbf{r}_\pi&=\mathbf{r}_\pi+(\boldsymbol{\beta}/\beta)[\mathbf{r}_\pi\cdot(\boldsymbol{\beta}/\beta)(\gamma-1)-\beta\gamma t],\\ \mathfrak{L}t&=\gamma(t-\boldsymbol{\beta}\cdot\mathbf{r}_\pi),\end{aligned}\right. \tag{40b}$$

$\gamma=(1-\beta^2)^{-\frac{1}{2}}$. The apparent difficulty of obtaining two different times for the two particles can be avoided in this case by applying the Lorentz transformation of time in such a way as if there were only one particle, the "generalized center-of-mass," with coordinate

$$\mathbf{R}=(E_{p1}\mathbf{r}_p+E_{\pi1}\mathbf{r}_\pi)/(E_{p1}+E_{\pi1}), \tag{41}$$

[i.e., a special choice of α_p, α_π in (29)], just as we consider only a single time for both particles; we then take

$$\mathfrak{L}t=\gamma(t-\boldsymbol{\beta}\cdot\mathbf{R}). \tag{40c}$$

The justification of this is mainly the following: if one Lorentz-transforms the coordinates of a two-particle plane wave

$$e^{i(\mathbf{p}_{p1}\cdot\mathbf{r}_p+\mathbf{p}_{\pi1}\cdot\mathbf{r}_\pi-E_1t)} \tag{42}$$

with $E_1=E_{p1}+E_{\pi1}$, one obtains the same result whether $\mathfrak{L}t$ is taken as in (40a,b) or as in (40c), namely another plane wave with inversely transformed momenta and energies,

$$\begin{aligned}\mathbf{p}_p&=\mathfrak{L}^{-1}\mathbf{p}_{p1}\\&=\mathbf{p}_{p1}+(\boldsymbol{\beta}/\beta)[\mathbf{p}_{p1}\cdot(\boldsymbol{\beta}/\beta)(\gamma-1)+\beta\gamma E_{p1}],\end{aligned} \tag{43a}$$

$$E_p=\mathfrak{L}^{-1}E_{p1}=\gamma(E_{p1}+\boldsymbol{\beta}\cdot\mathbf{p}_{p1}),$$

$$\begin{aligned}\mathbf{p}_\pi&=\mathfrak{L}^{-1}\mathbf{p}_{\pi1}\\&=\mathbf{p}_{\pi1}+(\boldsymbol{\beta}/\beta)[\mathbf{p}_{\pi1}\cdot(\boldsymbol{\beta}/\beta)(\gamma-1)+\beta\gamma E_{\pi1}],\end{aligned} \tag{43b}$$

$$E_\pi=\mathfrak{L}^{-1}E_{\pi1}=\gamma(E_{\pi1}+\boldsymbol{\beta}\cdot\mathbf{p}_{\pi1}),$$

or again

$$E=\mathfrak{L}^{-1}E_1=\gamma[E_1+\boldsymbol{\beta}\cdot(\mathbf{p}_{p1}+\mathbf{p}_{\pi1})]. \tag{43c}$$

In our case, the velocity of the transformation $\boldsymbol{\beta}$ is found from $\mathbf{p}_{p1}+\mathbf{p}_{\pi1}=0$, and is

$$\boldsymbol{\beta}=\mathbf{P}/E, \tag{44}$$

with $E=E_p+E_\pi$. So much for the transformation of the plane-wave exponential in (37); there is, further, a transformation of the proton spinor. If the Lorentz transformation is written as

$$(\mathfrak{L}x)_\mu=a_{\mu\nu}x_\nu, \tag{45}$$

then the spinor transforms according to

$$\psi'(x)=S_L(a)\psi(a^{-1}x), \tag{46}$$

where

$$a=(a_{\mu\nu}),\quad \gamma_4S_L^\dagger\gamma_4S_L=1,$$

and the matrix S_L is found from the condition that the Dirac equation should be invariant; this gives[14]

$$a_{\mu\nu}\gamma_\mu=S_L\gamma_\nu S_L^{-1}. \tag{47}$$

One can solve this equation and determine S_L; it comes out as follows[15]:

$$\begin{aligned}S_L&=\exp[-i\tfrac{1}{2}\chi(\boldsymbol{\beta}/\beta)\cdot\boldsymbol{\gamma}\gamma_4]\\&\equiv\cosh\tfrac{1}{2}\chi-i(\boldsymbol{\beta}/\beta)\cdot\boldsymbol{\gamma}\gamma_4\sinh\tfrac{1}{2}\chi;\end{aligned} \tag{48}$$

the relation between β and χ is given by

$$\cosh\chi=\gamma,\quad \sinh\chi=\beta\gamma. \tag{49}$$

The wave function to be Lorentz transformed is (37),

[14] W. Pauli, *Die allgemeinen Prinzipien der Wellenmechanik* (J. W. Edwards, Ann Arbor, Michigan, 1947).

[15] H. A. Tolhoek and F. W. Lipps, Johns Hopkins University Report PhLU m 5069, 1953 (unpublished); in this report, a misprint in the corresponding special equation of Pauli's[14] article is pointed out.

or if written out in some detail using (24):

$$\Psi^{(-)}(p_{p1},0,0,\lambda)$$
$$=\left(\frac{E_{p1}+M}{2E_{p1}}\right)^{\frac{1}{2}}\Bigg\{\begin{pmatrix}1\\ \mathbf{p}_{p1}\cdot\boldsymbol{\sigma}/(E_{p1}+M)\end{pmatrix}\chi_{p_{p1},\lambda}e^{i\mathbf{p}_{p1}\cdot\mathbf{r}}$$
$$+\sum_{J\lambda'}\frac{2J+1}{4\pi}\{\tfrac{1}{2}[e^{-2i\delta_{l+}}+(-1)^{1-\lambda-\lambda'}e^{-2i\delta_{(l+1)-}}]-\delta_{\lambda\lambda'}\}$$
$$\times\int e^{i(\lambda-\lambda')\phi'}d_{\lambda\lambda'}{}^{J}(\theta)e^{i\lambda'\phi'}\begin{pmatrix}1\\ \mathbf{p}_{p1}'\cdot\boldsymbol{\sigma}/(E_{p1}+M)\end{pmatrix}$$
$$\times\chi_{p_{p1}',\lambda'}e^{i\mathbf{p}_{p1}'\cdot\mathbf{r}}d\Omega_{-}'\Bigg\}e^{-iE_1t},\quad(50)$$

with $\mathbf{p}_{p1}=(p_{p1},0,0)$ and $\mathbf{p}_{p1}'=(p_{p1},\theta',\phi')$, and $d\Omega_{-}'$ meaning that only the ingoing-wave parts of $\exp(i\mathbf{p}_{p1}'\cdot\mathbf{r})$ should be taken. Lorentz transformation of this consists in application of S_L and in transforming the coordinates according to (40); (43) gives $p_p'=\mathcal{L}^{-1}p_{p1}'$, etc. The transformed $\Psi^{(-)}$ is to be used in (13b), implying an integration over $d^3r_pd^3r_\pi=d^3rd^3R$, with the $\mathbf{R}$ of (41). Note, however, that the outgoing wave parts in the integral of (50), defined such as to give asymptotically $(1/r)\exp(ip_{p1}'r)$, must be selected in the center-of-momentum system of the scattering, and the Lorentz transformation removes us from this system. We help ourselves by performing a three-dimensional transformation of the relative coordinate (integration variable!) back into the original relative coordinate system before the Lorentz transformation, $\mathbf{r}\rightarrow\mathbf{r}'$, by

$$\mathbf{r}'=\mathbf{r}+\boldsymbol{\beta}\mathbf{r}\cdot\boldsymbol{\beta}(\gamma-1)/\beta^2.\quad(51)$$

