Regge Poles and Resonances in Strong Interaction

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The scattering of two particles via the exchange of a resonance system is considered. The exchanged resonance is approximated by a pole in the Regge representation. The composite nature of the resonance is taken into account in the Regge-pole description so that one does not encounter divergence difficulties even though the exchanged resonance may be in a $J=1$ state.

The exchange of a $\pi\pi$ ($I=1$, $J=1$) resonance in pion-pion scattering is considered in some detail. It is shown that this exchange mechanism produces a strong attractive force in the $I=1$ scattering system. Whether this exchange force can reproduce a resonance at the observed energy depends on a dimensionless parameter in the representation. This dimensionless parameter is restricted to be positive and of order unity on the basis of a crude uncertainty-principle argument. The value of the parameter required to reproduce a resonance does fall within this range. However, the width of the calculated resonance is broader than the observed width by a factor of 2 to 3. This discrepancy is not unexpected since all other forces in the problem have not been considered.

I. INTRODUCTION

DURING the past year, the $\pi\pi$, πK, $\pi\Lambda$, $\pi\Sigma$, and the 3π resonances have been observed in high-energy experiments.[1] An immediate question one might ask is whether these resonances are produced by some exchange forces or whether they must be regarded as fundamental particles whose masses and lifetimes are arbitrary constants. The purpose of this paper is to explore the former possibility using Regge's representation for resonances.[2]

Since the observed resonances have widths that are characteristic of strong interactions, there is little doubt that the exchange of these resonating systems will themselves provide a substantial exchange force. It would be even more interesting if such exchange mechanisms were responsible for the existence of the resonances. Several attempts have been made along this direction.[3] Unfortunately, in previous work one encounters divergence difficulties in handling the exchange force when the resonance is in a $J=1$ state. These divergence difficulties are associated with the unrenormalizability of massive vector fields in the Lagrangian theory or, equivalently, the necessity of a cutoff in the dispersion theory. We shall show that such difficulties do not arise if one associates the exchanged resonance system with a pole in the Regge representation of the S matrix.[2]

In essence, Regge represents a resonance system in terms of a superposition of all integer angular momentum states which combine to form a pole of the S matrix in the complex angular-momentum plane. This superposition reduces to a single-integer angular-momentum amplitude only at two conjugate points of the complex-energy plane where the integer angular-momentum amplitude has poles associated with the resonance. If the energy of the system is in the neighborhood of the resonance energy, this way of representing the resonance is almost identical to the usual consideration of a fixed-integer angular-momentum amplitude. However, the analytic continuation outside of the resonance region becomes somewhat different. In fact, the continuation using Regge's representation gives a convergent result even though the system under consideration is primarily a $J=1$ amplitude in the resonance region. The example of the pion-pion $I=1$ resonance is used to illustrate the general approach using Regge's representation.

II. PION-PION RESONANCE

The pion-pion scattering amplitude is considered as a function of the familiar scalar variables s, t and u; $s=4(q^2+1)$, $t=-2q^2(1-\cos\theta)$, $u=-2q^2(1+\cos\theta)$. The P-wave amplitude is defined by

$$A_1(s)\equiv-\frac{1}{s-4}\int_0^{-(s-4)}dt\,P_1\left(1+\frac{2t}{s-4}\right)A^{(1)}(s,t),\quad(1)$$

where $A^{(1)}(s,t)$ is the $I=1$ amplitude in the s channel. One can write the dispersion relation for $A_1(s)$ in the form of an integral equation:

$$A_1(s)=B_1(s)+\frac{(s-4)}{\pi}\int_4^\infty ds'\,\frac{[(s'-4)/s']^{\frac{1}{2}}|A_1(s')|^2}{(s'-4)(s'-s)},\quad(2)$$

where $B_1(s)$ is the generalized "potential term" which is regular for $s\geq4$ and has branch points only along the negative real axis.[4] If $B_1(s)$ is bounded in the physical region $s\geq4$, then $A_1(s)$ can be solved by the usual N/D method. Here, we shall find it more convenient to convert the N/D equations into an integral equation for N

[1] References to these experiments can be found in N. H. Xuong and G. R. Lynch, Phys. Rev. Letters **7**, 327 (1961); M. Alston, L. W. Alvarez, P. Eberhard, M. L. Good, W. Graziano, H. K. Ticho, S. G. Wojcicki, Phys. Rev. Letters **6**, 300 (1961); E. Pickup, D. K. Robinson and E. O. Salant, Phys. Rev. Letters **7**, 192 (1961); M. Alston, L. W. Alvarez, P. Eberhard, M. L. Good, W. Graziano, H. K. Ticho, S. G. Wojcicki, Phys. Rev. Letters **6**, 698 (1961).

