*Cerium:* Cerium(IV) iodate was precipitated from $4M$ $HNO_3$. About 8 ml of irradiated solution containing about 1 g of $UO_2^{++}$, 0.5 mg each of $Ba^{++}$ and $La^{3+}$ carriers, and $La^{140}$ tracer was added to 21 ml of $9M$ $HNO_3$ containing 30 mg of $Ce^{3+}$ carrier. To the resulting solution was added, first, 5 ml of warm $1M$ $NaBrO_3$ solution, which rapidly oxidized Ce(III) to Ce(IV), and then 15 ml of warm $0.35M$ $HIO_3$. After being stirred for about 15 sec, a portion of the mixture was filtered, and the $Ce(IO_3)_4$ washed with a solution $0.1M$ in $HIO_3$ and $5M$ in $HNO_3$. The fraction of lanthanum remaining with the precipitate varied from 0.5 to 1.5% and was determined from the fraction of $La^{140}$ activity remaining with the precipitate.

*Niobium:* Barium fluorozirconate was precipitated from $3M$ HCl, $5M$ HF solution, leaving niobium in solution. Irradiated UNH or an aluminum recoil catcher was dissolved in 30 ml of $4M$ HCl containing 20 mg of Zr(IV) carrier and 100 mg of Nb(V) carrier present as the oxalate complex. The solution was divided into two approximately equal portions, and 3 ml of $30M$ HF was added to one. After the resulting solution had been stirred about 1 min, $BaCl_2$ solution was added to precipitate $BaZrF_6$. The supernatant was separated by centrifugation, and the precipitation repeated two or three more times. Tracer experiments with $Zr^{97}$ showed that 97 to 98% of the zirconium was removed by the first precipitation, and 99.97% in a total of three precipitations. The effective separation time was taken as the mean time of the first centrifugation, which lasted 40 sec. In estimating errors, we have considered that separation of niobium from zirconium could have occurred as early as the time at which $BaZrF_6$ formed or as late as the end of centrifugation.

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# Nuclear Rotational Spectra, the Elliott Model, and the $P_2$ Force*

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The essential features of the Elliott model with the momentum-dependent quadrupole-quadrupole operator which leads to $L(L+1)$ rotational spectra are reviewed. The operator is reduced to a momentum-independent residual interaction which differs somewhat from the $P_2$ interaction. The model Hamiltonian is separated into a rotational Hamiltonian, a deformed "intrinsic" Hamiltonian, and a perturbation term. The eigenfunctions and eigenvalues of the "intrinsic" Hamiltonian are found and used in Inglis' cranking model formula to calculate the moment of inertia. The model is modified, in a simple configuration, by taking a mixture of the long-range "$P_2$" interaction with the short-range $\delta$-function force. For an intermediate mixture the spectrum obtained resembles the spectrum predicted by the collective vibrational Model. Finally, the implications of a $P_2$ residual interaction for direct-interaction inelastic scattering processes are considered. The question is discussed whether one can actually see the $P_2$ residual interaction in rotational nuclei, and, if so, whether the strength of the $P_2$ interaction determined from such scattering experiments is consistent with the strength determined from the observed rotational spectra. Within the rough approximations made, the few experimental results available are not inconsistent with the calculation.

## I. INTRODUCTION

THE purpose of this paper is to explore some consequences and modifications of the nuclear model discussed by Elliott.[1]

In the simplest version of the shell model, all nucleons move independently in a central potential which is supposed to represent the averaged effect of all interactions between the particles. We take the average central potential to be a spherical harmonic oscillator potential. We neglect the spin-orbit force and work in $LS$ coupling:

$$H_0=\sum_{i=1}^{\nu} H_0(i)=\sum_{i=1}^{\nu} \tfrac{1}{2}(p_i^2+r_i^2) \qquad (1.1)$$

(in units $\hbar=\omega_0=M=1$). $\nu$ is the number of particles in the outermost shell. (The closed shells do not enter into the calculation and we drop them from the Hamiltonian.) In $LS$ coupling a shell-model orbital state or configuration is specified as $l_a^{\nu_a} l_b^{\nu_b}\cdots$, where $\nu_a+\nu_b+\cdots=\nu$. The oscillator potential has the well known degeneracy of single-particle levels

$$|1s|1p|2s\ 1d|2p\ 1f|3s\ 2d\ 1g|\cdots,$$

i.e., $l=N,\ N-2,\ \cdots,$ 1, or 0 where $E_N=N+\frac{3}{2}$. Thus, all configurations for which $l_a, l_b, \cdots$ belong to the same degenerate oscillator shell will be degenerate.

Elliott[1] has developed a coupling scheme appropriate for this situation which groups together states of angular momenta characteristic of a rotational band. He showed the similarity between the states determined by this coupling scheme and the states appearing in the

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[1] J. P. Elliott, Proc. Roy. Soc. (London) **A245**, 128, 562 (1958).

collective rotational model of Bohr and Mottelson[2] by showing that the different states of a given band could be obtained as the different angular momentum states projected out of a single "intrinsic" function and by showing that the quadrupole moments calculated for different states of the same band were related in a manner similar to that found in the strong-coupling collective model of Bohr and Mottelson.

The next problem is to find a residual interaction which, when added to the Hamiltonian (1.1), will remove the degeneracy present in the scheme described above and lead to rotational spectra, i.e., spectra of the form

$$E=\text{const}+L(L+1)/2I.$$

This question has received some discussion in the literature[3–5] and a momentum-dependent residual interaction has been found which has the desired effect.

After reviewing a simple way the method of obtaining rotational spectra, our first point is to obtain the correct form of the "$P_2$" force. We next define an intrinsic Hamiltonian in the most straightforward way possible and check the moment of inertia obtained from a cranking-model calculation[6] with that obtained directly from the rotational spectra. We next study the consequences, for a simple configuration, of mixing a short-range $\delta$-function force with the long-range "$P_2$" force. Finally, we consider some possible consequences of the model for direct-interaction inelastic scattering processes.

## II. ROTATIONAL SPECTRA

We introduce the operators[1]

$$\begin{aligned} &L_\tau=\textstyle\sum_i L_\tau(i),\\ &Q_\tau=\textstyle\sum_i Q_\tau(i)=\sum_i [r_i^2 C_{2\tau}(\hat{r}_i)+p_i^2 C_{2\tau}(\hat{p}_i)], \end{aligned} \tag{2.1}$$

where

$$\begin{gathered} L_{+1}=-(1/\sqrt{2})(L_x+iL_y),\quad L_0=L_z,\\ L_{-1}=(1/\sqrt{2})(L_x-iL_y),\\ C_{2\tau}(\hat{r}_i)=(4\pi/5)^{\frac{1}{2}}Y_{2\tau}(\theta_i,\varphi_i), \end{gathered}$$

e.g.,

$$Q_0(i)=\tfrac{1}{2}(2z_i^2-x_i^2-y_i^2)+\tfrac{1}{2}(2p_{z_i}^2-p_{x_i}^2-p_{y_i}^2),$$

which obey the commutation rules

$$\begin{aligned} &[H_0,L_\tau]=0,\quad [H_0,Q_\tau]=0,\\ &[L_\tau,L_{\tau'}]=-(2)^{\frac{1}{2}}\langle 1\tau 1\tau'|111\tau+\tau'\rangle L_{\tau+\tau'},\\ &[Q_\tau,L_{\tau'}]=-(6)^{\frac{1}{2}}\langle 2\tau 1\tau'|212\tau+\tau'\rangle Q_{\tau+\tau'},\\ &[Q_\tau,Q_{\tau'}]=3(10)^{\frac{1}{2}}\langle 2\tau 2\tau'|221\tau+\tau'\rangle L_{\tau+\tau'}. \end{aligned} \tag{2.2}$$

If one writes out the explicit formulas for the Clebsch-Gordan coefficients, one sees that the second line contains the usual angular momentum commutation relations and that the third line contains the commutation relations which characterize the $Q_\tau$ as a second-rank spherical tensor.[7]

We next construct an operator

$$T=Q^2+\lambda L^2 \tag{2.3}$$

($\lambda$=arbitrary constant), where

$$L^2=\sum_{i,j=1}^{\nu}\mathbf{L}(i)\cdot\mathbf{L}(j)=\sum_{i,j}[\sum_\tau(-1)^\tau L_{-\tau}(i)L_\tau(j)],$$

$$Q^2=\sum_{i,j=1}^{\nu}\mathbf{Q}(i)\cdot\mathbf{Q}(j)=\sum_{i,j}[\sum_\tau(-1)^\tau Q_{-\tau}(i)Q_\tau(j)].$$

$L^2$ commutes with $\mathbf{L}$, and since $Q^2$ is constructed as a scalar operator it also commutes with $\mathbf{L}$. Thus,

$$[T,L_\tau]=0. \tag{2.4}$$

Since all the $Q_\tau$ and $L_\tau$ commute with $H_0$, so do $Q^2$, $L^2$, and $T$. Thus, we may simultaneously diagonalize the operators $H_0$, $T$, $L^2$, $L_0$.