This gives a Jacobian $d^3r=(1/\gamma)d^3r'$, and, with the choice of $\mathbf{R}$ in (41), leads us back to the exponential $\exp(i\mathbf{p}_{p1}'\cdot\mathbf{r}')$, as desired. What is, however, the transformation of

$$e^{i(\mathbf{p}_p'\cdot\mathbf{r}_p+\mathbf{p}_\pi'\cdot\mathbf{r}_\pi)}\quad(52)$$

in (13b) under (51)? As far as the relative coordinate is concerned, we find

$$\mathbf{p}'\cdot\mathbf{r}=\mathbf{p}_1'\cdot\mathbf{r}',\quad(53)$$

where according to (41),

$$\mathbf{p}'=(E_{\pi1}\mathbf{p}_p'-E_{p1}\mathbf{p}_\pi')/E_1,\quad(54)$$

and according to (51),

$$\mathbf{p}_1'=\mathbf{p}'+\boldsymbol{\beta}(\gamma^{-1}-1)\mathbf{p}'\cdot\boldsymbol{\beta}/\beta^2.\quad(55)$$

Finally, after performing the integration d^3r' in (13b), we obtain

$$G_{\lambda'}{}^{(-)\dagger}(\mathbf{p}_p',\mathbf{p}_\pi')=(E_p/E_{p1})^{\frac{1}{2}}\delta_{\mathbf{p}_p'+\mathbf{p}_\pi',\mathbf{P}}$$
$$\times\{\delta_{\mathbf{p}_p',\mathbf{p}_p}\bar{\chi}_{p_p,\lambda}{}^{\dagger}\chi_{p_p,\lambda'}+g_{\lambda'\lambda}(\mathbf{p}_p',\mathbf{p}_p)\}\quad(56)$$

with the Lorentz-transformed Pauli spinor

$$\bar{\chi}_{p_p,\lambda}=[(E_p+M)(E_{p1}+M)]^{-\frac{1}{2}}(a-i\sigma_2b)\chi_{p_{p1},\lambda},\quad(57a)$$

$$\left.\begin{aligned}a&=(E_p+M)\cosh\tfrac{1}{2}\chi-\mathbf{p}_p\cdot(\boldsymbol{\beta}/\beta)\sinh\tfrac{1}{2}\chi,\\ b&=[p_p{}^2-(\mathbf{p}_p\cdot\boldsymbol{\beta}/\beta)^2]^{\frac{1}{2}}\sinh\tfrac{1}{2}\chi,\end{aligned}\right\}\quad(57b)$$

and with

$$g_{\lambda'\lambda}(\mathbf{p}_p',\mathbf{p}_p)=e^{i\lambda'\phi'}\left(\frac{E_p'+M}{2E_p}\right)^{\frac{1}{2}}\left(\frac{E_{p1}+M}{2E_p}\right)^{\frac{1}{2}}\frac{i}{\gamma}\frac{4\pi^{\frac{1}{2}}}{p_{p1}{}^2-p_1'^2}$$
$$\times\sum_l\frac{p_1'^{\,l}}{p_{p1}{}^{l+1}}[(l+1)^{\frac{1}{2}}\Delta_l{}^{(+)}\mathcal{Y}_{j=l+\frac{1}{2},\lambda}{}^{l\dagger}(\hat{p}_1')\bar{\chi}_{p_p',\lambda'}{}^{(+)}$$
$$+l^{\frac{1}{2}}\Delta_l{}^{(-)}(-1)^{\lambda+\frac{1}{2}}\mathcal{Y}_{j=l-\frac{1}{2},\lambda}{}^{l\dagger}(\hat{p}_1')\bar{\chi}_{p_p',\lambda'}{}^{(-)}].\quad(58)$$

Further notations are $E_p'=(p_p'^2+M^2)^{\frac{1}{2}}$, and

$$\bar{\chi}_{p_p',\lambda'}{}^{(\pm)}=\left(1,\ a_{\pm}\frac{\mathbf{p}_1'\cdot\boldsymbol{\sigma}}{E_{p1}+M}\right)$$
$$\times S_L{}^{\dagger}\begin{pmatrix}1\\ \mathbf{p}_p'\cdot\boldsymbol{\sigma}/(E_p'+M)\end{pmatrix}\chi_{p_p',\lambda'},\quad(59a)$$

$$a_+=1,\quad a_-=(p_{p1}/p_1')^2.\quad(59b)$$

From this and (14), (15), the cross section may be found in the following form. Express $\mathbf{B}$ in (14) as

$$\mathbf{B}=-e(2\pi/k)^{\frac{1}{2}}\delta_{\mathbf{P}+\mathbf{k},0}(E_p/E_{p1})^{\frac{1}{2}}(\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3);\quad(60)$$

the δ is obtained due to total momentum conservation. We find then

$$\mathbf{C}_1=-\mathbf{p}_0(E_{\pi0}E_{\pi k0})^{-\frac{1}{2}}g_{\lambda_0\lambda}(\mathbf{p}_0,\mathbf{p}_p),\quad(61a)$$

$$\mathbf{C}_2=-\mathbf{p}_p(E_\pi E_{\pi k})^{-\frac{1}{2}}\sum_{\lambda'}f_{\lambda'\lambda_0}(\mathbf{p}_p,\mathbf{p}_0)\bar{\chi}_{p_p,\lambda}{}^{\dagger}\chi_{p_p,\lambda'},\quad(61b)$$

$$\mathbf{C}_3=\sum_{\mathbf{p}_\pi'\lambda'}\mathbf{p}_\pi'(E_\pi'E_{\pi k}')^{-\frac{1}{2}}f_{\lambda'\lambda_0}(-\mathbf{p}_\pi'-\mathbf{k},\ \mathbf{p}_0)$$
$$\times g_{\lambda'\lambda}(-\mathbf{p}_\pi'-\mathbf{k},\ \mathbf{p}_p);\quad(61c)$$

$$\left.\begin{aligned}&E_{\pi0}=(p_0{}^2+m^2)^{\frac{1}{2}},\quad E_\pi=(p_\pi{}^2+m^2)^{\frac{1}{2}},\\ &E_\pi'=(p_\pi'^2+m^2)^{\frac{1}{2}},\\ &E_{\pi k0}=[(\mathbf{p}_0+\mathbf{k})^2+m^2]^{\frac{1}{2}},\quad E_{\pi k}=[(\mathbf{p}_\pi+\mathbf{k})^2+m^2]^{\frac{1}{2}},\\ &E_{\pi k}'=[(\mathbf{p}_\pi'+\mathbf{k})^2+m^2]^{\frac{1}{2}}.\end{aligned}\right\}\quad(62)$$

The three $\mathbf{C}$'s may be identified as follows: The G's of Eqs. (33) and (56) both consist of two terms, corresponding to plane waves and to scattered waves of the πp system. When inserted in $\mathbf{B}$, the product of two plane waves is forbidden by energy and momentum

conservation, in accordance with the fact that a free particle cannot radiate. $\mathbf{C}_1$ corresponds to the overlap between the initial plane wave and the scattered part of the final wave, the latter representing a pion off the energy shell. $\mathbf{C}_2$ has the final plane wave and the scattered part of the initial state (off the energy shell), and $\mathbf{C}_3$ is the overlap of both scattered waves. We shall neglect this term, however, for the following reason. The experimental elastic $\pi-p$ scattering cross sections are strongly forward peaked (indicating that the scattering is mostly caused by diffraction). The initial scattered wave amplitude is therefore concentrated in the forward direction. The final pion is observed in the forward direction also; it is associated with an ingoing scattered wave, coming from the backward direction. Thus, there will be no overlap between the two scattered waves. If the wave functions correspond to scattering below 1 Bev, there may be considerable backward scattering of the pions, and the above argument may become invalid; but, in the numerical example discussed by us, this is certainly not the case,