[2] T. Regge, Nuovo cimento **18**, 947 (1960).

[3] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960). References to related topics can be found in a review article: S. Mandelstam, Reports on Progress in Physics (to be published).

[4] G. F. Chew and S. C. Frautschi, Phys. Rev. **124**, 264 (1961).

since this equation only requires $B_1(s)$ for $s\geq 4$.[5] Thus Eq. (2) becomes

$$A_1(s)\equiv N_1(s)/D_1(s),$$

$$N_1(s)=B_1(s)+\frac{(s-4)}{\pi}\int_4^\infty ds'\frac{B_1(s')-B_1(s)}{(s'-4)(s'-s)}\times\left(\frac{s'-4}{s'}\right)^{\frac{1}{2}}N_1(s'),$$

$$D_1(s)=1-\frac{(s-4)}{\pi}\int_4^\infty ds'\frac{1}{(s'-4)(s'-s)}\times\left(\frac{s'-4}{s'}\right)^{\frac{1}{2}}N_1(s'). \tag{2'}$$

With the aid of the crossing symmetry condition, $B_1(s)$ can be calculated directly from the scattering amplitude in the t and u channels. The crossing symmetry condition implies[3]

$$A^{(1)}(s,t)=-A^{(1)}(s,u)=\tfrac{1}{2}A^{(1)}(t,s)+\tfrac{1}{3}A^{(0)}(t,s)-\tfrac{5}{6}A^{(2)}(t,s), \tag{3}$$

where the first variable designates the channel in which this variable is the center-of-mass energy squared and the second variable is the momentum transfer squared.

Since we are only concerned with the $I=1$ amplitude in the t and u channels, Eqs. (3) and (1) gives

$$B_1(s)\simeq-\frac{1}{s-4}\int_0^{-(s-4)}dt\,P_1\left(1+\frac{2t}{s-4}\right)\tfrac{1}{2}A_R{}^{(1)}(t,s)-\frac{1}{s-4}\int_0^{-(s-4)}du\,P_1\left(1+\frac{2u}{s-4}\right)\tfrac{1}{2}A_R{}^{(1)}(u,s), \tag{4}$$

where $A_R{}^1(t,s)[A_R{}^1(u,s)]$ denotes that part of $A^{(1)}(t,s)$ $[A^{(1)}(u,s)]$ which is regular for negative $t(u)$ and has branch points along the positive t axis (u axis). For example, if one keeps only the P-wave amplitude in $A^{(1)}(t,s)$ and $A^{(1)}(u,s)$, then the above argument yields

$$B_1(s)\simeq-\frac{1}{s-4}\int_0^{-(s-4)}dt\,P_1\left(1+\frac{2t}{s-4}\right)\frac{3(t-4)}{\pi}\times\int_4^\infty dt'\frac{[(t'-4)/t']^{\frac{1}{2}}|A_1(t')|^2}{(t'-4)(t'-t)}P_1\left(1+\frac{2s}{t-4}\right), \tag{5}$$

Of course, this result can also be obtained by the more conventional procedure of first calculating the left-hand discontinuity and then integrating over the left-hand branch cut. Incidentally, if one makes the further approximation of replacing $|A_1(t')|^2$ by a δ function, then Eq. (5) gives exactly the Born term for the exchange of a vector meson.

It is easily seen that $B_1(s)$, calculated from Eq. (5), diverges at least logarithmatically for large s. Such asymptotic behavior makes the integral equation (2) insoluble. Thus we are lead to conclude that replacing $A^{(1)}(t,s)$ and $A^{(1)}(u,s)$ by a P-wave amplitude is not a satisfactory approximation.

