New Class of Classical Uncertainty Relations Giving Uncertainty for Long and Certainty for Short Times

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Bohr's concept of complementarity enters any statistical description of physical phenomena. In quantum theory the complementary quantities are dynamical variables. In classical theory complementarity exists between dynamical and statistical variables. Slowing down of a "large" particle of mass m by multiple collisions with a gas of "small" molecules leads to certainty for "short times." For "long times," the multiple collisions introduce statistics leading to uncertainty. An uncertainty relation has been derived for the coordinate q as the dynamical and the drift momentum p_d as the statistical complementary variable: $\Delta p_d \Delta q \geq mD(1-e^{-t/\tau})$, where D is a diffusion coefficient and τ a relaxation time. This relation gives uncertainty for "long" and certainty for "short" times.

The existence of such a classical uncertainty relation poses the question of whether a generalization of (relativistic) quantum theory is possible, such that for "high energies" $\Delta p \Delta q \to 0$, while for low energies $\Delta p \Delta q \geqq \hbar/2$. This question is not answered in this note; however, it is pointed out that the telegrapher's equation (which classically implies the aforementioned generalized uncertainty principle) with $\tau = i\hbar/2mc^2$ is satisfied by the Klein-Gordon (also Dirac) wave function for a free particle if the time dependence of the rest mass is split off from it.

BOHR's concept of complementarity is not restricted to quantum theory, but enters any statistical description of physical phenomena. Thus, it enters also classical statistical physics. However, the complementary quantities have a different character in quantum and in classical theory. In quantum theory, the complementary quantities are canonically conjugate like coordinates and momenta. In classical theory, one of the complementary quantities is defined as a statistical average like (coordinate and) average momentum. In both cases, quantum and classical, the complementary quantities will be subject to uncertainty relations. However, the quantities of classical theory do not have (linear) operator character. Thus, there are no classical analogs to quantum commutation relations.

In this note we shall derive a classical uncertainty relation giving uncertainty for long and certainty for short times. The complementary quantities will be the coordinate and an average momentum. The physically simplest and clearest example for our new uncertainty relations comes from the theory of Brownian motion, particularly Rayleigh's problem (one dimensional). We consider the slowing down of a "large" particle of a given velocity by multiple collisions with a gas of "small" molecules. Clearly, for "short times," e.g., before any collisions occur, there is no statistics; we have certainty. For "long times," the multiple collisions introduce statistics; we have uncertainty.

This slowing down process can be described by the Kramers equation[1] (p: momentum; m: mass)

$$\frac{\partial f}{\partial t}+\frac{p}{m}\frac{\partial f}{\partial q}-\frac{1}{\tau}\frac{\partial}{\partial p}(pf)=0. \tag{1}$$

The third term is due to "frictional forces":

$$\tau = m/\mu, \tag{1a}$$

where μ is the "frictional constant." Taking the first two moments with respect to p of Eq. (1), we obtain a system of two first-order differential equations:

$$\begin{aligned}\partial n/\partial t+\partial j/\partial q&=0,\\ \tau\partial j/\partial t+D\partial n/\partial q+j&=0.\end{aligned} \tag{2}$$

Here we put

$$n(q,t)=\int f dp,\quad j(q,t)=\int\frac{pfdp}{m};\quad D=\frac{\tau}{m^2}\int\frac{p^2fdp}{n}, \tag{2a}$$

and the (approximative) assumption has been made that the second factor of D is independent of q.

Both n and j satisfy the telegrapher's equation in the form

$$\left[\tau\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}-D\frac{\partial^2}{\partial x^2}\right]\begin{bmatrix}n\\ j\end{bmatrix}=0. \tag{3}$$

[For the three-dimensional case the system (2) applies also if in the first equation $\partial j/\partial q$ is replaced by div$\mathbf{j}$ ($\mathbf{j}$ being a vector) and in the second equation $\partial n/\partial q$ is replaced by grad$n$. We have then

$$\begin{aligned}\partial n/\partial t+\mathrm{div}\mathbf{j}&=0,\\ \tau\partial\mathbf{j}/\partial t+D\,\mathrm{grad}n+\mathbf{j}&=0.\end{aligned} \tag{2b}$$

Here only n satisfies the telegrapher's equation, while $\mathbf{j}$ obeys a somewhat different equation because div grad$\neq$grad div.]