Inserting the $\mathbf{B}$ of (60) into the cross section (15), we find

$$d\sigma=\frac{e^2}{vk(2\pi)^4}\int p_\pi^2dp_\pi d\Omega_\pi k^2dkd\Omega_k\frac{E_p}{E_{p1}}\delta(E_p+E_\pi+k-E_{p0}-E_{\pi0})\tfrac{1}{2}\sum_{\lambda\lambda_0}\{|\mathbf{C}_1|^2-|\mathbf{C}_1\cdot\hat{k}|^2+|\mathbf{C}_2|^2-|\mathbf{C}_2\cdot\hat{k}|^2+2\,\mathrm{Re}(\mathbf{C}_1^*\cdot\mathbf{C}_2-\mathbf{C}_1^*\cdot\hat{k}\mathbf{C}_2\cdot\hat{k})\}. \quad (63)$$

VI. PHASE SPACE

The argument of the δ function in (63) will be considered as a function of p_π, and the integration over dp_π eliminated. The result is

$$\frac{d\sigma}{dkd\Omega_kd\Omega_\pi}=\frac{e^2}{v(2\pi)^4}\frac{E_p}{E_{p1}}\frac{E_\pi(E_0-k-E_\pi)p_\pi^2k}{p_\pi(E_0-k)+E_\pi\hat{p}_\pi\cdot\mathbf{k}}\times\tfrac{1}{2}\sum_{\lambda\lambda_0}\{|\mathbf{C}_1|^2-|\mathbf{C}_1\cdot\hat{k}|^2+|\mathbf{C}_2|^2-|\mathbf{C}_2\cdot\hat{k}|+2\,\mathrm{Re}(\mathbf{C}_1^*\cdot\mathbf{C}_2-\mathbf{C}_1^*\cdot\hat{k}\mathbf{C}_2\cdot\hat{k})\}=\Phi_1+\Phi_2+\mathrm{Re}\,\Phi_{12}, \quad (64)$$

where Φ_1 is the term containing $|\mathbf{C}_1|^2-|\mathbf{C}_1\cdot\hat{k}|^2$, etc., and with $E_0=E_{p0}+E_{\pi0}$. The actual value of p_π is obtained from solving the energy and momentum conservation equations:

$$E_p+E_\pi+k=E_0, \quad (65a)$$

$$\mathbf{p}_p+\mathbf{p}_\pi+\mathbf{k}=0. \quad (65b)$$

We eliminate p_p contained in E_p using (65b), and solve (65a) for p_π, with the following result: Calling $w=\hat{p}_\pi\cdot\hat{k}$, then in order to have a real solution p_π, k is limited by

$$0\leqslant k\leqslant\bar{k},$$

$$\bar{k}=\frac{1}{2}\frac{1}{E_0^2-m^2(1-w^2)}\{E_0(E_0^2-M^2-m^2)-m[4E_0^2M^2+(1-w^2)((E_0^2-M^2-m^2)^2-4M^2m^2)]^{\frac{1}{2}}\}. \quad (66)$$

There are, moreover, two regions of k: In the first one,

$$0\leqslant k\leqslant\bar{k}_{\min}\equiv\bar{k}(w=0)=\frac{(E_0-m)^2-M^2}{2(E_0-m)},$$

the solution is

$$p_\pi=\frac{1}{2}\frac{1}{E^2-k^2w^2}\{E[((E^2-\nu)^2-4E^2m^2)+4m^2k^2w^2]^{\frac{1}{2}}-kw(E^2-\nu)\}, \quad (67a)$$

where

$$\nu=M^2-m^2+k^2.$$

In the second region,

$$\bar{k}_{\min}\leqslant k\leqslant\bar{k}_{\max}\equiv\bar{k}(w=1)=[E_0^2-(M+m)^2]/2E_0,$$

there are two solutions

$$p_\pi=\frac{1}{2}\frac{1}{E^2-k^2w^2}\{\pm E[((E^2-\nu)^2-4E^2m^2)+4m^2k^2w^2]^{\frac{1}{2}}-kw(E^2-\nu)\}, \quad (67b)$$

meaning: All other quantities (especially emission angles) being the same, there will be two groups of pions with different momenta emitted. In this case, there is, however, a limitation in the w to certain negative values,

$$-1\leqslant w\leqslant-[4E^2m^2-(E^2-\nu)^2]^{\frac{1}{2}}/2mk, \quad (68)$$

for reasons of reality again. Note that all the quantities are in the center-of-momentum system.

Finally, to complete the cross section (64), we obtained using (61):

$$\tfrac{1}{2}\sum_{\lambda\lambda_0}\{|\mathbf{C}_1|^2-|\mathbf{C}_1\cdot\hat{k}|^2\}=\frac{p_0^2-(\mathbf{p}_0\cdot\hat{k})^2}{p_{10}^2E_{\pi 0}E_{\pi k0}E_{p0}E_p\gamma^2}\frac{4\pi^3}{(p_{p1}^2-p_{10}^2)^2}\Bigg\{\!\!\Bigg\{p_{10}^2(E_{p0}'+M)(E_{p1}+M)$$

$$\times\left\{\left|\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\frac{(l+1)\Delta_l^{(+)}+l\Delta_l^{(-)}}{(2l+1)^{\frac{1}{2}}}\mathcal{P}_{l0}(\hat{p}_{10})\right|^2+\left|\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}(\Delta_l^{(+)}-\Delta_l^{(-)})\mathcal{P}_{l1}(\hat{p}_{10})\right|^2\right\}+\frac{E_{p0}'-M}{E_{p0}+M}\left\{\left|\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\right.\right.$$

$$\left.\left.\times\frac{p_{10}^2(l+1)\Delta_l^{(+)}+p_{p1}^2l\Delta_l^{(-)}}{(2l+1)^{\frac{1}{2}}}\mathcal{P}_{l0}(\hat{p}_{10})\right|^2+\left|\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}(p_{10}^2\Delta_l^{(+)}-p_{p1}^2\Delta_l^{(-)})\mathcal{P}_{l1}(\hat{p}_{10})\right|^2\right\}$$

$$+2\mathbf{p}_0'\cdot\mathbf{p}_{10}\,\mathrm{Re}\left\{\left[\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\frac{(l+1)\Delta_l^{(+)}+l\Delta_l^{(-)}}{(2l+1)^{\frac{1}{2}}}\mathcal{P}_{l0}(\hat{p}_{10})\right]^*\left[\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\frac{p_{10}^2(l+1)\Delta_l^{(+)}+p_{p1}^2l\Delta_l^{(-)}}{(2l+1)^{\frac{1}{2}}}\mathcal{P}_{l0}(\hat{p}_{10})\right]\right.$$

$$\left.+\left[\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}(\Delta_l^{(+)}-\Delta_l^{(-)})\mathcal{P}_{l1}(\hat{p}_{10})\right]^*\left[\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}(p_{10}^2\Delta_l^{(+)}-p_{p1}^2\Delta_l^{(-)})\mathcal{P}_{l1}(\hat{p}_{10})\right]\right\}$$

$$+2(p_{p1}^2-p_{10}^2)\beta\gamma\frac{(\mathbf{p}_0-\mathbf{p}_p)\cdot\boldsymbol{\beta}+E_{p0}+E_p}{p_{10}[1-(\hat{p}_{10}\cdot\hat{p}_{p1})^2]^{\frac{1}{2}}}[(\boldsymbol{\beta}\cdot\mathbf{p}_{10}/\beta)(\mathbf{p}_0\cdot\hat{p}_{p1})-(\mathbf{p}_0\cdot\mathbf{p}_{10})(\boldsymbol{\beta}\cdot\hat{p}_{p1}/\beta)]$$

$$\times\mathrm{Re}\left\{\left[\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}\Delta_l^{(-)}\mathcal{P}_{l1}(\hat{p}_{10})\right]^*\left[\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\frac{l+1}{(2l+1)^{\frac{1}{2}}}\Delta_l^{(+)}\mathcal{P}_{l0}(\hat{p}_{10})\right]\right.$$