In Regge's representation, the resonance in the t channel is associated with a term of the form

$$A^{(1)}(t,s)\simeq\frac{\beta(t)}{\sin\pi\alpha(t)}\times\frac{1}{2}\left[P_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right)-P_{\alpha(t)}\left(1+\frac{2s}{t-4}\right)\right], \tag{6}$$

where $\alpha(t)$ is the position of the pole of the S-matrix in the complex angular momentum plane and $\beta(t)$ is the corresponding residue.[2] By definition, $\alpha=1$ at the two conjugate points in the unphysical sheet of the t-plane where the P-wave amplitude has poles associated with the resonance. At all other points of the t plane, α is in general not equal to one. In the neighborhood of the resonance, one can determine $\alpha(t)$ approximately by a method analogous to the argument given by Regge for potential scattering.[2]

In the neighborhood of the resonance, the P-wave amplitude may be expressed in the form

$$A_1(t)\equiv\left(\frac{t}{t-4}\right)^{\frac{1}{2}}e^{i\delta_1}\sin\delta_1\simeq\frac{\Gamma(t-4)}{t_r-t-i\Gamma(t-4)^{\frac{3}{2}}t^{-\frac{1}{2}}}, \tag{7}$$

where t_r and Γ are, respectively, the position and the width of the resonance. It is easily seen that $A_1(t)$ contains a conjugate pair of poles in the unphysical sheet of the t plane. Since the observed width is fairly narrow ($\Gamma\lesssim 0.3$), the locations of these poles are approximately given by

$$(t-4)^{\frac{1}{2}}\simeq\pm(t_r-4)^{\frac{1}{2}}-\tfrac{1}{2}i\Gamma(t_r-4)t_r^{-\frac{1}{2}}. \tag{8}$$

At these points, $\alpha=1$. Now, we expand $\alpha(t)$ in a Taylor series about these zeros of $(\alpha-1)$:

$$\alpha(t)\simeq 1-d_1[t_r-t-i\Gamma(t_r-4)t_r^{-\frac{1}{2}}(t-4)^{\frac{1}{2}}]. \tag{9}$$

An estimation of d_1 may be made according to the following crude argument.

First of all, the usual symmetry property $A_1{}^*(t^*)=A_1(t)$ implies that d_1 is real. Now, at $t=t_r$, $\alpha(t)$ acquires an imaginary part,

$$\text{Im}\alpha=d_1\Gamma(t_r-4)^{\frac{3}{2}}t_r^{-\frac{1}{2}}. \tag{10}$$

This would cause the system to decay after an angular displacement of

$$\Delta\phi\simeq 1/\text{Im}\alpha=d_1{}^{-1}\Gamma^{-1}(t_r-4)^{-\frac{3}{2}}t_r{}^{\frac{1}{2}}. \tag{11}$$

On the other hand, the total angular displacement

[5] The N/D equations in the form (2′) have been derived by J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).

within the lifetime τ is given by

$$\Delta\phi \simeq (d\phi/dT)\tau \simeq (2/R^2)\tau, \tag{12}$$

where R is the expectation value of the distance between the two resonating pions. From Eq. (8), one finds

$$\tau \simeq 2\Gamma^{-1}(t_r-4)^{-\frac{3}{2}}t_r. \tag{13}$$

We now equate (11) and (12). This gives

$$d_1 \simeq \tfrac{1}{2}R^2 t_r^{-\frac{1}{2}}. \tag{14}$$

Of course, the above argument is valid only for a nonrelativistic system. However, we believe that at least the sign and the order of magnitude of d_1 may be inferred from expression (14). We now relate R to the position and width of the resonance by using a P-wave effective range formula of the form given by Ross and Shaw[6]:

$$\left(\frac{t}{4}-1\right)^{\frac{3}{2}}\cot\delta_1 \simeq -\frac{3}{2R}\left(\frac{t}{4}-\frac{t_r}{4}\right). \tag{15}$$

By comparing Eq. (15) with Eq. (7), one obtains

$$R \simeq 3\Gamma t_r^{-\frac{1}{2}}, \quad d_1 \simeq (\tfrac{3}{2}\Gamma)^2 t_r^{-\frac{3}{2}}. \tag{16}$$