We define now the drift velocity and momentum which are clearly average quantities and are complementary to the space coordinate q:

$$v_d=j/n;\quad p_d=mv_d. \tag{4}$$

We have, with n as our distribution function

$$\begin{aligned}\Delta v_d\Delta q&=[\langle(p_d-\bar{p}_d)^2\rangle_{\mathrm{av}}]^{\frac{1}{2}}[\langle(q-\bar{q})^2\rangle_{\mathrm{av}}]^{\frac{1}{2}}\\ &=\left[m\int dq(j^2/n^2)n\right]^{\frac{1}{2}}\left[\int dq(q-\bar{q})^2n\right]^{\frac{1}{2}}.\end{aligned} \tag{5}$$

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[1] H. A. Kramers, Physica 7, 284 (1940).

Here we can assume that

$$\bar{p}_d=\int dq\, p_d n=0. \tag{5a}$$

Applying the Schwarz inequality

$$\left[\int dq\, F^2 \int dq\, G^2\right]^{\frac{1}{2}} \geqslant \int dq\, FG \tag{6}$$

to Eq. (4), we obtain

$$\Delta p_d \Delta q \geqslant m \int dq\, j(q-\bar{q}). \tag{7}$$

Now the integral $I=\int dq\, jq$ obeys the differential equation

$$\tau\partial I/\partial t+I-D=0, \tag{8a}$$

wherefrom, with $I(0)=0$,

$$I=D(1-e^{-t/\tau}). \tag{8}$$

Hence, we obtain as our final result

$$\Delta p_d \Delta q \geqslant mD(1-e^{-t/\tau}). \tag{9}$$

The choice $I(0)=0$ is necessary because we know that for $t=0$, $\Delta p_d \Delta q=0$, giving certainty. For long times Eq. (8) reduces to

$$\Delta p_d \Delta q \geqslant mD, \tag{9a}$$

a relation which has been obtained previously by Fürth.[2]

Now D is inversely proportional to a macroscopic (transport) cross section which itself is proportional to Avogadro's number N. Hence, D, and thus the product of uncertainty is proportional to $1/N$. This feature is characteristic of classical uncertainty relations. N molecules form, so to say, a critical domain.

For $(\Delta q)^2=\langle q^2\rangle_{\text{av}}$, we obtain Ornstein's formula:

$$\langle q^2\rangle_{\text{av}}=2D\tau[(t/\tau)-1+e^{-t/\tau}], \tag{10}$$

which gives for long times ($t\gg\tau$), Einstein's formula:

$$\langle q^2\rangle_{\text{av}}=2Dt, \tag{10a}$$

while for short times ($t\ll\tau$) it leads to

$$\langle q^2\rangle_{\text{av}}=(D/\tau)t^2, \tag{10b}$$

as one should expect. Eq. (10) follows from the (exact) Kramers Eq. (1) if one averages over the initial positions and velocities. The higher moments will be, of course, different for Eq. (1) and our approximate system Eq. (2). Equations (10a) and (10) and also (9a) and (9) correspond respectively to the Einstein-Smoluchowski and Ornstein-Uhlenbeck processes.

These new types of uncertainty relations are, perhaps, of sufficient interest in a purely classical context. Actually, however, they have been developed in an attempt to generalize quantum theory. Quantum fluctuations at high energies lead, as is well known, to divergencies. Most of these fluctuations could, perhaps, be avoided if it would be possible to formulate a generalized quantum theory in which for "high energies" $\Delta p\Delta q\rightarrow 0$ while for "low energies"[3] $\Delta p\Delta q\geqslant\hbar/2$, e.g., Heisenberg's relation is valid. Our classical considerations could, of course, be used only as a heuristic guide in such a program. Here, we shall restrict ourselves to two simple comments.