$$\left.+\left[\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\frac{l}{(2l+1)^{\frac{1}{2}}}\Delta_l^{(-)}\mathcal{P}_{l0}(\hat{p}_{10})\right]^*\left[\sum_l\frac{p_{10}^l}{p_{p1}^{l+1}}\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}\Delta_l^{(+)}\mathcal{P}_{l1}(\hat{p}_{10})\right]\right\}\Bigg\}\!\!\Bigg\}, \quad (69)$$

where

$$\mathbf{p}_{10}=\mathbf{p}_0+(\boldsymbol{\beta}/\beta)[\mathbf{p}_0\cdot(\boldsymbol{\beta}/\beta)(\gamma^{-1}-1)-\beta E_{p1}],\quad \mathbf{p}_0'=\mathbf{p}_0+(\boldsymbol{\beta}/\beta)[\mathbf{p}_0\cdot(\boldsymbol{\beta}/\beta)(\gamma-1)+\beta\gamma E_{p0}],\quad E_{p0}'=\gamma(E_{p0}+\boldsymbol{\beta}\cdot\mathbf{p}_0),$$

and in Rose's[8] notation,

$$\mathcal{P}_{lm}(\theta)=e^{-im\phi}Y_{lm}(\theta,\phi);$$

the arguments are here characterized by the corresponding vectors, e.g., if $\hat{p}_{10}=(\theta_{10},\phi_{10})$, then $Y_{lm}(\hat{p}_{10})\equiv Y_{lm}(\theta_{10},\phi_{10})$. Further:

$$\tfrac{1}{2}\sum_{\lambda\lambda_0}\{|\mathbf{C}_2|^2-|\mathbf{C}_2\cdot\hat{k}|^2\}=4\pi^3\frac{p_p^2-(\mathbf{p}_p\cdot\hat{k})^2}{E_\pi E_{\pi k}}\frac{E_{p0}+M}{E_{p0}}\frac{E_p+M}{E_p}\left(\frac{E_{p0}+E_p}{p_0^2-p_p^2}\right)^2$$

$$\times\left\{\left|\sum_l\frac{p_p^l}{p_0^{l+1}}\frac{1}{(2l+1)^{\frac{1}{2}}}\left[\frac{(l+1)\Delta_l^{(+)}}{E_{p0}+M}+\frac{l\Delta_l^{(-)}}{E_p+M}\right]Y_{l0}(\hat{p}_0\cdot\hat{p}_p)\right|^2\right.$$

$$+\tfrac{1}{2}\left|\sum_l\frac{p_p^l}{p_0^{l+1}}\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}\left[\frac{\Delta_l^{(+)}}{E_{p0}+M}-\frac{\Delta_l^{(-)}}{E_p+M}\right]\sum_m D_{m1}{}^l(\hat{p}_0)Y_{lm}(\hat{p}_p)\right|^2$$

$$\left.+\tfrac{1}{2}\left|\sum_l\frac{p_p^l}{p_0^{l+1}}\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}\left[\frac{\Delta_l^{(+)}}{E_{p0}+M}-\frac{\Delta_l^{(-)}}{E_p+M}\right]\sum_m D_{m1}{}^{l*}(\hat{p}_0)Y_{lm}{}^*(\hat{p}_p)\right|^2\right\}, \quad)70)$$

where we called $D_{m'm}{}^{l}(\hat{p}_0)=D_{m'm}{}^{l}(\phi_0,\theta_0,0)$ if $\hat{p}_0=(\theta_0,\phi_0)$. The $Y_{l0}(\hat{p}_0\cdot\hat{p}_p)$ means $\sim P_l(\cos(\hat{p}_0,\hat{p}_p))$. Lastly,

$$\tfrac{1}{2}\sum_{\lambda\lambda_0} 2\,\mathrm{Re}(\mathbf{C}_1{}^*\cdot\mathbf{C}_2-\mathbf{C}_1{}^*\cdot\hat{k}\mathbf{C}_2\cdot\hat{k})=\frac{\mathbf{p}_0\cdot\mathbf{p}_p-\mathbf{p}_0\cdot\hat{k}\mathbf{p}_p\cdot\hat{k}}{(E_{\pi0}E_{\pi k0}E_\pi E_{\pi k})^{\frac{1}{2}}}\frac{8\pi^{7/2}}{\gamma p_{10}{}^2E_{p0}E_p}\frac{E_{p0}+E_p}{E_{p1}+M}\frac{1}{p_{p1}{}^2-p_{10}{}^2}\frac{1}{p_0{}^2-p_p{}^2}$$

$$\times 2\,\mathrm{Re}\Bigg\{\Bigg\{S_0{}^{(+)}\sum_{l_0\mu}\frac{p_p{}^{l_0}}{p_0{}^{l_0+1}}\Bigg\{T_{l_0\mu}{}^{(+)}(\theta)\Bigg[\left(\frac{l_0+\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}(A_+\,\mathrm{Re}Y_{l_0,\mu-\frac{1}{2}}(\hat{p}_0)+\mathbf{B}_+\cdot\hat{p}_{p1}\,\mathrm{Im}Y_{l_0,\mu-\frac{1}{2}}(\hat{p}_0))$$

$$+\left(\frac{l_0-\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}(\mathbf{B}_+\times\hat{Q}_\mu)\cdot\hat{p}_{p1}\mathcal{P}_{l_0,\mu+\frac{1}{2}}(\theta_0)\Bigg]+T_{l_0\mu}{}^{(-)}(\theta)\Bigg[\left(\frac{l_0+\mu-\frac{1}{2}}{2l_0-1}\right)^{\frac{1}{2}}\mathbf{V}_+\cdot\hat{p}_{p1}\,\mathrm{Re}Y_{l_0-1,\mu-\frac{1}{2}}(\hat{p}_0)$$

$$+\left(\frac{l_0-\mu-\frac{1}{2}}{2l_0-1}\right)^{\frac{1}{2}}\mathbf{V}_+\cdot\hat{Q}_\mu\mathcal{P}_{l_0-1,\mu+\frac{1}{2}}(\theta_0)\Bigg]\Bigg\}+S_0{}^{(-)}\sum_{l_0\mu}\frac{p_p{}^{l_0}}{p_0{}^{l_0+1}}\Bigg\{T_{l_0\mu}{}^{(+)}(\theta)\Bigg[\left(\frac{l_0+\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}(A_-\,\mathrm{Re}Y_{l_0,\mu-\frac{1}{2}}(\hat{p}_0)$$

$$+\mathbf{B}_-\cdot\hat{p}_{p1}\,\mathrm{Im}Y_{l_0,\mu-\frac{1}{2}}(\hat{p}_0))+\left(\frac{l_0-\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}(\mathbf{B}_-\times\hat{Q}_\mu)\cdot\hat{p}_{p1}\mathcal{P}_{l_0,\mu+\frac{1}{2}}(\theta_0)\Bigg]$$

$$+T_{l_0\mu}{}^{(-)}(\theta)\Bigg[\left(\frac{l_0+\mu-\frac{1}{2}}{2l_0-1}\right)^{\frac{1}{2}}\mathbf{V}_-\cdot\hat{p}_{p1}\,\mathrm{Re}Y_{l_0-1,\mu-\frac{1}{2}}(\hat{p}_0)+\left(\frac{l_0-\mu-\frac{1}{2}}{2l_0-1}\right)^{\frac{1}{2}}\mathbf{V}_-\cdot\hat{Q}_\mu\mathcal{P}_{l_0-1,\mu+\frac{1}{2}}(\theta_0)\Bigg]\Bigg\}$$