We now introduce the new Hamiltonian:

$$H'=H_0-T, \tag{2.5}$$

$$H'|E_N t\gamma LM\rangle=E'|E_N t\gamma LM\rangle=(E_N-t)|E_N t\gamma LM\rangle, \tag{2.6}$$

where $E_N$, $t$, $L$, $M$ are the eigenvalues of the operators $H_0$, $T$, $L^2$, $L_0$, respectively, and $\gamma$ stands for all additional unspecified quantum numbers needed to specify distinctly each degenerate eigenstate of $H'$.[8] Thus, the set $H_0$, $T$, $\Gamma$, $L^2$, $L_0$ is a complete commuting set of operators. We want to show that the Hamiltonian $H'$ is degenerate in $L$. This can be done as follows: Using the commutation relations (2.2) we can calculate the commutators of the $Q_\tau$ with $T$. The $Q_\tau$ do not commute with $Q^2$ or with $L^2$. However, for a particular choice $\lambda$ the $Q_\tau$ do commute with $T$.

$$[T,Q_\tau]=[Q^2+\lambda L^2,Q_\tau]=0 \quad \text{for} \quad \lambda=3. \tag{2.7}$$

The existence of the operators $Q_\tau$ which commute with $H'$ but not with $L^2$ indicates that $H'$ is at least partially $L$ degenerate.[9] To see this choose any eigenfunction $|E_N t\gamma LM\rangle$ of $H'$ and operate on it with one of the $Q_\tau$. The resulting function can be expanded in the complete set of states

$$\begin{aligned} &Q_\tau|E_N t\gamma LM\rangle=\sum_{N't'\gamma'L'M'}a_{N't'\gamma'L'M'}|E_{N'}t'\gamma'L'M'\rangle,\\ &a_{N't'\gamma'L'M'}=\langle E_{N'}t'\gamma'L'M'|Q_\tau|E_N t\gamma LM\rangle. \end{aligned} \tag{2.8}$$

[2] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).
[3] H. J. Lipkin, Nuclear Phys. **8**, 421 (1958).
[4] S. A. Moszkowski, Phys. Rev. **110**, 403 (1958).
[5] V. Bargmann and M. Moshinsky, Nuclear Phys. **18**, 697 (1960).
[6] D. Inglis, Phys. Rev. **96**, 1059 (1954).
[7] See, e.g., A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).
[8] For only a few particles (small $\nu$), or for low energy shells (small $N$), the quantum numbers $E_N$, $t$, $L$, $M$ are sufficient to specify every eigenstate of $H'$. For more complicated cases another quantum number is needed. See reference 1.
[9] This point has been made by H. J. Lipkin, see reference 3, and Ann. Phys. **9**, 272 (1960).

Since all the $Q_\tau$ commute with $H_0$ and $T$, the matrix element must vanish unless $E_{N'}=E_N$, $t'=t$. Thus,

$$Q_\tau|E_Nt\gamma LM\rangle=\sum_{\gamma'L'M'}a_{\gamma'L'M'}|E_Nt\gamma'L'M'\rangle. \quad (2.8')$$

We can see that there must exist at least one term in the sum for which $L'\neq L$ as follows:

$$0\neq(L^2Q_\tau-Q_\tau L^2)|E_Nt\gamma LM\rangle$$
$$=L^2Q_\tau|E_Nt\gamma LM\rangle-L(L+1)Q_\tau|E_Nt\gamma LM\rangle.$$

Thus, $Q_\tau|E_Nt\gamma LM\rangle$ cannot be an eigenfunction of $L^2$ with eigenvalue $L(L+1)$ except in the trivial case $L=0$ or the exceptional case $Q_\tau|E_Nt\gamma LM\rangle=0$.

This argument proves that $H'$ is partially $L$ degenerate. It does not give the extent of the degeneracy, i.e., the number of different angular momentum states which can be obtained by repeated application of the $Q_\tau$ operators to a given angular momentum state. This is the problem of finding which values of $L$ are compatible with a given $E_N$, $t$, and $\gamma$, and the answer is provided by Elliott's work.[1] We leave the details of this answer to the Appendix and use only the result that corresponding to each eigenvalue $E_N-t$ of $H'$ there will be a finite set of $L$-degenerate eigenfunctions:

$$H'|E_Nt\gamma LM\rangle=(E_N-t)|E_Nt\gamma LM\rangle, \quad L=L_0, L_1, \cdots, L_{\max}, \quad (2.9)$$

where the values of $L$ which appear depend on the value of $t$ and of $\gamma$.

The final step is to define another Hamiltonian:

$$H=H'+\lambda L^2=H_0-T+\lambda L^2=H_0-Q^2. \quad (2.10)$$

Since the set of operators $H_0$, $T$, $\Gamma$, $L^2$, $L_0$ commutes with $H'$ and with $L^2$, the whole set commutes with $H$ as well. Thus, the eigenfunctions of $H'$ are simultaneously eigenfunctions of $H$ with eigenvalues $E=E_N-t+\lambda L(L+1)$:

$$H|E_Nt\gamma LM\rangle=E|E_Nt\gamma LM\rangle$$
$$=[E_N-t+\lambda L(L+1)]|E_Nt\gamma LM\rangle, \quad (2.11)$$

and all those states of fixed $t$ and $L=L_0, L_1, \cdots, L_{\max}$ which are degenerate eigenstates of $H'$ form a rotational band of eigenstates of $H$, i.e.,

$$E=\text{const}+\lambda L(L+1) \quad \text{for} \quad L=L_0, L_1, \cdots, L_{\max},$$

and different bands are separated by different values of $t$.

We now reduce $Q^2$ to a momentum-independent residual interaction. We split $Q^2$ into a "self-energy" correction to the single-particle central field and a sum of two-particle interactions.

$$Q^2=\sum_i[\sum_\tau(-1)^\tau Q_{-\tau}(i)Q_\tau(i)]+\sum_{i,j\ (i\neq j)}[\sum_\tau(-1)^\tau Q_{-\tau}(i)Q_\tau(j)]. \quad (2.12)$$

Due to the symmetry between the coordinates $\mathbf{y}_i$ and momenta $\mathbf{p}_i$ in the harmonic oscillator, the matrix elements (ME's hereafter) of $2r^2C_{2\tau}(\hat{r})$ within a given oscillator shell are the same as the ME's of the operator $Q_\tau$. Thus, the interaction term is equivalent to

$$4\sum_{i,j\ (i\neq j)}r_i^2r_j^2\mathbf{C}_2(\hat{r}_i)\cdot\mathbf{C}_2(\hat{r}_j)=4\sum_{i,j\ (i\neq j)}r_i^2r_j^2P_2(\cos\theta_{12}), \quad (2.13)$$

within a given oscillator shell. The single-particle term is not so simple since the operator $4r_i^4\mathbf{C}_2(\hat{r}_i)\cdot\mathbf{C}_2(\hat{r}_i)=4r_i^4$ has ME's within a given oscillator shell which are different from the ME's of $Q(i)^2$. One way to find the equivalent coordinate operator for this term is to calculate the single-particle self energy.

$$\epsilon_{Nl}=\langle Nl|Q(i)^2|Nl\rangle$$
$$=4\sum_{l'm'}{}'\sum_\tau|\langle Nl'm'|r_i^2\mathbf{C}_{2\tau}(\hat{r}_i)|Nlm\rangle|^2$$
$$=\frac{4}{2l+1}\sum_{l'}{}'|\langle Nl'||r_i^2\mathbf{C}_2(\hat{r}_i)||Nl\rangle|^2,$$

where the prime on the summation sign means a sum over different angular momentum states in the same oscillator shell. The reduced matrix elements appearing in the sum are:

$$\langle Nl||r^2\mathbf{C}_2||Nl\rangle^2=(N+\tfrac{3}{2})^2[l(l+1)(2l+1)/(2l-1)(2l+3)],$$
$$\langle Nl+2||r^2\mathbf{C}_2||Nl\rangle^2=(N-l)(N+l+3)\tfrac{3}{2}[(l+1)(l+2)/(2l+3)],$$
$$\langle Nl-2||r^2\mathbf{C}_2||Nl\rangle^2=(N-l+2)(N+l+1)\tfrac{3}{2}[(l-1)l/(2l-1)].$$

The result of the sum is

$$\epsilon_{Nl}=4N(N+3)-3l(l+1).$$

Comparing this result with

$$\langle Nl|r^4|Nl\rangle=\tfrac{3}{2}N(N+3)-\tfrac{1}{2}l(l+1)+15/4,$$

we see that the single-particle term is equivalent to

$$6r_i^4-5(N+\tfrac{3}{2})^2-45/4 \quad (2.14)$$

within an oscillator shell. Putting these results together, we have

$$Q^2\doteq\sum_{i,j\ (i\neq j)}4r_i^2r_j^2P_2(\cos\theta_{ij})+\sum_i[6r_i^4-5(N+\tfrac{3}{2})^2-45/4], \quad (2.15)$$

where $\doteq$ means within an oscillator shell, i.e., all ME's of the operator $Q^2$ between any two states of $\nu$ particles in the $N$ shell are equal to the same ME's of the modified "$P_2$" force appearing in Eq. (2.15). Strict equality does not hold because the operator $Q^2$ which commutes with all the $H_0(i)$ has no nonzero ME's con-