First, we observe that the telegrapher's equation (3) with $\tau=i\hbar/2mc^2$ is satisfied by the Klein-Gordon (also Dirac) wave function for a free particle, if the factor $\exp[-(i/\hbar)mc^2t]$ is split off from it.[4] Thus the well-known formal transition from the classical diffusion equation to the Schrödinger equation by introducing an imaginary diffusion coefficient $D=i\hbar/2m$ persists: The time-lag according to relativity can be expressed by an imaginary relaxation time τ. (All this is true, of course, for the three-dimensional case also.)

The analog of the system of first-order equations (2) is the system

$$\begin{aligned}\frac{\partial\psi_1}{\partial t}+\frac{\partial\psi_2}{\partial q}+\left(\frac{i}{\hbar}\right)mc^2\psi_1=0,\\ \frac{\partial\psi_2}{\partial t}+c^2\frac{\partial\psi_1}{\partial q}-\left(\frac{i}{\hbar}\right)mc^2\psi_2=0.\end{aligned} \tag{11}$$

Both ψ_1 and ψ_2 contain the time-dependence of the rest energy and satisfy the (second-order) Klein-Gordon equation:

$$\left[\frac{\partial^2}{\partial q^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\left(\frac{mc}{\hbar}\right)^2\right]\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}=0. \tag{12}$$

This system (11) may be contrasted with the usual "Hamiltonian form" of the Klein-Gordon equation, which is first order in $\partial/\partial t$, but second order in $\partial/\partial q$:

$$\begin{aligned}\frac{\partial\varphi_1}{\partial t}-\frac{i\hbar}{2m}\frac{\partial^2}{\partial q^2}(\varphi_1+\varphi_2)+\frac{i}{\hbar}mc^2\varphi_1=0,\\ \frac{\partial\varphi_2}{\partial t}+\frac{i\hbar}{2m}\frac{\partial^2}{\partial q^2}(\varphi_1+\varphi_2)-\frac{i}{\hbar}mc^2\varphi_2=0.\end{aligned} \tag{13}$$

Here

$$\begin{aligned}\varphi_1+\varphi_2=\sqrt{2}\psi,\\ \varphi_1-\varphi_2=\sqrt{2}\frac{i\hbar}{2m}\frac{\partial\psi}{\partial t};\end{aligned} \tag{13a}$$

ψ satisfies the Klein-Gordon equation.

There is, of course, no Klein-Gordon analog to the three-dimensional system (2a). The three-dimensional

[2] R. Fürth, Z. Physik **81**, 143 (1933).

[3] "High energies" and "low energies" correspond for our classical case to "short times" and "long times," respectively.

[4] This simple fact has been overlooked in the literature. Some otherwise excellent books discuss the telegrapher's and Klein-Gordon equations separately.

form of (13) results, simply, replacing $\partial^2/\partial q^2$ by ∇^2. These remarks indicate the limitations of the formal analogy between classical transport theory and (relativistic) wave mechanics.

Second, a simple generalization of the Dirac equation to contain a shorter "shortest time" τ (in addition to the "built-in" time $\hbar/2mc^2$) leads to the equation

$$(\gamma_\mu \partial/\partial x_\mu + c\tau\Box + mc/\hbar)\psi = 0, \tag{14}$$

where, perhaps, for example, $c\tau \sim (e^2/mc^2)(e^2/\hbar c)$. Eq. (14) corresponds to a two-mass Dirac equation studied by Pais and Uhlenbeck[5] and others.

Notes added in proof. I. Classical uncertainty relations of the type we derived in this paper can also be obtained, more generally, from a Boltzmann equation. In fact, our starting point was the query: "Since the Schrödinger equation is formally related to the diffusion equation and since the diffusion equation follows, by a series of approximations, from a Boltzmann equation, is there any Boltzmann-analog generalization of the Schrödinger equation?"