$$+S_1{}^{(+)}\sum_{l_0\mu}\frac{p_p{}^{l_0}}{p_0{}^{l_0+1}}\Bigg\{T_{l_0\mu}{}^{(+)}(\theta)\Bigg[\left(\frac{l_0-\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}(A_+\hat{Q}_\mu\cdot\hat{q}-\mathbf{B}_+\cdot\hat{p}_{p1}(\hat{q}\times\hat{Q}_\mu)\cdot\hat{p}_{p1})\mathcal{P}_{l_0,\mu+\frac{1}{2}}(\theta_0)$$

$$+\left(\frac{l_0+\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}(\mathbf{B}_+\cdot\hat{q}\,\mathrm{Im}Y_{l_0,\mu-\frac{1}{2}}(\hat{p}_0)+(\hat{q}\times\mathbf{B}_+)\cdot\hat{p}_{p1}\,\mathrm{Re}Y_{l_0,\mu-\frac{1}{2}}(\hat{p}_0))\Bigg]$$

$$+T_{l_0\mu}{}^{(-)}(\theta)\Bigg[-\left(\frac{l_0-\mu-\frac{1}{2}}{2l_0-1}\right)^{\frac{1}{2}}\mathbf{V}_+\cdot\hat{p}_{p1}\hat{Q}_\mu\cdot\hat{q}\mathcal{P}_{l_0-1,\mu+\frac{1}{2}}(\theta_0)+\left(\frac{l_0+\mu-\frac{1}{2}}{2l_0-1}\right)^{\frac{1}{2}}(\mathbf{V}_+\cdot\hat{q}\,\mathrm{Re}Y_{l_0-1,\mu-\frac{1}{2}}(\hat{p}_0)$$

$$-(\hat{q}\times\mathbf{V}_+)\cdot\hat{p}_{p1}\,\mathrm{Im}Y_{l_0-1,\mu-\frac{1}{2}}(\hat{p}_0))\Bigg]\Bigg\}-S_1{}^{(-)}\sum_{l_0\mu}\frac{p_p{}^{l_0}}{p_0{}^{l_0+1}}\Bigg\{T_{l0}{}^{(+)}(\theta)\Bigg[\left(\frac{l_0-\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}(A_-\hat{Q}_\mu\cdot\hat{q}$$

$$-\mathbf{B}_-\cdot\hat{p}_{p1}(\hat{q}\times\hat{Q}_\mu)\cdot\hat{p}_{p1})\mathcal{P}_{l_0,\mu+\frac{1}{2}}(\theta_0)+\left(\frac{l_0+\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}(\mathbf{B}_-\cdot\hat{q}\,\mathrm{Im}Y_{l_0,\mu-\frac{1}{2}}(\hat{p}_0)$$

$$+(\hat{q}\times\mathbf{B}_-)\cdot\hat{p}_{p1}\,\mathrm{Re}Y_{l_0,\mu-\frac{1}{2}}(\hat{p}_0))\Bigg]+T_{l_0\mu}{}^{(-)}(\theta)\Bigg[-\left(\frac{l_0-\mu-\frac{1}{2}}{2l_0-1}\right)^{\frac{1}{2}}\mathbf{V}_-\cdot\hat{p}_{p1}\hat{Q}_\mu\cdot\hat{q}\mathcal{P}_{l_0-1,\mu+\frac{1}{2}}(\theta_0)$$

$$+\left(\frac{l_0+\mu-\frac{1}{2}}{2l_0-1}\right)^{\frac{1}{2}}(\mathbf{V}_-\cdot\hat{q}\,\mathrm{Re}Y_{l_0-1,\mu-\frac{1}{2}}(\hat{p}_0)-(\hat{q}\times\mathbf{V}_-)\cdot\hat{p}_{p1}\,\mathrm{Im}Y_{l_0-1,\mu-\frac{1}{2}}(\hat{p}_0))\Bigg]\Bigg\}\Bigg\}\Bigg\} \quad (71)$$

with

$$S_0{}^{(+)}=\sum_l\frac{p_{10}{}^l}{p_{p1}{}^{l+1}}\Delta_l{}^{(+)*}\frac{l+1}{(2l+1)^{\frac{1}{2}}}\mathcal{P}_{l0}(\hat{p}_{10}),$$

$$S_0{}^{(-)}=\sum_l\frac{p_{10}{}^l}{p_{p1}{}^{l+1}}\Delta_l{}^{(-)*}\frac{l}{(2l+1)^{\frac{1}{2}}}\mathcal{P}_{l0}(\hat{p}_{10}), \quad (71a)$$

$$S_1{}^{(\pm)}=\sum_l\frac{p_{10}{}^l}{p_{p1}{}^{l+1}}\Delta_l{}^{(\pm)*}\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}\mathcal{P}_{l1}(\hat{p}_{10});$$

further,

$$\hat{p}_0=(\theta_0,\phi_0),\quad \hat{p}_p=(\theta,0), \quad (71b)$$

and

$$T_{l_0\mu}{}^{(+)}(\theta)=\left[a\left(\frac{l_0+\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}\mathcal{P}_{l_0,\mu-\frac{1}{2}}(\theta)+b\left(\frac{l_0-\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}\mathcal{P}_{l_0,\mu+\frac{1}{2}}(\theta)\right]\frac{\Delta_{l_0}{}^{(+)}}{E_{p0}+M},$$

$$T_{l_0\mu}{}^{(-)}(\theta)=\left[-a\left(\frac{l_0-\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}\mathcal{P}_{l_0,\mu-\frac{1}{2}}(\theta)+b\left(\frac{l_0+\mu+\frac{1}{2}}{2l_0+1}\right)^{\frac{1}{2}}\mathcal{P}_{l_0,\mu+\frac{1}{2}}(\theta)\right]\frac{\Delta_{l_0}{}^{(-)}}{E_p+M}, \quad (71c)$$

with a, b of Eq. (57b); moreover,

$$\hat{Q}_\mu=(\cos(\mu+\tfrac{1}{2})\phi_0,\ \sin(\mu+\tfrac{1}{2})\phi_0,\ 0),\qquad \hat{q}=(\cos\phi_{10},\ \sin\phi_{10},\ 0), \tag{71d}$$

and lastly,

$$\begin{aligned}A_+&=p_{10}^2\{[(E_{p0}+M)(E_{p1}+M)+\mathbf{p}_0\cdot\mathbf{p}_{10}]\cosh\tfrac{1}{2}\chi\\&\quad+[(E_{p0}+M)\hat{P}\cdot\mathbf{p}_{10}+(E_{p1}+M)\hat{P}\cdot\mathbf{p}_0]\sinh\tfrac{1}{2}\chi\},\\A_-&=[p_{10}^2(E_{p0}+M)(E_{p1}+M)+p_{p1}^2\mathbf{p}_0\cdot\mathbf{p}_{10}]\cosh\tfrac{1}{2}\chi\\&\quad+[p_{p1}^2(E_{p0}+M)\hat{P}\cdot\mathbf{p}_{10}\\&\quad+p_{10}^2(E_{p1}+M)\hat{P}\cdot\mathbf{p}_0]\sinh\tfrac{1}{2}\chi;\end{aligned} \tag{71e}$$

$$\begin{aligned}\mathbf{B}_\pm&=(\hat{P}\times\mathbf{p}_0)B_\pm,\\B_+&=p_{10}^2[E_{p0}+E_p+\boldsymbol{\beta}\cdot(\mathbf{p}_0-\mathbf{p}_p)]\sinh\tfrac{1}{2}\chi,\\B_-&=\{(p_{p1}^2-p_{10}^2)(E_{p1}+M)\\&\quad+p_{p1}^2[E_{p0}+E_p+\boldsymbol{\beta}\cdot(\mathbf{p}_0-\mathbf{p}_p)]\}\sin\tfrac{1}{2}\chi;\end{aligned} \tag{71f}$$

$$\mathbf{V}_\pm=\hat{P}p_0B_\pm-\hat{p}_0(A_\pm+\hat{P}\cdot\mathbf{p}_0B_\pm). \tag{71g}$$

The magnetic quantum numbers m, μ go over the full range allowed by the first index of the $\mathcal{P}$ or Y [unlike in (64), e.g., where λ_0 and λ are just $\pm\frac{1}{2}$ corresponding to the proton spin]. These formulas seem to depend on our choice of z axis, but they can all be expressed by rotation-invariant scalar products of vectors.