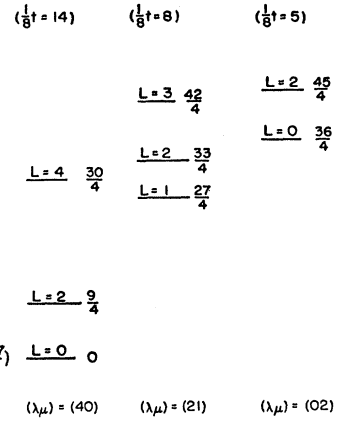


FIG. 1. Energy spectrum of two particles in $2s\ 1d$ shell with "$P_2$" interaction.

necting states of all $\nu$ particles in the $N$ shell with states in which some of the particles are in another shell, while the "$P_2$" operator which does not commute with the $H_0(i)$ does have such nonzero ME's. The term $\sum_i[5(N+\frac{3}{2})^2-45/4]=\nu(5N^2+15N+45/2)$ is a constant within an oscillator shell. Thus, up to this constant, we may write

$$Q^2\doteq\sum_{i<j}8r_i^2r_j^2P_2(\cos\theta_{ij})+\sum_i 6r_i^4, \tag{2.16}$$

which means that the spectrum of $Q^2$ is the same, up to the constant, as the spectrum of the modified "$P_2$" force diagonalized within an oscillator shell.

We can check that (2.16) is the form of the "$P_2$" force which gives the rotational spectrum (2.11) by explicitly diagonalizing the energy matrix for the simple case of two particles in the $N=2$ shell. We take[10]

$$H=H_0-\tfrac{1}{8}Q^2=H_0+V. \tag{2.17}$$

For two particles $V$ is

$$V=V(12)=-r_1^2r_2^2P_2(\cos\theta_{12})-\tfrac{3}{4}r_1^4-\tfrac{3}{4}r_2^4. \tag{2.18}$$

$H_0=H_0(1)+H_0(2)$ just adds a constant energy $2(N+\frac{3}{2})=7$ to all levels, so we drop it and diagonalize $V$. Since $V$ is a scalar under rotation and symmetric under particle interchange we can partially diagonalize the matrix by choosing angular momentum eigenfunctions of definite symmetry as a basis. A complete set of orbital states for two particles in the $|2s\ 1d|$ shell is:

$s^2(0)$, $d^2(0)$, $s^1d^1(2)$, $d^2(2)$, $d^2(4)$, symmetric in $1\leftrightarrow 2$;
$d^2(1)$, $s^1d^1(2)$, $d^2(3)$, antisymmetric in $1\leftrightarrow 2$.

[10] Clearly we can multiply the operator $T$ by an arbitrary constant $g$ without affecting the above development. The only modification is in the Hamiltonian (2.10) which becomes

$$H=H'+g\lambda L^2=H_0-gT+g\lambda L^2=H_0-gQ^2, \tag{2.10'}$$

which has energy levels

$$E=E_N-gt+g\lambda L(L+1). \tag{2.11'}$$

In Eq. (2.17) we choose $g=\frac{1}{8}$ for convenience.

The ME's of (2.18) are calculated from

$$\begin{aligned}&\langle Nl_1'l_2'L'M'|r_1^2r_2^2P_2(\cos\theta_{12})|Nl_1l_2LM\rangle\\&\quad=\langle Nl_1'|r_1^2|Nl_1\rangle\langle Nl_2'|r_2^2|Nl_2\rangle\\&\qquad\times\langle l_1'l_2'L'M'|P_2(\cos\theta_{12})|l_1l_2LM\rangle\\&\langle Nl_1'l_2'L'M'|r_1^4+r_2^4|Nl_1l_2LM\rangle\\&\quad=\delta_{L'L}\delta_{M'M}\delta_{l_1'l_1}\delta_{l_2'l_2}\\&\qquad\times[\langle Nl_1|r_1^4|Nl_1\rangle+\langle Nl_2|r_2^4|Nl_2\rangle].\end{aligned} \tag{2.19}$$

The radial integrals are

$$\begin{aligned}&\langle Nl|r^2|Nl\rangle=N+\tfrac{3}{2},\\&\langle Nl+2|r^2|Nl\rangle=[(N-l)(N+l+3)]^{\frac{1}{2}},\\&\langle Nl|r^4|Nl\rangle=\tfrac{3}{2}N(N+3)-\tfrac{1}{2}l(l+1)+15/4.\end{aligned} \tag{2.20}$$

The angular integrals are

$$\begin{aligned}&\langle l_1'l_2'L'M'|P_2(\cos\theta_{12})|l_1l_2LM\rangle\\&\quad=(-1)^{l_1+l_2+L}\delta_{L'L}\delta_{M'M}\begin{Bmatrix}l_1l_2L\\l_2'l_1'2\end{Bmatrix}\\&\qquad\times\langle l_1'||\mathbf{C}_2||l_1\rangle\langle l_2'||\mathbf{C}_2||l_2\rangle,\end{aligned} \tag{2.21}$$

where the curly bracket symbol is the well-known Wigner 6-$j$ symbol. Using these formulas we calculate the ME's and diagonalize the submatrices. The lowest eigenvalue is one of the $L=0$ eigenvalues. Its value is $-257/8$. Taking this as an arbitrary zero, the spectrum is as shown in Fig. 1.[11]

We see that there are three rotational bands, the levels of each of which fit Eq. (2.11')

$$\begin{aligned}&E=-gt+g\lambda L(L+1), \quad \lambda=3,\ g=\tfrac{1}{8};\\&E=\text{const}+L(L+1)/2I, \quad I=\tfrac{4}{3}.\end{aligned} \tag{2.22}$$

The eigenfunctions for the $t/8=14$ states are:

$$\begin{aligned}&\psi(L=0)=(5/9)^{\frac{1}{2}}s^2(0)+(4/9)^{\frac{1}{2}}d^2(0),\\&\psi(L=2)=(7/9)^{\frac{1}{2}}s^1d^1(2)-(2/9)^{\frac{1}{2}}d^2(2),\\&\psi(L=4)=d^2(4).\end{aligned}$$

The eigenvalues $t$ of $T(T=Q^2+3L^2)$ were calculated directly from the absolute positions of the $L=0$ levels. The absolute value of the lowest $L=0$ eigenvalue is $-257/8+(1/8)\nu(5N^2+15N+45/2)=-14$, the second term coming from the constant dropped in Eq. (2.16).

## III. MOMENT OF INERTIA AND THE CRANKING MODEL

The Hamiltonian of the Bohr and Mottelson collective model in the strong coupling case is (approximately) separable into a rotational Hamiltonian and an intrinsic Hamiltonian. In the first approximation the intrinsic Hamiltonian is taken to be a deformed single-particle potential well. The moment of inertia appearing in the

[11] This $(\lambda\mu)$ notation refers to Elliott's group theoretical classification reviewed in the Appendix.

rotational term is a parameter in the model. Inglis[6] has suggested a method of calculating the moment of inertia. In this "cranking-model" calculation one assumes a deformed potential well filled with nucleons. One then takes the potential well to be rotating with a small angular velocity $\omega$ so that the nucleons can follow it adiabatically, and calculates the extra angular momentum the nucleons must have to follow the deformed well. The coefficient of $\omega$ appearing in this expression is taken to be the moment of inertia.

The Hamiltonian (2.10) which we have been discussing is exactly separable into a rotational term and a remainder:

$$H=H_{\rm rot}+H'=g\lambda L^2+H_0-gt. \tag{3.1}$$

$H'$, as it stands, cannot be taken as the intrinsic Hamiltonian appearing in the rotational model or the cranking model because it is spherically symmetric.[12] To obtain a deformed single-particle potential well from $H'$ we can carry out a Hartree self-consistent field calculation.

