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Proof of Partial-Wave Dispersion Relations in Perturbation Theory

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The partial-wave dispersion relation for the nucleon-nucleon scattering is proved in every order of perturbation theory. A partial-wave dispersion relation for the pion-nucleon scattering is proved only for the graphs in which the two external pion lines are attached to the nucleon open polygon.

DISPERSION relations for partial-wave amplitudes have usually been derived under the assumption of the Mandelstam representation,[1] whose validity is very questionable. In a previous work,[2] instead of the Mandelstam representation we have proved in every order of perturbation theory that the following integral representation holds for the scattering amplitudes in many practical cases:

$$\int_0^\infty d\alpha \int_0^1 dz \frac{\rho_{12}(\alpha,z)}{\alpha-zs-(1-z)t-i\epsilon} + \int_0^\infty d\beta \int_0^1 dz \frac{\rho_{23}(\beta,z)}{\beta-zt-(1-z)u-i\epsilon} + \int_0^\infty d\gamma \int_0^1 dz \frac{\rho_{31}(\gamma,z)}{\gamma-zu-(1-z)s-i\epsilon} \quad (1)$$

plus possible three single-dispersion terms, where s, t, and u are the three invariant squares. It is of some interest to see to what extent partial-wave dispersion relations can be proved from the integral representation (1).

Since the partial-wave dispersion relation for the pion-pion scattering is already proved,[3] we will first consider the nucleon-nucleon scattering.[4] In this case it was proved[2] that the weight functions ρ_{12}, ρ_{23}, and ρ_{31} vanish unless

$$\begin{aligned} &\alpha \geq 4M^2z+4\mu^2(1-z), \\ &\beta \geq \min[4\mu^2z+4M^2(1-z),\, 4M^2z+4\mu^2(1-z)], \\ &\gamma \geq 4\mu^2z+4M^2(1-z), \end{aligned} \quad (2)$$

respectively.

The invariant squares s, t, and u are expressed in terms of the 3-momentum, q, and the scattering angle, θ, in the center-of-mass system:

$$\begin{aligned} s&=4(q^2+M^2), \\ t&=-2q^2(1-\cos\theta), \\ u&=-2q^2(1+\cos\theta). \end{aligned} \quad (3)$$

The partial-wave amplitude $A_l(q^2)$ is defined by

$$A_l(q^2)=\frac{1}{2}\int_{-1}^{+1} d(\cos\theta) A(q^2,\cos\theta) P_l(\cos\theta), \quad (4)$$

where the scattering amplitude $A(q^2,\cos\theta)$ is given by (1) with (3). Therefore, $A_l(q^2)$ is represented as

$$A_l(q^2)=\int_0^1 dy\, P_l(1-2y) \int_0^1 dz \left[\int d\alpha \frac{\rho_{12}}{D_{12}-i\epsilon} + \int d\beta \frac{\rho_{23}}{D_{23}-i\epsilon} + \int d\gamma \frac{\rho_{31}}{D_{31}-i\epsilon}\right], \quad (5)$$

where

$$y=(1-\cos\theta)/2 \quad (6)$$

and

$$\begin{aligned} D_{12}&=\alpha-4zM^2-4\{z-y(1-z)\}q^2, \\ D_{23}&=\beta+4\{yz+(1-y)(1-z)\}q^2, \\ D_{31}&=\gamma-4(1-z)M^2-4\{(1-z)-(1-y)z\}q^2. \end{aligned} \quad (7)$$

From (2) we see that the denominator functions D_{ij} are non-negative definite for $-\mu^2 \leq q^2 \leq 0$. We, therefore, obtain[5] the partial-wave dispersion relation

$$A_l(q^2)=\frac{1}{\pi}\int_0^\infty d\lambda \frac{\mathrm{Im}A_l(\lambda)}{\lambda-q^2-i\epsilon} - \frac{1}{\pi}\int_{-\infty}^{-\mu^2} d\lambda \frac{\mathrm{Im}A_l(\lambda)}{\lambda-q^2+i\epsilon} \quad (8)$$

except for the contributions from the pole terms (one-pion exchange) which of course cause no trouble.

Next, we will consider the pion-nucleon scattering. The relations (3) are now replaced by

$$\begin{aligned} s&=[(q^2+M^2)^{\frac{1}{2}}+(q^2+\mu^2)^{\frac{1}{2}}]^2, \\ t&=-4yq^2, \\ u&=-4(1-y)q^2+[(q^2+M^2)^{\frac{1}{2}}-(q^2+\mu^2)^{\frac{1}{2}}]^2. \end{aligned} \quad (9)$$

Equations $D_{12}=0$, $D_{23}=0$, and $D_{31}=0$ have no complex

[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[2] N. Nakanishi, Progr. Theoret. Phys. (Kyoto) **26**, 337 (1961). Possibility of deriving partial-wave dispersion relations was pointed out in the added note of this paper.

[3] J. G. Taylor, Nuovo cimento **22**, 92 (1961); See also R. J. Eden, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), p. 219.

[4] After completion of this work, the author received a preprint of K. Yamamoto in which the same proof was given for the pion-pion and nucleon-nucleon scatterings. But he did not discuss the pion-nucleon scattering.

[5] See, for example, Theorem 14-1 of N. Nakanishi, Suppl. Progr. Theoret. Phys. (Kyoto) **18**, 1 (1961).

root if

$$\begin{aligned}
&[\alpha-2y(1-z)(M^2+\mu^2)]^2 \\
&\qquad +4y(1-z)[z-y(1-z)](M^2-\mu^2)^2\geq 0, \\
&\{\beta-2[yz+(1-y)(1-z)](M^2+\mu^2)\}^2 \\
&\qquad +4y(1-2z)[yz+(1-y)(1-z)](M^2-\mu^2)^2\geq 0, \\
&[\gamma-2(1-y)z(M^2+\mu^2)]^2 \\
&\qquad +4yz[(1-y)z-(1-z)](M^2-\mu^2)^2\geq 0,
\end{aligned} \tag{10}$$

respectively. Therefore, if the supports of the weight functions satisfy

$$\begin{aligned}
&\alpha\geq 4M^2(1-z), \\
&\beta\geq 4M^2 z, \\
&\gamma\geq M^2+\mu^2,
\end{aligned} \tag{11}$$

then we shall obtain a partial-wave dispersion relation, whose spectral function vanishes at least in a region[6]

$$-M^2<q^2<-\mu^2. \tag{12}$$

While the third inequality in (11) is always satisfied,[2] the other two inequalities in (11) are not always satisfied in general graphs. But we can easily prove that they are satisfied in the graphs in which the two external pion lines are attached to the nucleon open polygon. In order to prove the partial-wave dispersion relation for general graphs, it will be necessary to investigate detailed properties of the weight functions.

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[6] The region (12) corresponds to a circle $s=(M^2-\mu^2)e^{i\phi}$ on the s plane. If the Mandelstam representation is assumed, all complex singularities on the s plane lie on this circle [see S. W. MacDowell, Phys. Rev. **116**, 774 (1959)].