VII. PHASE-SHIFT ANALYSIS OF ELASTIC PION-PROTON SCATTERING

Our formulas for the differential cross section, Eq. (64) together with (69)–(71), depend on the phase shifts of the elastic pion-proton scattering through the quantities $\Delta_l^{(\pm)}$. We shall try in the following a simple-minded approach to obtain these from an analysis of the available experimental data. The elastic scattering cross section in the center-of-mass system is found by taking the asymptotic form of (25) for large relative coordinates $\mathbf{r}=(r,\theta_r,\phi_r)$,

$$\Psi^{(+)}(p,0,0,\lambda)\cong\Phi(p,0,0,\lambda)+\sum_{\lambda'}f_{\lambda\lambda'}(\theta_r,\phi_r)\frac{1}{r}\Phi(p,\theta_r,\phi_r,\lambda'), \tag{72}$$

where the scattering amplitude is

$$f_{\lambda\lambda'}(\theta_r,\phi_r)=(ip)^{-1}\sum_J(J+\tfrac{1}{2})\times\langle\lambda'|S^J(p)-1|\lambda\rangle e^{i(\lambda-\lambda')\phi_r}d_{\lambda\lambda'}{}^J(\theta_r); \tag{73}$$

to find the elastic scattering cross section

$$d\sigma/d\Omega=\textstyle\sum_{\lambda'}|f_{\lambda\lambda'}|^2, \tag{74}$$

we use (31) and (24) and get

$$d\sigma/d\Omega=|f(\theta_r)|^2+|g(\theta_r)|^2, \tag{75a}$$

with a non-spin-flip amplitude

$$f(\theta_r)=\frac{\pi^{\frac{1}{2}}}{ip}\sum_l(2l+1)^{-\frac{1}{2}}[(l+1)\Delta_l^{(+)}+l\Delta_l^{(-)}]\mathcal{P}_{l0}(\theta_r), \tag{75b}$$

and a spin-flip amplitude

$$g(\theta_r)=\frac{\pi^{\frac{1}{2}}}{ip}\sum_l\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}(\Delta_l^{(+)}-\Delta_l^{(-)})\mathcal{P}_{l1}(\theta_r), \tag{75c}$$

which are well-known expressions.[16] We shall try to fit (75) to the experimental data of Wang *et al.*[4h] and Chretien *et al.*[4c] at laboratory pion energies of $E_{\pi 0}{}^L=6.80$ Bev and $E_{\pi 1}{}^L=1.44$ Bev, respectively, which after conversion to the center-of-mass system by

$$p_{\pi 0,1}=Mp_{\pi 0,1}{}^L(M^2+m^2+2ME_{\pi 0,1}{}^L)^{-\frac{1}{2}}, \tag{76}$$

will correspond to center-of-mass energies of $E_{\pi 0}=1.73$ Bev (or $E_0=3.69$ Bev), and $E_{\pi 1}=0.724$ Bev, respectively. In general, several phase shifts ($\gtrsim 7$ and $\gtrsim 3$, respectively) will contribute, and they will be complex due to the large inelastic cross section at these energies. The elastic angular distribution is mostly forward and diffraction-like, and there are theoretical reasons[17] that the phase shifts will be almost purely imaginary at these high energies. That is the simplifying assumption we are going to make, namely, purely imaginary phase shifts and no spin-flip scattering, and this will permit us to obtain these phase shifts from the experimental angular distributions in a unique way.[18] It can also be shown[18] that this interpretation of pion-proton scattering does not lead to contradictions (which in proton-proton scattering is the case). If we call $\delta_l=i\eta_l$ (no suffix $\pm$, because no spin flip assumed), and

$$0\leqslant b_l=e^{-2\eta_l}\leqslant 1, \tag{77}$$

then the analysis is made by[19]

$$1-b_l=p\int_{-1}^{1}\left(\frac{d\sigma_{\mathrm{el}}}{d\Omega}\right)^{\frac{1}{2}}P_l(x)dx, \tag{78a}$$

which for large l (optical limit), setting $2l+1=2p\rho$, goes over into

$$1-b(\rho)=p\int_{-1}^{1}\left(\frac{d\sigma_{\mathrm{el}}}{d\Omega}\right)^{\frac{1}{2}}J_0(p\rho\sin\vartheta)d\cos\vartheta. \tag{78b}$$

The sign of the square root has to be appropriately chosen if there are nodes in the experimental cross section; p is $p_{\pi 0}$ or $p_{\pi 1}$ for our two cases. The results for $1-b_l$ are shown in the inserts of Figs. 1 and 2 [in the latter case, Eq. (78b) has been used for the analysis for $l\geqslant 5$]. At 6.80 Bev, the proton transparency increases smoothly to the edge, but at 1.44 Bev, taking

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[16] J. Ashkin, Suppl. Nuovo cimento **14**, 221 (1959).

[17] D. I. Blokhintsev, V. S. Barasenkov, and V. G. Grisin, Nuovo cimento **9**, 249 (1958); S. Z. Belenkii, Zhur. Eksp. i Teor. Fiz. **33**, 1051 (1957) [translation: Soviet Phys.—JETP **6**, 960 (1958)].

[18] Z. Koba, A. Krzywicki, R. Raczka, and Z. Chylinski, Nuclear Phys. **19**, 199 (1960).

[19] See reference 18; see also V. I. Veksler, in *Proceedings of the Ninth Annual International Conference on High-Energy Physics, Kiev, 1959* (Academy of Science, U.S.S.R., 1960).

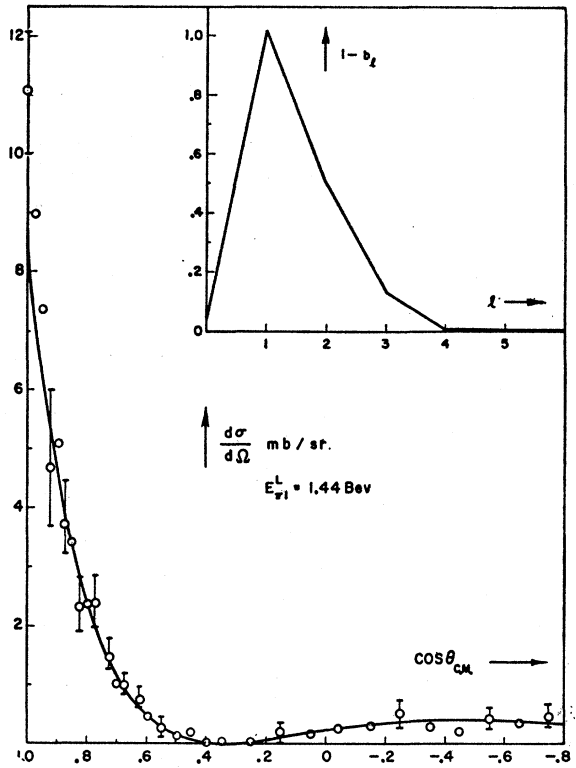


FIG. 1. Analysis of elastic pion-proton scattering at 1.44 Bev using imaginary phase shifts, no spin flip.