Using the commutation relations (2.2) and the symmetry properties of the Clebsch-Gordan coefficients, one can show that the operator $T$ can be written in the factored form

$$\begin{aligned}T=&\sum_{i,j}\{\sum_\tau(-1)^\tau[Q_{-\tau}(i)+i(\lambda)^{\frac{1}{2}}L_{-\tau}(i)]\\&\times[Q_\tau(j)-i(\lambda)^{\frac{1}{2}}L_\tau(j)]\},\\=&\sum_{i,j}\mathbf{X}(i)\cdot\mathbf{X}^*(j),\qquad(\tau=0,\pm1,\pm2;\ L_{\pm2}=0)\end{aligned} \tag{3.2}$$

where we have defined

$$\begin{aligned}X_\tau(i)&=Q_\tau(i)+i(\lambda)^{\frac{1}{2}}L_\tau(i),\\X_\tau^*(i)&=Q_\tau(i)-i(\lambda)^{\frac{1}{2}}L_\tau(i).\end{aligned} \tag{3.3}$$

Taking $T=\sum_{i,j}T(ij)=\frac{1}{2}\sum_{i,j}2T(ij)$, the Hartree Hamiltonian is

$$H'_{\rm Hart}=\sum_i[H_0(i)+U(i)], \tag{3.4}$$

$$\begin{aligned}U(i)&=-2g\sum_j\int\phi^*(j)T(ij)\phi(j)d\tau_j-\alpha_i\\&=-2g\sum_\tau(-1)^\tau X_{-\tau}(i)\langle\sum_j X_\tau^*(j)\rangle-\alpha_i,\end{aligned} \tag{3.5}$$

$$\alpha_i=-g\sum_\tau(-1)^\tau\langle X_{-\tau}(i)\rangle\langle\sum_j X_\tau^*(j)\rangle, \tag{3.6}$$

and

$$[H_0(i)+U(i)]\phi(i)=\epsilon_i\phi(i). \tag{3.7}$$

These equations differ from the usual form of the Hartree equations[13] in the addition of the $C$-number term $\alpha_i$ to $U(i)$ and has the consequence that the Hartree approximation to the energy $E'$ is just the sum of the single-particle energies:

$$\chi=\prod_i\phi(i), \tag{3.8}$$

$$H'_{\rm Hart}\chi=E'_{\rm Hart}\chi, \tag{3.9}$$

$$E'_{\rm Hart}=\sum_i\epsilon_i=\int\chi^*H'\chi d\tau. \tag{3.10}$$

By choosing appropriate single-particle functions we can immediately obtain self-consistent solutions to Eqs. (3.4) through (3.7). The appropriate single-particle functions are the eigenfunctions $\phi_{qK}(i)$ of the operators $Q_0(i)$ and $L_0(i)$. (We denote the eigenvalue of $L_0$ by $K$ instead of by $m$ to conform with the usual rotational model notation for the intrinsic angular momentum.) With this choice for the $\phi(i)$ we have

$$\begin{aligned}\langle X_\tau^*(j)\rangle&=\langle qK(j)|X_\tau^*(j)|qK(j)\rangle\\&=[q(j)-i(\lambda)^{\frac{1}{2}}K(j)]\delta_{\tau0},\end{aligned} \tag{3.11}$$

so

$$U(i)=-2g\textstyle\sum_j[q(j)-i(\lambda)^{\frac{1}{2}}K(j)]X_0(i)-\alpha_i \tag{3.12}$$

and

$$\alpha_i=-g\textstyle\sum_j[q(j)-i(\lambda)^{\frac{1}{2}}K(j)][q(i)+i(\lambda)^{\frac{1}{2}}K(i)]. \tag{3.13}$$

Since the functions $\chi=\prod_i\phi_{qK}(i)$ are eigenfunctions of $\sum_i X_0(i)$ and hence of $U(i)$, the fields $U(i)$ are self-consistent and the energy eigenvalues are given by

$$H'_{\rm Hart}\chi=\{E_N-g\sum_{i,j}[q(i)q(j)+\lambda K(i)K(j)]\}\chi. \tag{3.14}$$

Since the operator $X_0(i)$ is not spherically symmetric we can take $H'_{\rm Hart}$ as the "intrinsic" Hamiltonian $H_{\rm intr}$ to be used in the cranking-model calculation of the moment of inertia appearing in the Hamiltonian (3.1). Thus we have

$$H=g\lambda L^2+H_0-gT=H_{\rm rot}+H_{\rm intr}+(H'-H'_{\rm Hart}). \tag{3.15}$$

The cranking-model formula for the moment of inertia is

$$I=2\sum_{n\ (n\neq0)}\frac{|\langle0|L_x|n\rangle|^2}{E_n-E_0}, \tag{3.16}$$

where the eigenstates and eigenvalues appearing in the formula are those of the "intrinsic" Hamiltonian of Eq. (3.15).

[12] The Hamiltonian (3.1) differs fundamentally from the Bohr and Mottelson Hamiltonian in that there is no separation into intrinsic and collective coordinates. The angular momentum $L$ is the total (orbital) angular momentum of the particles outside of the closed shells and operates on the same coordinates that $H'$ operates on. There is no rotation of the core. The eigenfunctions (e.g. 2.23) of (3.1) do not separate into the product of a function of collective coordinates and a function of intrinsic particle coordinates as do the eigenfunctions of the Bohr and Mottelson Hamiltonian (see reference 2). [It is the primary point of Elliott's papers (reference 1) that these eigenfunctions, although apparently so different, have many similar properties.] Another difference lies in the absence of spin in the Elliott model. Only in case the total spin $S=0$ does the spectrum (2.11) have the form $J(J+1)/2I$ of the rotational model.

[13] See any standard reference on quantum mechanics, e.g., L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, New York, 1955).

In order to carry out the calculation and to estimate the magnitude of the neglected term $(H'-H'_{\text{Hart}})$ we need to know the eigenvalues $q$, $K$ and the single-particle eigenfunctions $\phi_{qK}$. The eigenvalues are

$$\begin{aligned} q(i)&=3N_z(i)-N, \\ K(i)&=\langle L_z(i)\rangle=m. \end{aligned} \tag{3.17}$$

Thus, a single particle in the $N=1$ shell has one quantum of energy which may be along the $z$ axis in which case $q=2$ and $K=0$, or in the $xy$ plane in which case $q=-1$ and $K=\pm 1$. Similarly, for the $N=2$ shell the possible single-particle states are:

| $N_z$ | $q$ | $K$ | No. of states |
|---|---|---|---|
| 2 | 4 | 0 | 1 |
| 1 | 1 | $\pm 1$ | 2 |
| 0 | $-2$ | $0, \pm 2$ | 3 |
| | | Total | 6 |

etc. The eigenfunctions $\phi_{qK}$ can be obtained by solving the Schrödinger equation for the harmonic oscillator in cylindrical coordinates since this choice of coordinates leaves $Q_0=3H_{0z}-H_0$ and $L_0=L_z$ constants of the motion.

The choice of the $z$ axis with respect to which the operators $Q_0$ and $L_0$ are defined is arbitrary since the original Hamiltonian (3.1) is spherically symmetric. This leads to degeneracy in the eigenvalues (3.14) of $H_{\text{intr}}$. For example, with one particle in the $N=1$ shell $E_{\text{intr}}=-4g$ [we neglect the $E_N$ which is constant for a given number of particles in a given shell and which drops out of the energy denominators in (3.16),] whether the quantum is along the $z$ axis or in the $xy$ plane. The distortion of the field, which is created by the particles themselves, is seen when a second particle is added. Then the lowest eigenvalue, $E_{\text{intr}}=-16g$, is obtained if both quanta are along the $z$ axis or if both are circulating in the $xy$ plane in the same sense ($K=+1$ or $K=-1$ for both quanta). If one quantum is along the $z$ axis and one is in the $xy$ plane or if both are in the $xy$ plane but are circulating in opposite directions ($K=+1$ for one, $-1$ for the other), then $E_{\text{intr}}=-4g$; and similarly for more particles or higher shells, the lowest eigenvalue occurs when all quanta are aligned along the $z$ axis or all quanta are circulating in the same sense in the $xy$ plane.

Thus we can form the ground state $|0\rangle$ by lining all the particles along some axis and calling this axis the $z$ axis. Then the ground state is

$$\begin{aligned} |0\rangle&=\prod_{i=1}^{\nu}\phi_{q_0K_0}(i), \\ q_0&=2N, \quad K_0=0. \end{aligned} \tag{3.18}$$

The states $|n\rangle$ having nonzero ME's with $|0\rangle$ through the operator $L_x=\sum_i L_x(i)$ are all those states with $\nu-1$ particles in the single-particle state $q_0$, $K_0$ and one particle in the state $q$, $K=K_0\pm 1$. There are $2\nu$ such states ($K=\pm 1$ for each particle). The ME's $\langle 0|L_x|n\rangle=\langle q_0K_0|L_x|qK\rangle$ can be calculated by expanding the functions $\phi_{qK}$ in terms of the spherical eigenfunctions $\psi_{lm}$. Since both the $\phi_{NqK}$ and the $\psi_{Nlm}$ are known the coefficients appearing in the expansion can be calculated by doing the integration $\int\phi_{qK}^*\psi_{lm}d\tau$.