Since pion-pion scattering does involve relativistic kinematics, we have no decisive way of calculating $\alpha(t)$ for the problem concerned. However, the above argument leads us to consider a one parameter formula

$$\alpha(t) = 1 - C(\tfrac{3}{2}\Gamma)^2 t_r^{-\frac{3}{2}}[t_r - t - i\Gamma(t_r-4)t_r^{-\frac{1}{2}}(t-4)^{\frac{3}{2}}], \tag{17}$$

where C is expected to be positive and of order unity. By substituting the observed values $\Gamma \simeq 0.3$ and $t_r \simeq 29$ into (17), one finds

$$\alpha(0) \simeq 1 - 0.042C. \tag{18}$$

This estimation shows that α remains close to unity even at a fair distance from the resonance. The important feature of $\alpha(t)$ given by Eq. (17) is that $\alpha(t)$ is pure real for $t<4$ and monotonically decreasing towards the left. We shall use this formula up to a point where $\alpha(t)$ becomes substantially different from unity and join the $\alpha(t)$ curve to a constant beyond such point. Fortunately, $B_1(s)$ is quite insensitive to where we join α to a constant.

From Eq. (17),

$$\alpha(-370) = 1 - \tfrac{1}{2}C, \quad \alpha(-770) = 1 - C, \quad \alpha(-1200) = 1 - \tfrac{3}{2}C. \tag{19}$$

We have considered each of these three points and joined α continuously to the constants indicated. The final solution for $A_1(s)$ as shown in the next section changes by less than 3% from one case to another. The reason for this insensitivity is due to the fact that all calculated values of $B_1(s)$ do not differ until $s>374$. But beyond $s\sim 374$, the main contribution to the integral in (4) again comes from small values of t since $P_{\alpha(t)}[-1-2s/(t-4)] \sim (-2s/t)^{\alpha(t)}$ and $\alpha(t)$ is monotonically nonincreasing. Hence our ignorance of $\alpha(t)$ for large t is not a severe handicap. We do assume however that the correct continuation of α along the negative t axis does not arise up to a large positive value at large negative t, for this would cause an unreasonably singular backward scattering at high energy (not to be confused with the backward peak which must be associated with the u channel).

We shall now find a representation of $[\beta(t)/\sin\pi\alpha(t)]$ which appears in Eq. (6). As emphasized before, we are only concerned with the contribution of the Regge pole term to $B_1(s)$ for $s\geq 4$. Such contribution must come from the regular part of the Regge pole term for $t\leq 0$.

Let us denote the regular part of $(\beta/\sin\pi\alpha)$ for $t\leq 0$ by $(\beta/\sin\pi\alpha)_R$ and express it in terms of an integral over a right-hand cut. Since $-\frac{1}{3}(\beta/\sin\pi\alpha)$ is identical to the P-wave amplitude in the limit of t approaching the resonance poles of the amplitude, we will approximate the discontinuity of $-\frac{1}{3}(\beta/\sin\pi\alpha)$ in the physical region $(t'\geq 4)$ by the imaginary part of the P-wave amplitude. Hence,

$$-\left[\frac{\beta(t)}{\sin\pi\alpha(t)}\right]_R \simeq \frac{3(t-4)}{\pi} \times \int_4^\infty dt' \frac{[(t'-4)/t']^{\frac{1}{2}}|A_1(t')|^2}{(t'-4)(t'-t)}. \tag{20}$$

Again, Eq. (20) may be continued to the negative t region until α becomes substantially different from one. A test of joining $(\beta/\sin\pi\alpha)_R$ to a constant is made at the point where α is joined to a constant. As is expected, $B_1(s)$ is again insensitive to such maneuverings.

Before we write down the final formula for $B_1(s)$, we note that there is one slight complication in the Regge pole term due to the Pauli principle. The first P_α term in (6) is regular for negative t but the second P_α term has a branch point at $t=-(s-4)$ with a discontinuity of $-2i\sin\pi\alpha P_\alpha(-1-2s/(t-4))$ in the range $-(s-4)\leq t\leq 4$. In order to obtain $A_R^{(1)}(t,s)$ from (6), we must remove this branch point while maintaining the given branch cuts of α and $(\beta/\sin\pi\alpha)_R$ along the positive t-axis $(t\geq 4)$.