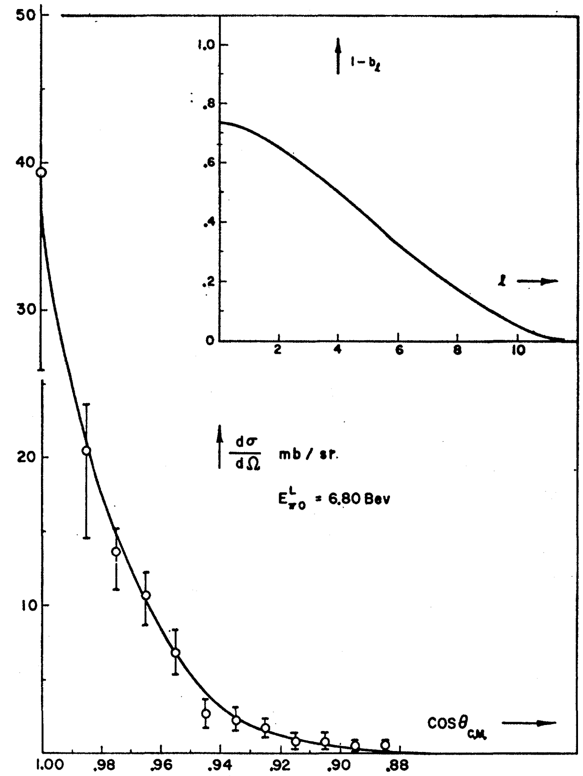


FIG. 2. Analysis of elastic pion-proton scattering at 6.80 Bev using imaginary phase shifts, no spin flip.

the results seriously, the proton has a transparent center. This is largely connected with fitting the backward tail of the angular distribution at this energy (the main parts of Figs. 1 and 2 show our fits to the experimental points with the b's of the inserts), and the fit is only barely possible due to $1-b_1$ being 1.02. The model seems hardly realistic at this energy, and we have indeed tried to represent the b_l at 1.44 Bev by a curve analogous to that of Fig. 2 at 6.8 Bev and explaining the difference by real parts of the phase shifts, assumed small so that an iteration procedure could be used, but this attempt failed. We should therefore point out that the use of phases as given by b_l of Fig. 1 to evaluate the bremsstrahlung cross section is largely thought of as an illustration, of value for showing the magnitude and general features. A more accurate evaluation awaits a better understanding of the elastic scattering.

TABLE I. Absorption coefficients at 6.8 Bev.

l	$1-b_l$	l	$1-b_l$
0	0.742	6	0.327
1	0.715	7	0.238
2	0.662	8	0.163
3	0.590	9	0.099
4	0.500	10	0.050
5	0.419	11	0.015

The values of b_l taken are listed in Tables I and II. The points at $\cos\theta_{\text{c.m.}}=1$ in Figs. 1 and 2 are not experimental, but were obtained from the forward scattering amplitudes of Cronin,[20] calculated from dispersion relations and from the optical theorem, partly by extrapolation. Writing

$$d\sigma(0)/d\Omega=D^2+I^2,$$

we find imaginary parts $I^2=0.98$ and 3.80 (fermi)^2 in the c.m. system at 1.44 and 6.80 Bev laboratory energies, respectively, and small dispersive parts $D^2=0.128$ and 0.130 (fermi)^2, showing that the scattering is actually mainly absorptive.

VIII. NUMERICAL RESULTS

It has been stated in the older literature (reference 3; first of references): (1) that the spatial extension of the

TABLE II. Absorption coefficients at 1.44 Bev.

l	$1-b_l$
0	0.04
1	1.02
2	0.50
3	0.13

[20] J. W. Cronin, Phys. Rev. **118**, 824 (1960).

pion charge will modify the results of a bremsstrahlung calculation with Eq. (10) taken as the interaction; the pion-photon vertex is expected to have a structure, expressed by a factor

$$F\left(\frac{p_i^2}{m^2}-1,\frac{p_f^2}{m^2}-1,\frac{k^2}{m^2}\right), \tag{79}$$

which is a function of the squares of initial and final pion and photon four-momenta, with

$$F(0,0,0)=1. \tag{80}$$

Denoting the deviations from the energy shell of the intermediate pion by

$$\begin{aligned} z_1&=-2(E_{\pi 0}k-\mathbf{p}_{\pi 0}\cdot\mathbf{k})/m^2,\\ z_2&=2(E_\pi k-\mathbf{p}_\pi\cdot\mathbf{k})/m^2, \end{aligned} \tag{81}$$

this would give for the cross section:

$$\frac{d\sigma}{dkd\Omega_k d\Omega_\pi}=\Phi_1|F(0,z_1,0)|^2+\Phi_2|F(z_2,0,0)|^2 +\mathrm{Re}\Phi_{12}F^*(0,z_1,0)F(z_2,0,0). \tag{82}$$

Using some results of a calculation of Low,[21] it turns

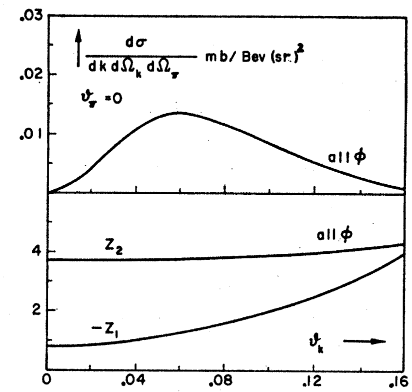
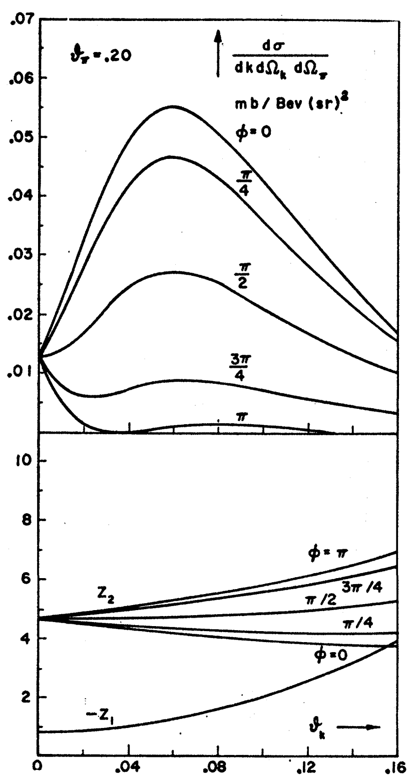


FIG. 3. Differential pion bremsstrahlung cross section for $E_{\pi 0}{}^L=6.80$ Bev, $k=1.364$ Bev, $\vartheta_\pi=0$, and z_1, z_2.



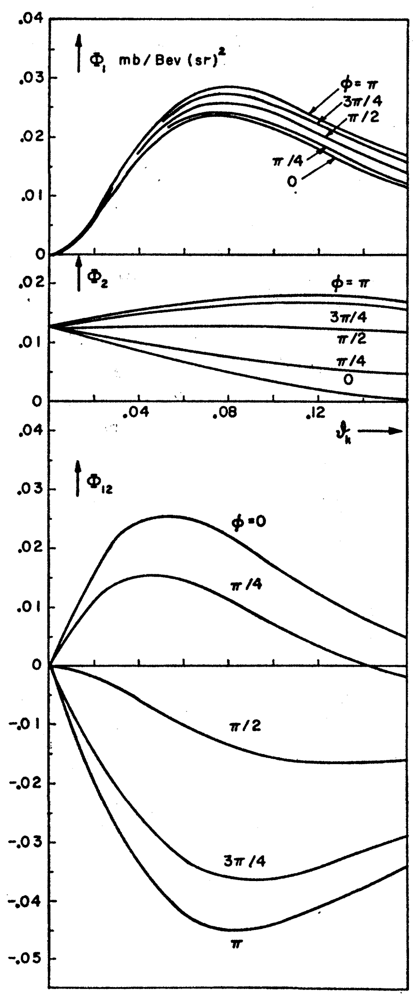


FIG. 4. Differential pion bremsstrahlung cross section for $E_{\pi 0}{}^L=6.80$ Bev, $k=1.364$ Bev, $\vartheta_\pi=0.20$ rad, and its constituents, and z_1, z.