In the $N=1$ shell the problem is particularly simple because each function $\phi_{qK}$ must be equal to the function $\psi_{l=1,m=K}$ up to a phase. Then using the intrinsic energy eigenvalues given by Eq. (3.14) we can carry out the calculation of $I$ by (3.16). If it were permissible to put arbitrarily many particles in the ground state (3.18), then the result would be

$$I=2\frac{2\nu|1/\sqrt{2}|^2}{12g(\nu-1)}\xrightarrow[\nu\to\infty]{}\frac{1}{6g}=I_{\text{eff}}. \tag{3.19}$$

In the $N=2$ shell a similar result is obtained

$$I=2\frac{2\nu|\pm 1|^2}{12g(2\nu-1)}\xrightarrow[\nu\to\infty]{}\frac{1}{6g}=I_{\text{eff}}. \tag{3.20}$$

In fact, the Pauli principle allows no states in which more than four nucleons have the same "orbital" quantum numbers $q_0$, $K_0$. Thus, (3.18) can be taken as the ground state only for $\nu\leq 4$, and Eqs. (3.19) and (3.20) also are correct only for $\nu\leq 4$. For these values of $\nu$, $I$ ranges from $2I_{\text{eff}}$ to $(4/3)I_{\text{eff}}$ for the $N=1$ shell and from $2I_{\text{eff}}$ to $(8/7)I_{\text{eff}}$ for the $N=2$ shell, except for $\nu=1$ in the $N=1$ shell. In this case, the states $q=1$, $K=\pm 1$ are degenerate with the state $q_0$, $K_0=0$ so the formula is inapplicable. For more than four particles in a shell, one can form the appropriate ground state by multiplying (3.18) by additional single-particle functions with different $q$, $K$ and carry out the calculations equivalent to (3.19) and (3.20). In the $N=1$ shell one finds values of $I$ ranging from $(2/3)I_{\text{eff}}$ to $(8/3)I_{\text{eff}}$ for $4<\nu<12$.

Finally, to get an estimate of the magnitude of the neglected correction term in (3.15), we note that the magnitude of the ground state eigenvalue of $H'_{\text{Hart}}$ is given by (3.14) as $4\nu^2N^2$. The ground state eigenvalue of $H'$ is $t_{\max}=4\nu^2N^2+12\nu N$.[14] If we take these numbers as giving a measure of the relative magnitudes of $H'$ and $H'_{\text{Hart}}$ we have

$$\frac{H'-H'_{\text{Hart}}}{H'_{\text{Hart}}}\sim\frac{1}{\nu}\quad(\nu\to\infty). \tag{3.21}$$

Thus, in the same limit in which $I\to I_{\text{eff}}$, the correction term to the Hamiltonian (3.15) becomes small compared to the term used in the calculation.

[14] This expression is obtained by writing out $T=Q^2+3L^2$ in terms of creation and destruction operators and considering the case $\nu N=N_z$, $N_x=N_y=0$. Since we do not use this result elsewhere we omit the calculation. We note that this expression gives the result obtained at the end of Sec. II.

## IV. δ-FUNCTION INTERACTION

In Sec. II we discussed the general problem of $\nu$ particles in the $N$th oscillator shell interacting through the "$P_2$" force. When one is dealing with particles within a shell, the "$P_2$" force represents the long-range part of the multipole expansion of an arbitrary residual interaction.[15] We saw that this long-range force led to rotational spectra, a result usually explained as a collective effect. It is of some interest to consider also a short-range force, typified by a δ-function interaction, or a mixture of long- and short-range interactions. The simple methods used in Sec. II are not directly applicable to an interaction other than the "$P_2$" interaction; however, we did see that the energy matrix could easily be directly diagonalized for a simple configuration. We can carry out an exactly similar calculation for two particles in the $2s$ $1d$ shell interacting through a mixture of "$P_2$" and δ-function forces if we make a multipole expansion of the δ-function interaction.

$$V(\mathbf{r}_1,\mathbf{r}_2)=-g'[r_1^2r_2^2P_2(\cos\theta_{12})+\tfrac{3}{4}r_1^4+\tfrac{3}{4}r_2^4] -a\delta(\mathbf{r}_1-\mathbf{r}_2). \quad (4.1)$$

(We take $g'=8g$ for convenience.) For the δ-function interaction we have

$$a\delta(\mathbf{r}_1-\mathbf{r}_2)=\frac{a}{2\pi}\frac{\delta(r_1-r_2)}{r_1^2}\delta(1-\cos\theta_{12}), \quad (4.2)$$

$$\delta(\mathbf{r}_1-\mathbf{r}_2)=\sum_\kappa\frac{2\kappa+1}{4\pi}\frac{\delta(r_1-r_2)}{r_1^2}P_\kappa(\cos\theta_{12}). \quad (4.3)$$

The ME's of the "$P_2$" part of the interaction have already been calculated in Sec. II. The ME's of the δ-function interaction are

$$\langle Nl_1'l_2'L'M'|a\delta(\mathbf{r}_1-\mathbf{r}_2)|Nl_1l_2LM\rangle =a\sum_\kappa\frac{2\kappa+1}{4\pi}\langle Nl_1'l_2'|\frac{\delta(r_1-r_2)}{r_1^2}|Nl_1l_2\rangle \times\langle l_1'l_2'L'M'|\mathbf{C}_\kappa(1)\cdot\mathbf{C}_\kappa(2)|l_1l_2LM\rangle. \quad (4.4)$$

The radial integral reduces to

$$\int_0^\infty R_{Nl_1'}(r)R_{Nl_2'}(r)R_{Nl_1}(r)R_{Nl_2}(r)r^2dr. \quad (4.5)$$

The angular ME is

$$\langle l_1'l_2'L'M'|\mathbf{C}_\kappa(1)\cdot\mathbf{C}_\kappa(2)|l_1l_2LM\rangle =\delta_{L'L}\delta_{M'M}(-1)^{l_1+l_2+L}\begin{Bmatrix} l_1 & l_2 & L\\ l_2' & l_1' & \kappa\end{Bmatrix} \times\langle l_1'||\mathbf{C}_\kappa(1)||l_1\rangle\langle l_2'||\mathbf{C}_\kappa(2)||l_2\rangle, \quad (4.6)$$

[15] See the article by B. R. Mottelson, in *The Many Body Problem*, edited by C. Dewitt (John Wiley & Sons, Inc., New York, 1959).

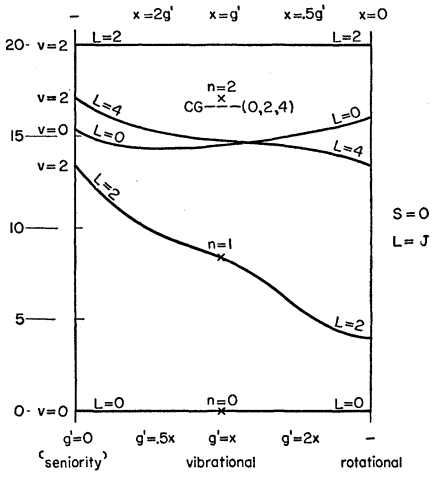


FIG. 2. Energy spectrum for two particles in $2s$ $1d$ shell.
$V(12)=-g'[r_1^2r_2^2P_2(\cos\theta_{12})+\frac{3}{4}r_1^4+\frac{3}{4}r_2^4]-x(40\pi)\delta(\mathbf{r}_1-\mathbf{r}_2)$,
where $x=a/40\pi$. We have arbitrarily normalized $g'$ and $x$ so that the total separation of the lowest $L=0$ and the highest $L=2$ levels is constant at 20 (arbitrary units). Thus there is one other parameter, the ratio, for the three remaining levels. The $S=1$ levels have been omitted.

which has the selection rules

$$\Delta(l_1'\,\kappa\,l_1),\quad \Delta(l_2'\,\kappa\,l_2)$$
$$l_1'+\kappa+l_1=\text{even},\quad l_2'+\kappa+l_2=\text{even}.$$

In the $2s$ $1d$ shell this restricts $\kappa$ to at most 0, 2, 4. We calculate only the symmetric orbital states for which $S=0$ and $L=J$. We omit the triplet states. We diagonalize the matrices for different values of the parameters $g'$ and $a$ and obtain Fig. 2.

In the limit of a pure δ-function force we see a large gap between the lowest $L=0$ level and the rest of the levels. This is characteristic of the spectra obtained in the seniority scheme of $jj$-coupling with a δ-function interaction, in which there is a low-lying $v=0$, $j^2(0)$ level and $j-\frac{1}{2}$ closely grouped higher levels having $v=2$ and $J=2, 4, \cdots, 2j-1$. The spectrum in Fig. 2 differs from this in having two seniority zero, $L=0$ states. This comes about because we have mixed two degenerate $l$ shells rather than working with a single $j$ shell. The two different $L=0$ states are essentially from two different pure $l^2$ configurations, i.e., the overlap between the calculated eigenfunction for the lowest $L=0$ level and a pure $d^2(0)$ wave function is 92%. (The two $L=2$ levels are well mixed between $l=0$ and $l=2$. The $l=4$ level is, of course, pure $d^2$.)

In the limit of pure "$P_2$" force we see the exact rotational spectrum calculated previously.

In the region intermediate between the pure δ-function force and the pure "$P_2$" force we have a spectrum which, except for the high lying $L=2$ level, closely resembles the vibrational spectrum predicted by the vibrational form of the collective model. The vibrational model predicts evenly spaced levels with $J=0$ lowest,

$J=2$ next, and a triply degenerate $J=0$, 2, 4 level after that. In the figure we have labeled $n=0$, 1, 2 for zero, one, and two vibrational quanta. The dotted line labeled CG refers to the "center of gravity" (average energy weighted by statistical factor $2L+1$) of the $L=0$, 2, 4 levels obtained from our calculation.