The method of removing the branch cut of P_α is shown below. The result is unique to within a subtraction constant which, in any case, gives no contribution to $B_1(s)$.

$$A_R^{(1)}(t,s) = \frac{3}{2}\left[\frac{\beta(t)}{\sin\pi\alpha(t)}\right]_R \left[P_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right) - P_{\alpha(t)}\left(1+\frac{2s}{t-4}\right)\right] - \frac{3}{2\pi}\int_{-s+4}^{4} dt' \left[\frac{\beta(t')}{\sin\pi\alpha(t')}\right]_R \times \frac{\sin\pi\alpha(t')P_{\alpha(t')}(-1-2s/(t'-4))}{(t'-t)}. \tag{21}$$

[6] M. Ross and G. Shaw, Ann. Phys. (New York) 13, 147 (1961).

In practice, the integral term gives very small contribution to $B_1(s)$ due to the factor $\sin[\pi\alpha(t')]$.

Finally, we substitute Eqs. (20) and (21) into Eq. (4) and obtain

$$B_1(s)=-\frac{1}{s-4}\int_0^{-(s-4)} dt\, P_1\left(1+\frac{2t}{s-4}\right)\times$$
$$\left\{\frac{3(t-4)}{\pi}\int_4^\infty dt' \frac{[(t'-4)/t']^{\frac{1}{2}}|A_1(t')|^2}{(t'-4)(t'-t)}\right.$$
$$\times\left[-\tfrac{1}{2}P_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right)+\tfrac{1}{2}P_{\alpha(t)}\left(1+\frac{2s}{t-4}\right)\right]$$
$$-\frac{1}{2\pi}\int_{-s+4}^{4} dt' \frac{\sin\pi\alpha(t')P_{\alpha(t')}(-1-2s/(t'-4))}{t'-t}$$
$$\left.\times\frac{3(t'-4)}{\pi}\int_4^\infty dt''\frac{[(t''-4)/t'']^{\frac{1}{2}}|A_1(t'')|^2}{(t''-4)(t''-t')}\right\}. \quad (22)$$

An over-all factor of 2 has been inserted to account for the contribution from the u channel.

It is easily seen that Eq. (22) reduces to Eq. (5) when α is replaced by one. In fact, $B_1(s)$ calculated by (22) does not differ substantially from the $\alpha=1$ case for small values of s. However, $B_1(s)$ given by Eq. (22) vanishes at infinity in contrast to Eq. (5) which produces a logarithmatic divergence. It is the convergent asymptotic behavior which allows a solution of the dispersion integral equation (2).

Although the explicit formulas we used for $\alpha(t)$ and $[\beta(t)/\sin\pi\alpha(t)]_R$ are by no means exact, we would like to stress that $B_1(s)$ will approach zero for large s as long as $\alpha(t)<1$ for $t<4$. This feature of $\alpha(t)$ is quite independent of the details of our derivation. Thus, in principle, Regge's representation of the pion-pion $I=1$ resonance gives an exchange force in the pion-pion system without producing a divergent result.

III. NUMERICAL RESULTS

At present, we would consider Eq. (22) as a semi-phenomenological representation for $B_1(s)$ and solve Eq. (2) using this $B_1(s)$. Aside from the question of principle discussed above, we would like to point out some practical advantages of the present representation over the conventional cutoff theory. Firstly, the result of the cutoff theory depends strongly on both the position and the shape of the cutoff; whereas there is only one corresponding parameter C in the present formula. Secondly, the parameters in the cutoff theory vary over a very wide range, whereas the order of magnitude of C here is restricted by the uncertainty principle argument.