[21] F. E. Low, Phys. Rev. **110**, 974 (1948); for a derivation of the relevant equation of Low, Eq. (2.5), see Y. Takahashi, Nuovo cimento **6**, 371 (1957), and N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954). I am indebted to Professor M. L. Goldberger for directing my attention to these papers.

out, however, that the renormalization of the off-energy-shell pion propagator [which in our calculation arises from the integral of Eq. (30)] exactly cancels out all effects of a pion charge extension; if $\Delta_{Fc}(q)$ is the complete renormalized pion propagator and $\Gamma_\mu(q,q')$ the renormalized vertex operator, then

$$\Delta_{Fc}(q+k)\Gamma_\mu(q+k,q)=\Delta_F(q+k)\gamma_\mu(q+k,q),$$

where $q^2=m^2$, $k^2=0$, $\Delta_F=[(q+k)^2-m^2]^{-1}$, and $\gamma_\mu=(2q+k)_\mu$. Accordingly, Eq. (64) rather than (82) is the correct expression for the bremsstrahlung cross section, which is then exactly valid for a dressed pion everywhere in the range of validity of our approximation, i.e., where the radiation is "external" only.

Our numerical results were obtained under the following assumptions: The initial pion laboratory energy was taken as $E_{\pi0}{}^L=6.80$ Bev, the final pion energy in the laboratory system of the elastic scattering $E_{\pi1}{}^L=1.44$ Bev; the connection to $E_{\pi1}$ is by Eq. (76), and to our bremsstrahlung center-of-mass final energy:

$$E_{\pi1}=E(E_\pi+\beta p_\pi w)/(E^2-k^2)^{\frac{1}{2}}, \qquad (83)$$

which leads to a relation *independent of angles*,

$$E_{\pi1}=(\gamma/2E)(E^2-\nu). \qquad (84)$$

By this fortuitous circumstance, we are enabled to use the phase shifts at one and the same final energy of the elastic scattering, no matter what emission angles we consider. These energies satisfy Eqs. (3); the choice of $E_{\pi1}{}^L$ also leads through (84) to the value (center-of-mass)

$$k=1.364 \text{ Bev}, \qquad (85)$$

which is conveniently below $\bar{k}_{\min}$, so the corresponding pions have a unique momentum. To satisfy further Eqs. (4), we have to restrict ourselves to center-of-mass angles

$$\vartheta_k\lesssim 0.10, \quad \vartheta_\pi\lesssim 0.36 \text{ rad}. \qquad (86)$$

Using also the b_l of Tables I and II, which lead to real Δ_l and real Φ's, we obtain the results of Figs. 3–5, i.e., cross sections plotted vs ϑ_k, with φ and ϑ_π as parameters. They show a peaking of the cross section at certain small angles ϑ_k; the maxima increase with ϑ_π up to $\vartheta_\pi\sim 0.36$ rad, the limit of validity of our approximation, but would decrease above that. Underneath, the deviations of the virtual pion from the energy shell, z_{12}, are

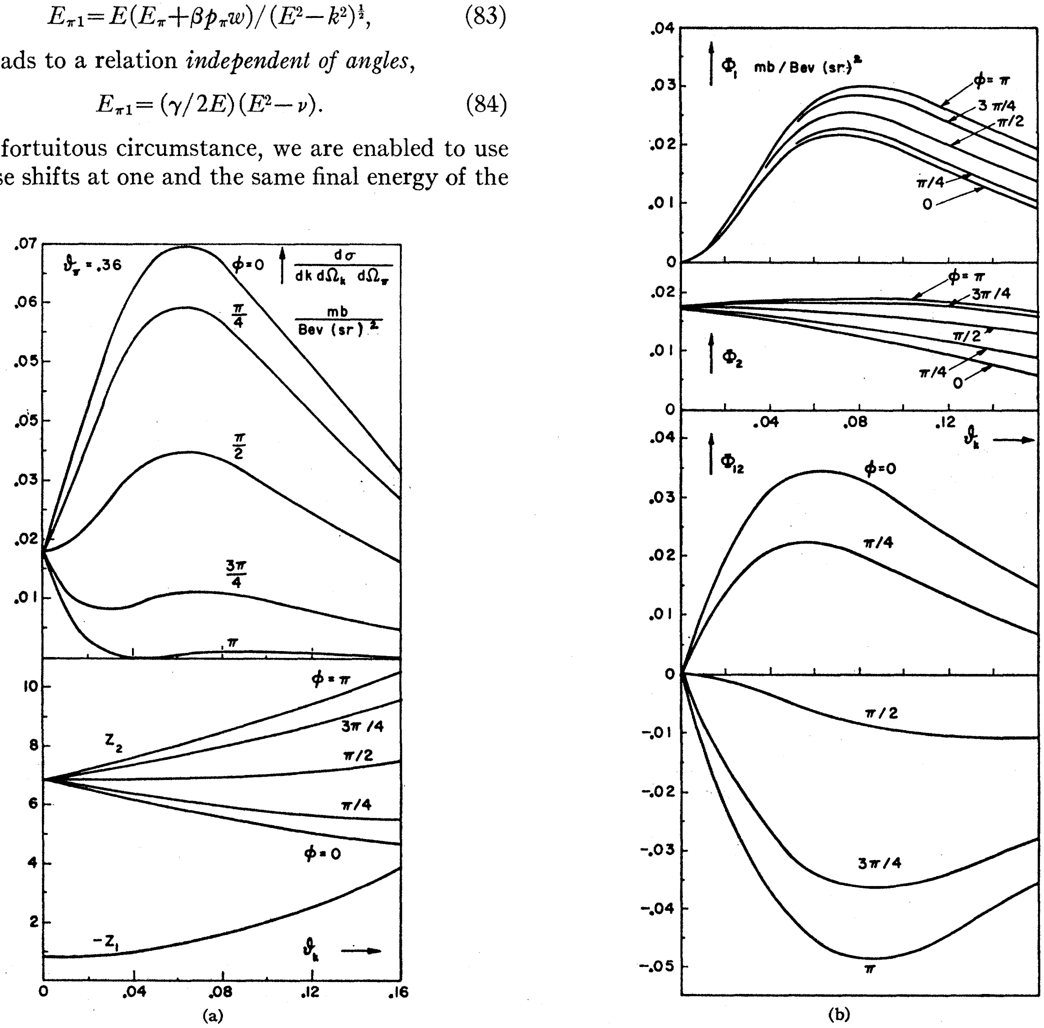


FIG. 5. Differential pion bremsstrahlung cross section for $E_{\pi0}{}^L=6.80$ Bev, $k=1.364$ Bev, $\vartheta_\pi=0.36$ rad, and its constituents, and z_1, z_2.

plotted, and we see that they may considerably differ from unity. In Figs. 4(b) and 5(b), the cross section is taken apart into its constituents Φ.

One may observe that our cross section has certain characteristic denominators possessing poles at four-momentum values of a real intermediate pion, which represent the free pion propagators. This is reminiscent of the results of peripheral-collision theory.[22] It would, however, not be sufficient, as is done in reference 22, to describe the process by *one* single-pion-exchange diagram, as according to Figs. 4(b) and 5(b), both our terms $\mathbf{C}_1$ and $\mathbf{C}_2$ produce contributions of a similar order of magnitude in the forward direction, with an important interference. Also, the magnitudes of z_{12} show that we are by no means dealing here with an approach to the poles.

In conclusion, it may be said that our numerical results illustrate the magnitude and behavior of the pion bremsstrahlung cross section, and that these results may be made quite accurate by using reliable phase shifts of the elastic scattering. A practical difficulty in the measurements might arise from photons originating in the decay of inelastically produced neutral pions.

[22] S. Drell, Revs. Modern Phys. **33**, 458 (1961).

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