In the standard form of the shell model, $O^{18}$ is described as a nucleus with two neutrons in the $d_{\frac{5}{2}}$ shell. In the "$P_2$" model where one ignores the spin-orbit energy and has the harmonic oscillator degeneracy, $O^{18}$ is described as a nucleus with two neutrons in the $2s$ $1d$ shell. In Fig. 3 we compare our calculation for $x=0.7g$ with the predictions of the vibrational model and with the experimentally observed level scheme.[16]

Although the vibrational model gives a good fit it does not predict the order of the close together $J=0$, 2, 4 levels. Our $LS$-coupling calculation with a mixture of "$P_2$" and $\delta$-function interactions, although placing the highest $J=2$ level too high, does predict the correct order of these three levels. However, the unjustified neglect of the spin-orbit energy and of the triplet spin states prevents one from taking this calculation seriously.

## V. THE "$P_2$" FORCE AND DIRECT INTERACTION SCATTERING

The Elliott model, or "$P_2$" model, involves two primary features. The first is the use of $LS$ coupling and harmonic oscillator wave functions with the consequent degeneracies and symmetry properties. The second is the residual "$P_2$" interaction, assumed to act only within an oscillator shell, which leads to $L(L+1)$ rotational spectra in the model. This interaction is characterized by a strength parameter $g'$, and the effective moment of inertia is inversely proportional to this parameter.

$$I_{\text{eff}}=4/3g'. \tag{5.1}$$

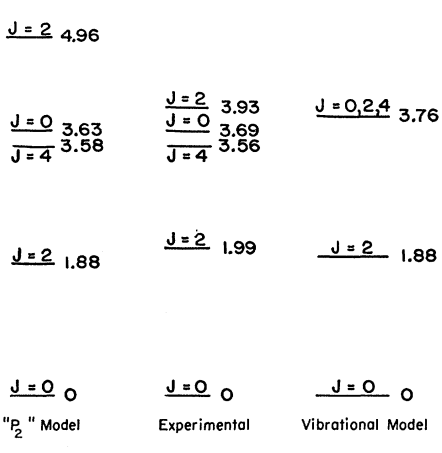


FIG. 3. Energy spectrum of $O^{18}$ (Mev). The $S=1$ levels have been omitted in the "$P_2$" model.

[16] N. Jarmie and M. G. Silbert, in *Proceedings of the International Conference on Nuclear Structure, Kingston, Canada* (University of Toronto Press, Toronto, 1960), p. 491.

We would like to have an independent method of determining this parameter.[17] Also, because the spin-orbit interaction is neglected, and, in particular, because the rotational spectra obtained from the model are of the form $L(L+1)$ rather than $J(J+1)$, the model is clearly not a realistic model of the nucleus. However, we might consider the "$P_2$" interaction apart from the specific Elliott model and without the restriction that it act only within a shell. An arbitrary shell model residual interaction will, in general, contain many multipoles in its multipole expansion. The results obtained from the "$P_2$" model *suggest* that in nuclei which exhibit rotational spectra the $P_2$ multipole may be the dominant multipole, and that the moments of inertia of these nuclei are related to the strength parameter $g'$ of the "$P_2$" interaction by Eq. (5.1). In addition, the "$P_2$" model suggests that the radial factor of the $P_2$ multipole is $r_i^2 r_j^2$ although this is not true for an arbitrary residual interaction.

Now consider the scattering of an incident nucleon from a target nucleus. For high enough energies ($E>10$ Mev), inelastic processes which leave the target nucleus in a resolved final state can be described as direct interaction processes.[18] Because of the short range of nuclear forces there is no interaction (except the Coulomb interaction for an incident proton) until the incident nucleon is very close to the surface of, or inside of, the target nucleus. Inside of the nucleus the incident nucleon interacts with all of the nucleons of the target nucleus. If the sum of these nucleon-nucleon interactions can be replaced by an average single-particle central potential which contributes only to the elastic scattering and a sum of two-particle residual interactions, then the interaction appearing in the Born approximation matrix element is the residual interaction. Then the Born approximation inelastic scattering amplitude is

$$f_{n0}(\mathbf{k}_f,\mathbf{k}_i) = -\frac{1}{4\pi}\frac{2M}{\hbar^2}\int e^{-i\mathbf{k}_f\cdot\mathbf{r}_0}\psi_n(\mathbf{r}_1\cdots\mathbf{r}_A)V(\mathbf{r}_0,\mathbf{r}_1,\cdots,\mathbf{r}_A) \times\psi_0(\mathbf{r}_1\cdots\mathbf{r}_A)e^{i\mathbf{k}_i\cdot\mathbf{r}_0}d\tau_0 d\tau_1\cdots d\tau_A, \tag{5.2}$$

where $\mathbf{r}_0$ is the coordinate vector of the incoming and

[17] In the collective model of rotational nuclei the moment of inertia is a parameter. In the "$P_2$" model of rotational nuclei the moment of inertia does not appear directly as a parameter. Its place is taken by the parameter $g$ (or $g'=8g$). It is difficult to obtain independent information about this parameter because it does not appear in the wave functions, i.e., the eigenfunctions of $H_0$ are simultaneously the eigenfunctions of $H_0-gQ^2$. [A particular example is the set of eigenfunctions explicitly calculated in Sec. II for $g=\frac{1}{8}$ which are the same as the eigenfunctions given by Elliott (reference 1) for $g=0$.] This has the consequence that properties of the nucleus, such as the quadrupole moment, which depend on the nuclear wave function are independent of $g$ as long as the residual interaction is pure "$P_2$" interaction.

[18] C. A. Levinson, in *Nuclear Spectroscopy*, edited by F. Ajzenberg-Selove (Academic Press Inc., New York, 1960), Part B, p. 670.

outgoing particle, $M$ is the reduced mass of the particle $\{M=[A/(A+1)]M_p\}$, and $\mathbf{r}_1, \cdots, \mathbf{r}_A$ are the coordinates of the nucleons in the target nucleus which is initially in the state $\psi_0$ and finally in the state $\psi_n$. The suggested residual interaction is:

$$V(\mathbf{r}_0,\mathbf{r}_1,\cdots,\mathbf{r}_A)=-g'\frac{M^2\omega_0^3}{\hbar}\sum_{j=1}^{A} r_0^2 r_j^2 P_2(\cos\theta_{0j}), \quad r_0\leq R \quad (5.3)$$

$$=0, \quad r_0>R.$$

The single-particle $r^4$ terms in the "$P_2$" force are dropped because they do not contribute to the inelastic scattering. The extra constants appear for dimensional reasons.[19] $R$ is some radial distance, somewhat larger than the nuclear radius $R_N$, to be determined by fitting the observed angular distributions of inelastically scattered nucleons. The scattering amplitude does not specify the initial or final spin states of the incident particle. Since the interaction is spin-independent, the sum and average over final and initial spin states of the incident particle just give a factor of 1.

In taking (5.2) to be the scattering amplitude we have ignored the requirement that the total wave function be antisymmetric. Since the nuclear wave functions $\psi_0$ and $\psi_n$ are assumed to be antisymmetric, this means that the actual scattering amplitude is a linear combination of (5.2) with all of the exchange amplitudes obtained from (5.2) by interchanging the coordinates $\mathbf{r}_0$ and $\mathbf{r}_j$ $(j=1, 2, \cdots, A)$ in the final wave function $\exp(i\mathbf{k}_f\cdot\mathbf{r}_0)\psi_n(\mathbf{r}_1\cdots\mathbf{r}_A)$. However, these exchange amplitudes, which are of the form

$$\int e^{i\mathbf{k}_f\cdot\mathbf{r}_j}\psi_n(\mathbf{r}_1,\cdots,\mathbf{r}_0,\cdots,\mathbf{r}_A)V(\mathbf{r}_0,\mathbf{r}_1,\cdots,\mathbf{r}_A) \times\psi_0(\mathbf{r}_1,\cdots,\mathbf{r}_j,\cdots\mathbf{r}_A)e^{-i\mathbf{k}_i\cdot\mathbf{r}_0}d\tau_0 d\tau_1\cdots d\tau_A,$$

are all quite small relative to (5.1), at least for small scattering angles, if $k_iR\gg1$. This is also a condition for the applicability of the Born approximation. In a typical experiment the incident energy is 14 Mev and $R$ is about $6\times10^{-13}$ cm, in which case $kR\approx5$. Taking into account the strength of the nuclear forces, this indicates appreciable distortion of the incident wave. Another problem, in the case of incident protons, is the neglect of the Coulomb interaction. The relevant parameter[20] is $\eta=Ze^2/\hbar v$ which is approximately $\frac{1}{2}$ for nuclei around $Z=10$ bombarded by 14-Mev protons, indicating appreciable Coulomb distortion of the incident wave. The Coulomb interaction has two effects. The first is to add the cross section for Coulomb excitation of a given nuclear level to the cross section for excitation by a direct nucleon-nucleon interaction. This is a small effect since the Coulomb cross section is small compared to the nuclear reaction cross section (an estimate for a specific case will be given below). The other effect is to elastically scatter nucleons away from the nucleus so that they do not undergo a nuclear reaction at all. The Born approximation, by ignoring this effect, overestimates the nuclear reaction cross section. In order to see what information might be extracted from inelastic scattering experiments, we go ahead and calculate the cross section from the amplitude (5.2), bearing in mind its inadequacies.