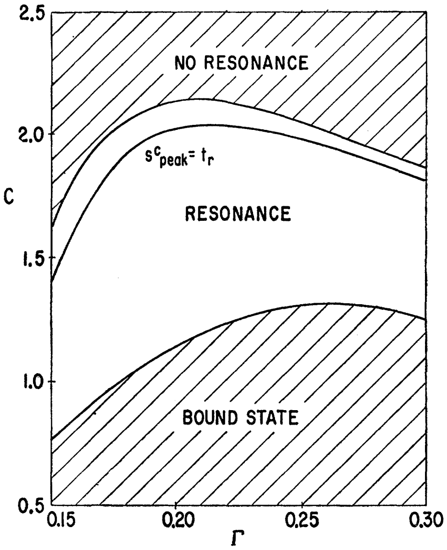

FIG. 1. Qualitative features of solutions to Eq. (2) are shown as a function of the width of the exchanged resonance (Γ) and the dimensionless parameter C. The curve inside the "resonance" region corresponds to values of Γ and C such that the calculated resonance energy in the s channel coincides with the energy of the resonance exchanged. The calculated *half*-width ($\frac{1}{2}\Gamma_{cal}$) on the low-energy side is approximately 0.2 along the curve. The half width on the high-energy side is considerably larger. The boundary lines for "bound state" and "no resonance" curve downward for large Γ due to the increasing slope of $\alpha(t)$.

In the numerical calculation of $B_1(s)$, we use an effective range formula for the absorptive part of the P-wave amplitude in the region $t'\geq 4$.[7]

$$\left(\frac{t'-4}{t'}\right)^{\frac{1}{2}}|A_1(t')|^2 \underset{t'\geq 4}{=} \mathrm{Im}\left\{\frac{(t'-4)\Gamma}{t_1-4-(t'-4)[1-L(t')]-i\Gamma(t'-4)[(t'-4)/t']^{\frac{1}{2}}}\right\},$$
$$L(t')=\frac{2}{\pi}\left(\frac{t'-4}{t'}\right)^{\frac{1}{2}}\ln[\tfrac{1}{2}(t'-4)^{\frac{1}{2}}+\tfrac{1}{2}t'^{\frac{1}{2}}]. \quad (23)$$

The two parameters t_1 and Γ are chosen to fit the observed data. We find that $t_1\simeq 25$ and Γ can vary between 0.15 and 0.30, all within the uncertainty of experimental results. (Note that t_1 is slightly below the resonance peak which is ~ 29.)

For each value of Γ, we calculate $\alpha(t)$ with C varied from 0.5 to 1.5. For each pair of Γ and C, $B_1(s)$ is calculated by Eq. (22), and Eq. (2) is solved for $A_1(s)$. The results are summarized in Fig. 1.

It is clear that the exchange of the $I=1$, $J\simeq 1$ resonance does produce a strong attractive force for the P-wave amplitude as long as $C\lesssim 2$. In fact, a P-wave resonance is found in the solutions corresponding to the

[7] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1961).

unshaded area in the figure. However, the calculated width of the resonance is in general broader than the observed resonance width by approximately a factor of 2. This discrepancy is not unexpected since we have not taken into account other forces in the problem such as the exchange of an $I=0$ or $I=2$ pair. Neither have we considered inelastic processes and the exchange of four pions, etc.

IV. REMARKS

At this stage, it is a matter of opinion whether Fig. 1 shows sufficient evidence that the exchange of the $I=1$ resonance is the primary mechanism by which the resonance itself is produced. A detailed study of the $I=0$, 2 forces as well as inelastic processes and the exchange of four pions may be necessary before we can arrive at a definite conclusion. The main point of this paper, however, is to demonstrate the calculation of scattering amplitudes using dispersion relations with Regge's description of composite systems. One could also regard the convergent result obtained here as a hint to the renormalization of a composite-vector field theory. Incidentally, we find it somewhat amusing that the crude formula for $\alpha(t)$ does predict the right order of magnitude for C. In fact, the range of C for producing resonances is much closer to unity than we could expect.

A number of problems including πK, πN, and NN scattering are being studied along the same line by a group in La Jolla. We feel that our present program is at least a more satisfactory semiphenomenological approach to the scattering problems compared with the cut off type methods. By calculating $B_1(s)$ directly from the crossing condition, we also avoided the evaluation of the "left-hand cuts" in the usual-dispersion calculations of partial wave amplitudes.[8] In fact, Regge's representation produces a divergent oscillating left-hand cut for $B_1(s)$. A complicated subtraction procedure would be needed if we were following the conventional procedure of first calculating the left-hand cut and then solving the N/D equations. We believe that our present program is much simpler and the crossing condition on physical amplitudes is somewhat more direct.

The CDC-1604 computer of the University of California at San Diego was used to obtain the numerical results in Fig. 1.

[8] The author has benefited by a discussion with G. F. Chew concerning this point.