II. Recent work by P. C. Hemmer [Det Fysiske Seminar i Trondheim No. 2 (1959)] and particularly by R. J. Rubin [*Proceedings of the International Symposium on Transport Processes in Statistical Mechanics*, edited by I. Prigogine (Interscience Publishers, Inc., New York, 1958), p. 155; J. Math. Phys. **1**, 309 (1960); **2**, 373 (1961)] on the Brownian motion of a particle coupled to a system of one-, two-, and three-dimensional oscillators contains implicitly uncertainty relations of the type we derived, though in a somewhat more complicated form.

III. The uncertainty relations derived in this note reduce for long times formally to the Bohr-Heisenberg relations if we use the replacements $\hbar \rightarrow mD$ and $p \rightarrow p_d$. We wish to point out that in a similar sense there is a classical analog to the relation

$$\Delta E \Delta t \sim \hbar. \tag{A}$$

One has to replace $\hbar \rightarrow -k$ and $t \rightarrow 1/T$ (k: Boltzmann's constant, T: absolute temperature) to obtain

$$\Delta E \Delta(1/T) = -k, \quad \text{or} \quad \Delta E \Delta T \sim kT^2. \tag{B}$$

The replacement used is, of course, the same which carries over the unitary operator $e^{(i/\hbar)Ht}$ into the Boltzmann factor $e^{-H/kT}$, or the Schrödinger into the Bloch equation. Here, as in thermalization, thermodynamics is involved. The particular case of slowing down, treated in this note, does not involve thermodynamics, though it does imply irreversibility. Fundamentally, our view of the physical significance of the relation (A) is essentially the same as that of Y. Aharonov and D. Bohm [Phys. Rev. **122**, 1649 (1961)].

[5] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).

IV. The formal *analogy* between diffusion and Schrödinger equation, and also between telegrapher's and Klein-Gordon equation is *based on group theory*. Still, we believe that this formal *analogy* might be of considerable heuristic help in the search toward a generalized quantum theory. W. Pauli [*Encyclopedia of Physics* (Springer-Verlag, Berlin, Germany, 1958), Vol. 1, p. 107, footnote 2] pointed out, that originally the commutation relations for the spin operators S_k were founded on the *analogy* to those for the angular momenta M_k, which, in turn, can be deduced from the canonical commutation relations for the p_k and q_k. The possibility of the kinematic derivation of the commutation relations for the S_k from the rotation group was first pointed out by J. von Neumann and E. Wigner [Z. Physik. **47**, 203 (1927)].

V. L. Rosenfeld kindly called my attention to a very interesting note of his in Nature **190**, 384 (1961) and a preprint of his lectures on "Questions of Irreversibility and Ergodicity," at the Varenna School, May 1960. His and our views on the role of complementarity in classical physics are essentially the same. In particular, Rosenfeld discusses the relation (B), however, without pointing out its formal connection with relation (A).

VI. D. Bohm kindly called my attention to his Chapter 6, "Hidden variables in quantum theory," in *Quantum Theory*, edited by D. R. Bates (Academic Press Inc., New York, 1962), Vol. III. There is some similarity between our views and Bohm's in that we both consider Heisenberg's relations to be weakened for short times and correspondingly short distances. However, we are working toward a generalization of quantum theory by the explicit introduction of one or more new parameters. We do not believe that it is fruitful to consider hidden variables in the present quantum theory. Bohm mentioned to us in conversation, that he wrote the above chapter more than two years ago. Some of his present views are outlined in a general way in a very interesting paper in the British Journal for the Philosophy of Science, **XII**, 103 (1961).

VII. The system (11) has another classical analog also, e.g., the linear system for the flow of electricity in cables from which the telegrapher's equations originate.

$$\begin{aligned} \partial I/\partial t + (1/L)(\partial V/\partial q) + (R/L)I &= 0, \\ \partial V/\partial t + (1/C)(\partial I/\partial q) + (G/C)V &= 0. \end{aligned} \tag{C}$$

Here: I and V are current density and potential, respectively; R and G are the resistance and leakage conductance of the line per unit length, respectively; C is the electrostatic capacity of the conductor.

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