The cross section is:

$$\frac{d\sigma(\theta)}{d\Omega}=\frac{k_f}{k_i}|f_{n0}(\mathbf{k}_f,\mathbf{k}_i)|^2 =\frac{4}{5}\frac{k_f}{k_i}g'^2\left(\frac{M\omega_0}{\hbar}\right)^6 \times\{|\langle J_f||\sum_{j\neq0} r_j^2\mathbf{C}_2(\hat{r}_j)||J_i\rangle|^2/(2J_i+1)\} \times R^{10}|f(qR)|^2, \quad (5.4)$$

$$f(qR)=j_3(qR)/qR. \quad (5.5)$$

The angular dependence is contained in the dimensionless factor $f(qR)$. The operator appearing in the nuclear matrix element differs from the electric quadrupole operator only by the constant factor $(1/e)(4\pi/5)^{\frac{1}{2}}$ and in that the sum is over all nucleons weighted equally rather than over protons and neutrons separately weighted by their effective charges. Thus the square of the nuclear matrix element divided by $2J_i+1$ is proportional to the reduced transition probability $B(E2,J_fJ_i)$ for the corresponding $E2$ transition, the constant of proportionality depending only on $Z$ and $A$.

$$\{|\langle J_f||\sum_{j\neq0} r_j^2\mathbf{C}_2(\hat{r}_j)||J_i\rangle|^2/(2J_i+1)\} =\beta B(E2,J_fJ_i)\uparrow, \quad \beta\simeq\frac{4\pi}{5}\left(\frac{1}{e}-\frac{Z}{A}\right)^2\approx\frac{\pi}{5e^2}. \quad (5.6)$$

Then the cross section (5.4) may be reduced to

$$\frac{d\sigma(\theta)}{d\Omega}=9.86\times10^{-6}\frac{k_f}{k_i}g'^2\left(\frac{A}{A+1}\right)^6 B'(E2)[\hbar\omega_0(\text{Mev})]^6 \times[R(\text{fermis})]^{10}|f(qR)|^2 \text{ mb/sr}, \quad (5.7)$$

$$B(E2)=(e^2\times10^{-48}\text{ cm}^4)\times B'(E2).$$

We now consider what can be learned from the cross section. The most striking point is that the operator in the nuclear matrix element is a second rank tensor. Thus, if we consider the scattering of an incident

[19] To go from our original units ($\hbar=M=\omega_0=1$), in which the oscillator Hamiltonian has the form $H=(p^2+r^2)/2$ and $E_N=N+\frac{3}{2}$ to an ordinary system of units in which $H=p^2/2M+M\omega_0^2r^2/2$, $r^2$ is replaced by $M\omega_0r^2/\hbar$ and $V$ is multiplied by the unit of energy $\hbar\omega_0$. In ordinary units, Eq. (5.1) has the form: $I_{\text{eff}}=(4/3g')(\hbar/\omega_0)$.

[20] K. Alder, A. Bohr, T. Huss, B. Mottelson, and A. Winther, Revs. Modern Phys. 28, 432 (1956).

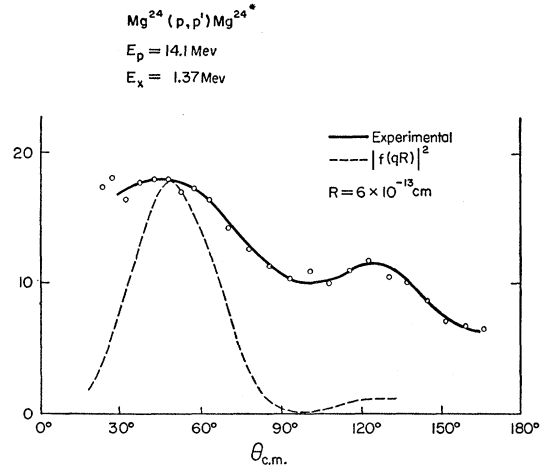


FIG. 4. Differential cross section for protons inelastically scattered from magnesium [experimental curve from Oda *et al.* (see reference 21)].

nucleon from an even-$A$ target nucleus with a rotational band $0^+$ (ground state), $2^+$, $4^+$, $\cdots$, Eq. (5.4) says that the cross section for direct nuclear excitation of the $4^+$ state (or any other higher states in the band) is zero. This will not be the case for an arbitrary residual interaction which contains many multipoles. In particular, a residual interaction which contained the $P_4$ multipole in the form $-g'\sum_{j\neq0} r_0^4 r_j^4 P_4(\cos\theta_{0j})$ would excite the $4^+$ state more strongly than the $2^+$ state if the estimate $B(E4)/B(E2)\approx R^4$ is accurate. Of course this is not a general form of the $P_4$ term which could appear in an arbitrary residual interaction, because we have chosen a particular form of the coefficient $f_4(r_0,r_j)$ appearing in the general multipole expansion. It is also quite possible that the estimate of $B(E4)/B(E2)$ is too large, i.e., the nuclear wave functions may describe a large quadrupole distortion but very little $2^4$-pole distortion, in which case the cross section for direct excitation of the $4^+$ level would be small.

The second point is an independent determination of the strength parameter $g'$ of the "$P_2$" interaction. One would like to compare the values of $g'$ obtained from measured differential cross sections and reduced $E2$ transition probabilities and Eq. (5.7) with the values of $g'$ obtained from observed rotational spectra and Eq. (5.1). There are several difficulties in addition to those involved in the Born approximation. One is the extreme sensitivity of the magnitude of the cross section to the interaction radius $R$ which is only crudely determined from the angular distribution. Another major difficulty is the strong dependence on the additional oscillator parameter $\omega_0$ which is not well known. Finally, there is the dearth of experimental data. Most experiments on inelastic scattering to resolved final nuclear states have been done on relatively light nuclei,[18] whereas most nuclei which exhibit rotational spectra are heavy ($150\lesssim A\lesssim190$, $220\lesssim A$). In the region $A\approx25$ there are some nuclei which exhibit rotational properties. Oda *et al.*[21] have measured differential cross sections for inelastic ($pp'$) scattering to the first-excited state of even-$A$ nuclei in this region. Of these nuclei $Mg^{24}$ shows a definitely rotational spectrum with $J=0$, 2, 4 levels at 0, 1.37, and 4.12 Mev,[22] and the spectrum of $Ne^{20}$ has recently been analyzed as a series of rotational bands although the ground state band is considerably distorted from a $J(J+1)$ spacing.[23]

Equation (5.7) was used to calculate the differential cross section for the inelastic scattering of 14.1-Mev protons exciting the 1.37-Mev level in $Mg^{24}$. An interaction radius $R=6\times10^{-13}$ cm was chosen to fit the angular distribution as shown in Fig. 4. The over-all fit is not good, but the positions of the observed maxima and minima are reproduced closely, and in the forward direction where the Born approximation is most likely to give accurate results the shape is not too bad. Using the experimental value[24] $B'(E2)=0.034$, the experimental peak height of 18 mb is reproduced with $g'=0.1095$ and $\hbar\omega_0=5.56$ Mev, where this value of $g'$ is obtained from Eq. (5.1')[19]:

$$I=(4/3g')(\hbar/\omega_0), \qquad (5.1')$$

taking $I=3\hbar^2/E_x$ with $E_x=1.37$ Mev. This value of $\hbar\omega_0$ is too small for the $2s\,1d$ shell. A more reasonable although perhaps still too small value of $\hbar\omega_0$ is 10 Mev. In this case the value of $g'$ obtained from (5.1') is $g'=0.0609$. Using these values of $g'$ and $\hbar\omega_0$ in Eq. (5.7) we obtain a peak height which is about ten times the observed peak height. If we take $\hbar\omega_0=14$ Mev, which is the value obtained from the Moszkowski estimate,[25] we find that the calculated peak height is about forty times the observed peak height when the consistent value of $g'$ is used. We have pointed out that due to the appreciable Coulomb distortion[26] of the incident plane wave the Born approximation will overestimate the inelastic cross section. Thus, if the effect of the nuclear distortion is small, or goes in the same direction, it is possible that the value of $g'$ obtained from this scattering experiment is consistent with the value obtained from the observed rotational spectrum. However, in the absence of a good

[21] Y. Oda, M. Takeda, N. Takano, T. Yamazaki, C. Hu, K. Kikuchi, S. Kobayashi, K. Matsuda, and Y. Nagakara, J. Phys. Soc. Japan **15**, 761 (1960).

[22] H. E. Gove, in *Proceedings of the International Conference on Nuclear Structure, Kingston, Canada* (University of Toronto Press, Toronto, 1960), p. 438.

[23] A. E. Litherland, J. A. Kuehner, H. E. Gove, M. A. Clark, and E. Almqvist, Phys. Rev. Letters **7**, 98 (1961).

[24] H. E. Gove and C. Broude, in *Proceedings of the Second Conference on Reactions between Complex Nuclei* (John Wiley & Sons, Inc., New York, 1960), p. 57. In this same Vol. I. Kh. Lemberg reports a value almost twice this large (p. 112). Our choice is arbitrary.

[25] S. A. Moszkowski, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 411.

[26] Although the Coulomb distortion of the incident wave is appreciable because $\eta\sim\frac{1}{2}$, the cross section for Coulomb excitation of the $2^+$ level has a peak height of only 2 or 3 mb and this is at less than 10°. This estimate is made using the formulas and tables in the review article by Alder *et al.* (see reference 20).

estimate of the nuclear distortion of the incident wave we are not justified in reaching any conclusion about the consistency of the difference values of $g'$.

On the other hand, our first statement that the higher states in the rotational band should be only weakly excited compared to the $2^+$ state if the $P_2$ multipole is the dominant multipole is not dependent on the use of the plane wave Born approximation. Even if "correct" distorted waves are used the selection rule holds for any factorable potential which contains only the $P_2$ multipole. The only approximation we have made which might upset this rule is in replacing the set of coupled equations[18] actually needed to describe the direct interaction process by a single scattering amplitude. Oda *et al.* do not give the cross section for inelastic scattering to the $4^+$ state. Cross and Clarke[27] report that in scattering 18-Mev neutrons from $Mg^{24}$ they observed less than one fourth as many neutrons scattered inelastically from the $4^+$ level as from the $2^+$ level. This observation is consistent with the hypothesis that the $P_2$ multipole is much stronger than the $P_4$ multipole in the residual interaction in $Mg^{24}$ (we can draw no conclusion about any other multipoles since they would not contribute to the $0^+ \rightarrow 4^+$ cross section in any event). In addition, we have pointed out that to conclude that the $P_2$ multipole is much stronger than the $P_4$ multipole we would need independent information on the magnitude of the reduced $E_4$ transition probability. In the case of $Ne^{20}$ such information is available. Litherland *et al.*[23] report that both $B(E2)$ and $B(E4)$ are roughly twenty times the single-particle estimates. In this case the estimate $B(E4)/B(E2) \approx R^4$ is correct and if the cross section for excitation of the $4^+$ level is found to be much smaller than the cross section for excitation of the $2^+$ level it can be attributed to the weakness of the $P_4$ multipole rather than to a lack of $2^4$-pole distortion in the wave functions.

In this section we have taken seriously the idea that the $P_2$ multipole is the dominant multipole in the residual interaction in rotational nuclei. This idea is suggested by the results obtained from the "$P_2$" model and also by Bohr and Mottelson's conjecture that it is this long-range part of the residual interaction which leads to the deformations characteristic of rotational nuclei. We have then suggested that direct-interaction inelastic scattering experiments might be used as a probe of the multipole structure of the residual interaction if independent information is available on the nuclear multipole matrix elements. In particular, for those excited states which have been investigated both by inelastic nucleon scattering and by Coulomb excitation or inelastic electron scattering, the reduced transition probabilities $B(E\lambda)$ can be determined from the latter and the inelastic nucleon scattering experiments can be used to investigate the multipole structure of the nuclear residual interaction. There remains the possibility that the residual interaction between the incident nucleon and a nucleon in the target nucleus may differ significantly from the residual interaction between two nucleons inside of the nucleus since the incident nucleon is in an unbound state, and the need to take into account the distortion of the incident wave and the effects of the strong coupling of different states in the same rotational band.

[27] W. G. Cross and R. L. Clarke, Bull. Am. Phys. Soc. 4, 258 (1959).


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## APPENDIX A

### Group-Theoretic Classification of Rotational States

Elliott[1] utilized the fact that the oscillator Hamiltonian,

$$H_0 = \sum_i \tfrac{1}{2}[\mathbf{a}(i)\cdot\mathbf{a}^\dagger(i) + \mathbf{a}^\dagger(i)\cdot\mathbf{a}(i)], \tag{A1}$$

is invariant under the group of three-dimensional unitary transformations among the components of the vector $\mathbf{a} = 1/\sqrt{2}(\mathbf{p} - i\mathbf{r})$. Thus, the eigenfunctions of $H_0$ may be classified according to their transformation properties under the group $U_3$, i.e., according to the irreducible representations of $U_3$. The group $U_3$ has nine infinitesimal operators. There are nine independent products:

$$a_r^\dagger a_s, \quad r, s = x, y, z, \tag{A2}$$

all of which commute with $H_0$. These operators, or linear combinations of them, are the infinitesimal operators of the group. In order to eliminate transformations which are only an over-all change of phase, Elliott makes the restriction to the unimodular three-dimensional unitary group $SU_3$ by removing the unit infinitesimal operator. Then the eight infinitesimal operators are the $L_\tau$ and $Q_\tau$. The irreducible representations of $SU_3$ can be labelled by two numbers,

$$SU_3: (\lambda\mu). \tag{A3}$$

$\lambda$, $\mu$ take on only positive integral values and are restricted by the condition

$$\lambda + 2\mu \leq \nu N. \tag{A4}$$

In fact, not all values of $\lambda, \mu$ allowed by these conditions occur. The problem of which $(\lambda\mu)$ do occur for a given number of particles $(\nu)$ in a given oscillator shell $(N)$ is solved by Elliott and tabulated for the $N=2$ and $N=3$ shells in his papers. The commutation relations (3) show that the group of three-dimensional rotations $R_3$ (infinitesimal operators $L_\tau$) is a subgroup of $SU_3$. Thus the eigenfunctions of $H_0$ may also be classified according to their transformation properties under the group $R_3$,

i.e., according to their angular momentum $L$. The problem of which irreducible representations of the subgroup $R_3$ occur in a given irreducible representation $(\lambda\mu)$ of $SU_3$ is a straightforward group theoretic problem which is also solved by Elliott. The result may be stated as follows:

$$L=K,\ K+1,\ K+2,\ \cdots,\ K+\max\{\lambda,\mu\},$$

where

$$K=\min\{\lambda,\mu\},\ \min\{\lambda,\mu\}-2,\ \cdots,\ 1,\text{ or }0$$

except when $K=0$, in which case

$$L=\max\{\lambda,\mu\},\ \max\{\lambda,\mu\}-2,\ \cdots,\ 1,\text{ or }0, \quad \text{(A5)}$$

e.g.

$$\text{for}\quad (\lambda\mu)=(\lambda 0),\quad L=\begin{cases}0,\ 2,\ 4,\ \cdots,\ \lambda, & \lambda\text{ even}\\ 1,\ 3,\ 5,\ \cdots,\ \lambda, & \lambda\text{ odd};\end{cases}$$

$$\text{for}\quad (\lambda\mu)=(\lambda 1),\quad L=1,\ 2,\ 3,\ \cdots,\ \lambda+1;$$

$$\text{for}\quad (\lambda\mu)=(22),\quad L=0,\ 2^2,\ 3,\ 4.$$

All of the $|E_N[\lambda\mu]LM\rangle$ eigenstates of $H_0$ classified by Elliott are degenerate, i.e., they are all states of energy $E_N$. Since the operator $T$ was explicitly constructed to commute with all the group operators $L_\tau$ and $Q_\tau$, the Hamiltonian $H'=H_0-T$ is also invariant under the group $SU_3$ and the subgroup $R_3$. Thus the eigenfunctions of $H'$ may also be classified by the quantum numbers $[\lambda\mu]$ and $L$, $M$. The states of same $[\lambda\mu]$ but different $L$, $M$ are still degenerate. This follows from the fact that given any state $|E_N[\lambda\mu]L'M'\rangle$ belonging to an irreducible representation $[\lambda\mu]$ we can generate all states $|E_N[\lambda\mu]LM\rangle$ belonging to the irreducible representation by operating on $|E_N[\lambda\mu]L'M'\rangle$ with all the group operators, i.e., by repeated application of the $L_\tau$ and $Q_\tau$ operators. Since these operators all commute with $H'$, the states obtained in this way are all eigenfunctions of $H'$ with the same energy. Thus all eigenfunctions belonging to the same representation $[\lambda\mu]$ are degenerate. However, states of different $[\lambda\mu]$ are no longer degenerate since the eigenvalue $t$ of $T$ will be different for different $[\lambda\mu]$:

$$H'|E_N[\lambda\mu]LM\rangle=(E_N-t_{[\lambda\mu]})|E_N[\lambda\mu]LM\rangle. \quad \text{(A6)}$$

If we now add the operator $\lambda L^2$ to $H'$ to obtain

$$H=H'+\lambda L^2=H_0-Q^2, \quad \text{(A7)}$$

we obtain the rotational spectra

$$H|E_N[\lambda\mu]LM\rangle = [E_N-t_{[\lambda\mu]}+\lambda L(L+1)]|E_N[\lambda\mu]LM\rangle. \quad \text{(A8)}$$